CHAPTER 4

Euclid Book I

Basic Geometry

4.1 The Overall Composition of Book I

Definitions 1–23	Basic concepts are described or defined
Postulates 1–5 Common Notions 1–5	Postulates and Common Notions are the axioms of plane geometry
1–26	A: Foundations of plane geometry without using parallels
27–32	B: The theory of parallel lines, angles in a triangle
33–45	C: The theory of parallelograms and their areas
46–48	D: The theorem of Pythagoras

B. Artmann, *Euclid—The Creation of Mathematics*

4.2 Definitions and Axioms

Euclid created the model of a mathematical text: Start with explicitly formulated definitions and axioms, then proceed with theorems and proofs. Unlike modern authors, who do not pretend to know what a set is, Euclid wants to say what he is talking about, or to give some sort of description of the objects of geometry. He does this in the first group of definitions, 1–9.

Definitions

Def. 1. *A point is that which has no part.*
Def. 2. *A line is breadthless length.*
Def. 3. *The extremities of a line are points.*
...
Def. 8. *A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*
Def. 9. *And when the lines containing the angle are straight, the angle is called rectilinear.*

It has often been observed that Euclid makes no use of these definitions in his subsequent proofs. They are explications that should clarify the significance of a term to the reader but play no formal rule in deductions. In Def. 8, the lines forming an angle may be curved. In Book III Euclid occasionally uses angles between circles and straight lines, but in our discussion of Proposition I.5 we will find indications for a greater popularity of angles between curved lines in pre-Euclidean times.

Most of the following definitions are abbreviations in the modern manner, for instance:

Definitions

Def. 19. ... *trilateral figures are those contained by three straight lines* ...
Def. 20. *Of trilateral figures, an equilateral triangle is that which has its three sides equal, an isosceles triangle which has two of its sides alone equal, and a scalene triangle that which has its three sides unequal.*

In a modern formal sense, an equilateral triangle is isosceles as well, but not so for Euclid. Similarly, in Def. 22, a rectangle (called "oblong" there) is not a square. This agrees with a more colloquial modern use: If a rectangle is spoken of, in most cases this means "not a square," because otherwise one could be more specific. Obviously, from a logical point of view, it is much more convenient to include the squares with the rectangles.

After the definitions, Euclid proceeds to state his famous postulates. Modern axioms of geometry resemble these postulates rather closely.

Postulates

1. *Let it be postulated to draw a straight line from any point to any point, and*
2. *to produce a limited straight line in a straight line,*
3. *to describe a circle with any center and distance,*
4. *that all right angles are equal to each other.*
5. *[The parallel postulate will be discussed below]*

Today Postulates 1 and 2 would be expressed in a way like "given any two distinct points, there is a unique line passing through them." Euclid's emphasis is more on construction than on "existence," more a difference in style than in substance.

The geometric postulates are followed by what Euclid calls "common notions." These are axioms about the behavior of general magnitudes, not only geometric objects.

1. *Things equal to the same thing are also equal to one another.*
2. *If equals are added to equals the wholes are equal.*
3. *If equals are subtracted from equals the remainders are equal.*
4. *Things which coincide with one another are equal to one another.*
5. *The whole is greater than the part.*

Many authors have noted the incompleteness of Euclid's axioms in comparison to modern foundations of geometry. The most obvious point is the absence of any thought of the ordering of points on a line or the concept of "betweenness." Euclid uses all assertions about ordering on an intuitive basis. These objections concern relatively minor points and do not in any way diminish Euclid's basic achievement: In mathematics, one has to start from explicitly stated first principles and deduce all following assertions from these principles. (About ordering, see Notes. About modern axiomatics, see Hartshorne [2000].)

Historically, the idea of stating axioms seems to be rather new in Euclid's time when compared to the definitions. In a very thorough and penetrating investigation, Mueller [1991b] examines the starting points of mathematical theories as preserved in the writings of Plato, Aristotle, and Euclid. Mueller summarizes on p. 63:

> However, if we look at the *Elements*, although we find at the beginning of book 1 definitions, postulates, and common notions ... at the beginning of the remaining books we find only definitions. I believe there are two related inferences we can draw from this: (1) Euclid did not believe that proportion theory, number theory, or solid geometry required its own postulates; (2) at the end of the fourth century there were no accepted presentations of these theories which included postulates, and probably no such presentations at all, presumably because no mathematician recognized the need for them. A further inference I draw is that the idea of such presentations of any mathematical theory was relatively new in Euclid's time, i.e., did not precede Plato's maturity. I believe the evidence suggests that Euclid himself is responsible for the postulates, but for the moment I will only say that, even if they are thought to predate, say, Plato's *Republic*, they should still be seen as the exception rather than the rule by Euclid's time.
>
> The rule in the *Elements* and, I am suggesting, earlier in the history of Greek mathematics is a theory, the only explicit starting points of which are definitions. These definitions are, for the most part, either explications, which perhaps clarify the significance of a term to the reader but play no formal role in subsequent arguments, or abbreviations in the modern manner.

In spite of this rather diverse historical picture, Euclid's axioms have been of utmost importance for the development of mathematics because, as it was said at the beginning, with them he created the model of a mathematical theory.

4.3 Book I, Part A: Foundations

The essential contents of Part A of Book I are first the basic congruence theorems and elementary constructions such as bisecting angles and segments, and second some propositions about "greater" relations of angles and sides of triangles, based on I.16 and culminating with the triangle inequality I.20.

The very first propositions show how to construct an equilateral triangle and how to copy segments without moving them. The delicate constructions in I.2, 3 are based directly on the Postulates 1, 2, and 3. Proposition I.4 is the first substantial theorem, the congruence theorem "side–angle–side," for short, SAS. Euclid states it like this:

Prop. I.4.
If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend.

For the proof see Fig. 4.1:

Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE and AC to DF, and the angle BAC equal to the angle EDF.

I say that the base BC is also equal to the base EF, the triangle ABC will be equal to the triangle DEF, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal side subtend, that is, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

Before looking into the proof, we observe some peculiarities of Euclid's style. He always states his theorems in two ways: at first in general words, and then a second time in a more specific way indicating points, lines, angles, and so on by various letters.[1] Very often

[1]This is very much like today's usage: Theorem: A continuous real function maps closed intervals onto closed intervals. Let $[a, b]$ be a closed interval and $f : [a, b] \to \mathbb{R}$ be continuous ...

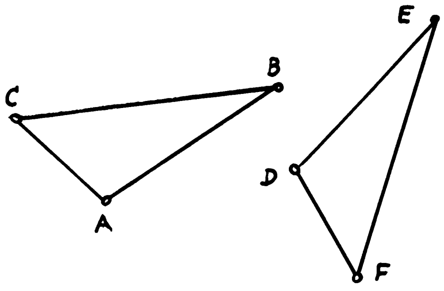

FIGURE 4.1

the theorem is accompanied by a suitable diagram. One particular phrase needs explanation: *"The triangle ABC will be equal to the triangle DEF."* This is clarified by later use of the same expression: It simply means "the triangles have equal areas." Euclid uses the word "area"(or its Greek equivalent) only occasionally.[2]

The Proof of Prop. I.4

The method of proof of I.4 stands in strong contrast to the meticulous constructions in I.1–3. Euclid just takes the triangle ABC and superimposes it on triangle DEF in a such way that A is placed on D, B on E, and C on F. From this he easily derives his assertions.

On the one hand, this method of superposition clearly has no basis in Euclid's axioms, but on the other hand, practically nothing can be done in elementary geometry without the congruence theorems. (For the congruence theorem side-side-side, SSS, in I.8 he uses the same method.) In fact, what we see here is another axiom. Modern axiomatic studies by Hilbert and others have shown that there is no way to resolve this dilemma: Either SAS has to be used as an axiom or one has to use superposition in a modern version by postulating the existence of certain rigid motions of the plane. (For more details, see Hartshorne [2000], Section 17, esp. p. 154.)

[2]The Greeks knew perfectly well how to measure their properties, and they knew that Pharaoh's taxation office measured the fields of the Egyptian peasants to their disadvantage. In mathematics they avoid the concept of "area," using instead phrases like the one above, e.g., "this rectangle is equal to that rectangle" and similarly.

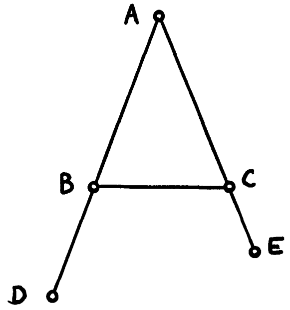

FIGURE 4.2

In the next pair I.5/6 of propositions, Euclid proves a fundamental lemma about isosceles triangles that is used frequently in Books I–VI. In the statement and proof of I.5 we ignore Euclid's assertion about outer angles. I.6 is the converse of I.5.

Prop. I.5

In isosceles triangles the angles at the base are equal to one another.

Let ABC be an isosceles triangle having the side AB equal to the side AC; and let the straight lines BD, CE be produced further in a straight line with AB, AC.

I say that the angle ABC is equal to the angle ACB and the angle CBD to the angle BCE [Fig. 4.2].

Prop. I.6.

If in a triangle two angles are equal to one another, the sides which subtend the equal angles will also be equal to one another.

For the proof of I.5, Euclid first constructs two auxiliary triangles *BFC* and *CGB* [Fig. 4.3]:

Let a point *F* be taken at random on *BD*; from *AE* the greater let *AG* be cut off equal to *AF* the less; and let the straight lines *FC*, *GB* be joined.

In the next two steps he first shows the congruence of the triangles $\triangle AFC$ and $\triangle AGB$ by using SAS, and then again by SAS the congruence $\triangle BFC \cong \triangle CGB$:

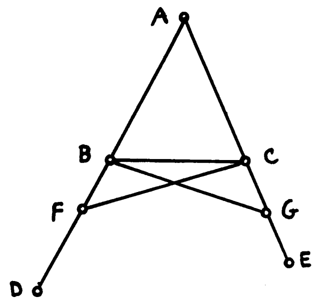

FIGURE 4.3

(1) We have $\angle FAC = \angle GAB$, and $AF = AG$ by construction, and $AC = AB$, hence $\Delta AFC \cong \Delta AGB$, and especially $BG = CF$ and $\angle BFC = \angle CGB$.

(2) From the construction we get $BF = CG$; moreover, we know from the outset $BC = CB$, and from (1) we have $\angle FBC = \angle CGB$; hence $\Delta CGB \cong \Delta BFC$ by SAS.

Now Euclid concludes:

> Therefore the angle *FBC* is equal to the angle *GCB*, and the angle *BCF* to the angle *CBG*. Accordingly, since the whole angle *ABG* was proved equal to the angle *ACF*, and in these the angle *CBG* is equal to the angle *BCF*, the remaining angle *ABC* is equal to the remaining angle *ACB*; and they are at the base of the triangle *ABC*. Q.E.D.

All the steps of this proof are justified by Euclid's axiomatic base. (For details, see the analysis in Hartshorne [2000], Section 10.)

We will direct our attention to another question, which has frequently baffled students of Euclid: How can anybody understand the introduction of his auxiliary points, lines, and triangles at the beginning of his proof?

In this particular instance, I.5, we are in the lucky position of having a historical predecessor of Euclid's proof that explains the initial construction. It comes from Aristotle's *Prior Analytics*, we quote it from Heath's commentary on I.5. It makes use of mixed angles between circular arcs and straight lines in the following way: (a) the angles of semicircles (called AC and BD), that is, between a diameter and the circumference, are equal, and (b) the two angles in a segment, that is, between a chord and the circumference, are equal. Aristotle uses the proof in his discussion of some logical points. (See Fig. 4.4.)

> For let A, B be drawn [i.e. joined] to the center.
> If then, we assumed (1) that the angle AC is equal to the angle BD without asserting generally that the angles of semicircles are equal, and again (2) that the angle C is equal to the angle D without making the further assumption that the two angles of all segments are equal, and if we then inferred, lastly, that, since the whole angles are equal, and equal angles are subtracted from them, the angles which remain, namely E, F are equal. We should commit a petitio principii, unless we assumed [generally] that, when equals are subtracted from equals, the remainders are equal. (*Prior Analytics* 41 b 13–22)

First observe a basic similarity in Aristotle's and Euclid's proofs: We have two equal big angles, from which two smaller equal angles are subtracted, resulting in the desired equality of the base angles.

In Euclid's time mixed angles were no longer acceptable; he does not use them save on a few minor occasions in Book III. A transition from Aristotle's proof to one without mixed angles can be explained in a plausible way. (We use Euclid's notation; see Fig. 4.5)

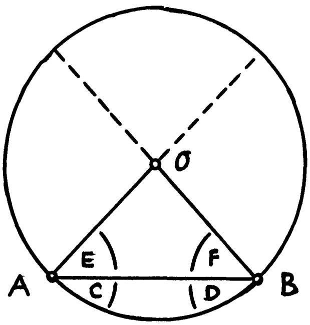

FIGURE 4.4

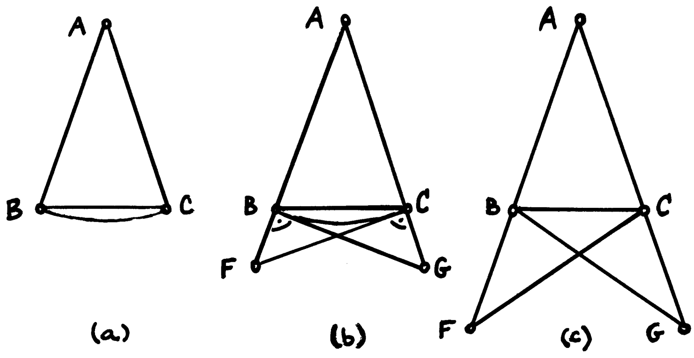

FIGURE 4.5

First replace the "angle of the semicircle" by the right angle between the radius AB and the tangent BG, similarly for AC and CF. The congruence theorem ASA (With α, side AC = side AB, and the right angles) would be needed in order to show $\triangle ABG \cong \triangle ACF$. Hence the "big" right angles $\angle ABG$ and $\angle ACF$ are equal, and the result would follow as in Euclid's proof. Note that the symmetrically situated small triangles $\triangle BFC$ and $\triangle CBG$ replace the intuitively symmetric segment.

However, Euclid cannot use tangents and ASA at this stage of Book I, so he disposes with the right angles between radii and tangents and cleverly provides himself with the equal sides AF and AG in a direct way so that he can use SAS instead of ASA. (E and D are merely auxiliary points for the prolongation of the sides.)

Here, I think, we have found a natural explanation for Euclid's construction, albeit a hypothetical one. Observe that Fig. 4.5 (b) appears in Book III.17 where Euclid constructs the tangent to a circle and, in a less obvious way, in I.2 as well.

Aristotle lived 384–322 and was a member of Plato's Academy 367–348 when Plato died. It seems very likely that he got his mathematical education in the Academy, and so it is possible that in looking at his proof we see a small fragment of Leon's "Elements," the textbook of Plato's Academy.

Propositions 7–15. In Propositions 7 and 8 Euclid proves the congruence theorem side–side–side (SSS), using the method of superposition for the second time. Propositions 9–15 are devoted

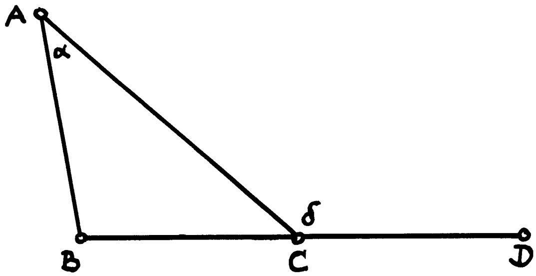

FIGURE 4.6

to the common auxiliary constructions and initial propositions of plane geometry: bisecting angles and segments, constructing perpendiculars, supplementary and vertical angles.

Prop. I.16.
If one of the sides of any triangle is produced, the exterior angle is greater than each of the interior and opposite angles.

Claim. angle α < angle δ (Fig. 4.6).

Construction. Bisect AC at E, draw BE and extend it to F such that $BE = EF$, join C and F, let $\alpha' =$ angle ECF (Fig. 4.7).

Proof

(i) Triangle ABE is congruent to triangle CFE by the congruence theorem SAS. Hence $\alpha = \alpha'$.

(ii) But α' is a part of δ. Hence $\alpha = \alpha' < \delta$ by common notion 5, Q.E.D.

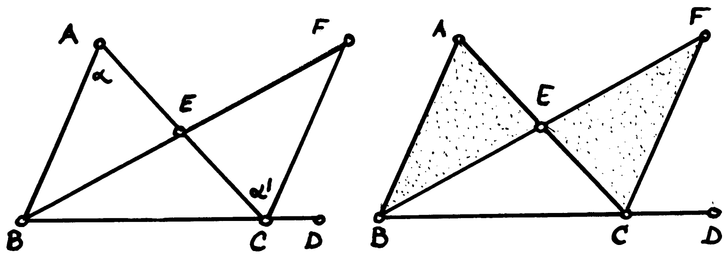

FIGURE 4.7

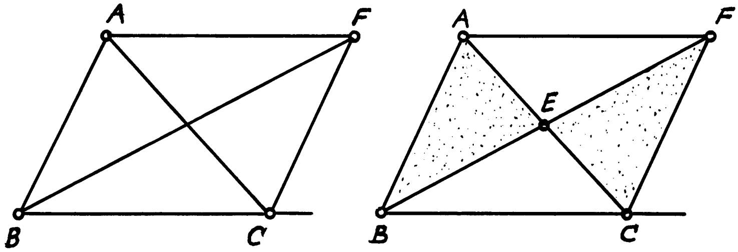

FIGURE 4.8

If Euclid had the theory of parallels at his disposal right here, the claim of I.16 would be a trivial consequence of I.32 about the sum of the angles in a triangle. He explicitly states in I.32 that the exterior angle is the sum of the two interior and opposite angles. Hence we see a conscious composition at work. Before discussing this, we will try to understand the genesis of the proof of I.16 with the help of parallels.

What can be said about this proof? It is ingenious, and one can see how its author hit upon his idea: Just add the line AF to the figure (Fig. 4.8).

All of a sudden, we see a parallelogram $ABCF$ "behind" the proof of I.16. At this stage, we may use parallels and have $\alpha = \alpha'$ because of alternate angles; AC is a transversal of the two parallel lines AF and BC. Furthermore, E will be the intersection of the diagonals of this parallelogram.

However, and this is the essential idea, in order to prove I.16 it is possible to avoid parallels and use the congruence theorem 1, 4 instead.

Further evidence of the mathematical competence of the author of I.16 is his ability to connect I.16 with its consequences, the important theorems I.20, the triangle inequality, and I.27, the existence of parallels. (After all, deductive structures are what mathematics is all about.)

On the other hand, there is a weak spot in the proof. The assertion "α' is a part of δ" has no base in Euclid's axioms. It is just read off from the diagram. This has often been observed: Compare, for instance, Heath's commentary. (Heath confuses the "Riemann hy-

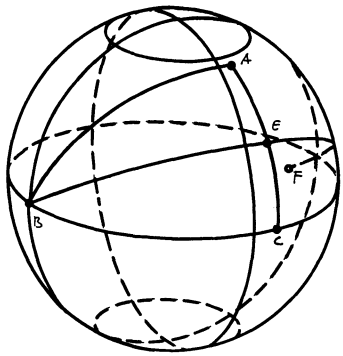

FIGURE 4.9

pothesis" with Riemannian non-Euclidean geometry, but otherwise he is mathematically correct.) Proposition I.16 is not true in the so-called elliptical (spherical) geometries, which satisfy all of Euclid's axioms except the parallel postulate. A counterexample to I.16 is easily drawn on the sphere. In Fig. 4.9, supplied by E. Hartmann, the point F will be in the southern hemisphere; hence $\alpha' > \delta$.

It should, however, be clear that any Greek mathematician would reply to this objection that he was dealing with plane, not spherical, geometry. Certainly a man like Menelaus of Alexandria (about 100 C.E.), who wrote about spherical geometry, knew the phenomenon. It seems that nobody noted the error before the end of the nineteenth century, when non-Euclidean geometries and order-relations in geometry came to the attention of mathematicians. The likely reason for Euclid's neglect of questions about the ordering of points on a line (or betweenness) may be that he regarded it as a part of logic—or just took it for granted. In fact, I.16 remains valid in the second class of non-Euclidean geometries, the so-called hyperbolic geometries, which can be defined over ordered fields. The reader interested in more details about order relations in geometry should consult Hartshorne, Chapters 1.3 and 3.15.

Propositions I.17–20. Proposition I.17 is a direct consequence of I.16. It is again a weak variant of I.32 about the sum of the angles in a triangle:

Prop. I.17.
In any triangle two angles taken together in any manner are less than two right angles.

Proposition I.18 says that in any triangle the greater side subtends the greater angle, and I.19 is its converse. These propositions lead to

Prop. I.20.
In any triangle two sides taken together in any manner are greater than the remaining one.

This is the famous triangle inequality. Proclus comments on this:

> The Epicureans are want to ridicule this theorem, say it is evident even to an ass and needs no proof ... they make [this] out from the observation that, if hay is placed at one extremity of the sides, an ass in quest of provender will make his way along the one side and not by way of the two others. (Proclus–Morrow p. 251)

(The Epicureans of today might as well add that one could see the proof on every campus where people completely ignorant of mathematics traverse the lawn in the manner of the ass.) Proclus replies rightly that a mere perception of the truth of a theorem is different from a scientific proof of it, which moreover gives reason why it is true. (Which means, to embed it into a deductive structure.) In the case of Euclid's geometry, the triangle inequality can indeed be derived from the other (equally plausible) axioms. On the other hand, the Epicureans win in the modern theory of metric spaces, where the triangle inequality is the fundamental axiom of the whole edifice.

Propositions I.21–26. Three of the remaining propositions of part A are about "greater" relations for sides and angles of triangles (21, 24, 25). Prop. 22 gives the construction of a triangle from its sides, provided that the triangle inequality is valid. Using this, Euclid shows in Prop. 23 how to copy an angle. The combined congruence theorems *ASA* and *AAS* are tagged on in I.26 as a sort of loose end.

4.4 Book I, Part B: The Theory of Parallels

Euclid defines in *Def.* I.23:

> *Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

For short, parallels in a plane are nonintersecting straight lines. This has remained unchanged in the modern theory of incidence geometry (cf. Hartshorne, [2000], Section 6 and many other places). In so-called affine planes, parallel lines are those that have no point in common. The modern parallel axiom in an affine plane is this:

> *Given a line g and a point P not on g, there exists one and only one line h passing through P that does not meet g.*

This axiom really has two parts:

(1) The parallel h to g through P exists.

(2) It is unique ("only one line").

(Part (2) is sometimes called "Playfair's axiom.")

Euclid's geometry is richer than the theory of affine planes: He has the congruence axioms and—implicitly—the conditions of ordering and betweenness. Via ordering he got I.16 and from this he derives the "existence" part for parallels in I.27. For uniqueness, he has to introduce a special axiom, the famous parallel postulate. He uses it in I.29 in order to prove a property of parallels that immediately provides uniquenes (without saying so). We quote Postulate 5 below, but first I.27 together with its proof.

Prop. I.27
If a straight line falling on two straight lines makes the alternate angles equal to one another, the straight lines will be parallel to one another.

For let the straight line *EF* falling on the two straight lines *AB*, *CD* make the alternate angles *AEF*, *EFD* equal to one another [Fig. 4.10]; I say that *AB* is parallel to *CD*.

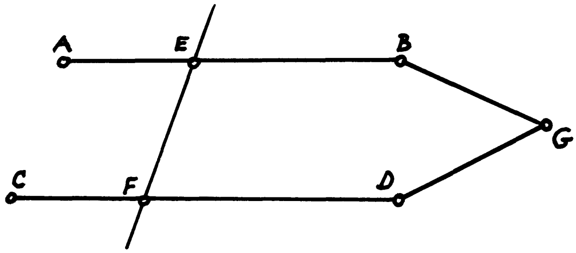

FIGURE 4.10

For, if not, AB, CD when produced will meet either in the direction of B, D or towards A, C.
Let them be produced and meet, in the direction of B, D at G.
Then, in the triangle GEF, the exterior angle AEF is equal to the interior and opposite angle EFG:
which is impossible. [I.16]
Therefore AB, CD when produced will not meet in the direction of B, D.
Similarly it can be proved that neither will they meet towards A, C.
But straight lines which do not meet in either direction are parallel; [Def. 23]
therefore AB is parallel to CD.

Before going on we introduce some convenient notation:

$g \parallel h$ for parallel lines g and h

and

$2R$ for two right angles (or 180°).

Proposition 28 is a useful variant of 27. It says, with the notation taken from Fig. 4.11,

$$\alpha = \gamma \Longrightarrow g \parallel h,$$
$$\beta + \gamma = 2R \Longrightarrow g \parallel h.$$

For the proof of Prop. 29, we need the parallel postulate.

Postulate 5. Let it be postulated:

That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines,

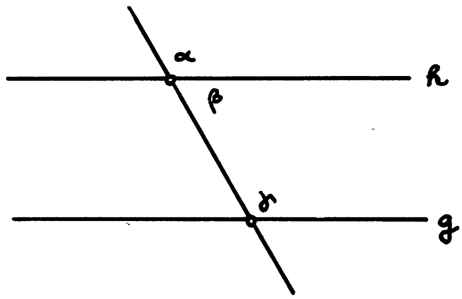

FIGURE 4.11

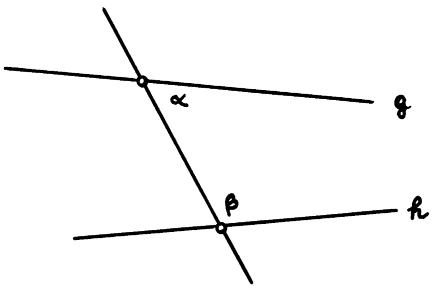

FIGURE 4.12

if produced indefinitely, meet on that side on which are the angles less than the two right angles. [Fig. 4.12]

Prop. I.29.
A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles.

We will abbreviate the proof by using the notation from Fig. 4.13(b) and dealing with the main case of alternating angles α, β only.

The claim is:

$$\text{If} \qquad g \parallel h, \qquad \text{then} \qquad \alpha = \beta.$$

This is the converse of I.27.

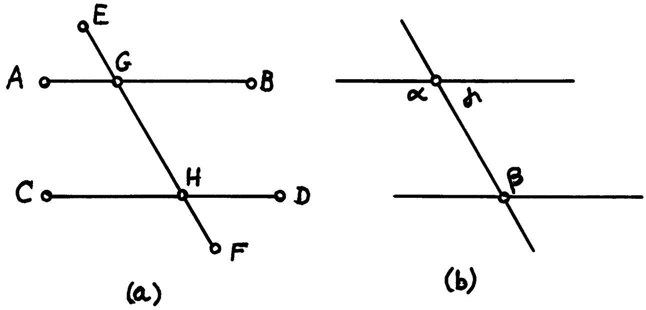

FIGURE 4.13

The proof is by contradiction, or rather by proving the logically equivalent statement

$$\text{If not} \quad \alpha = \beta, \quad \text{then not} \quad g \parallel h.$$

If not $\alpha = \beta$, then one of them is greater, say $\alpha > \beta$:

$$\beta < \alpha \Longrightarrow \beta + \gamma < \alpha + \gamma.$$

But

$$\alpha + \gamma = 2R,$$

whence

$$\beta + \gamma < 2R = \alpha + \gamma.$$

Now, Postulate 5 says that g and h have to meet, that is,

$$\beta + \gamma < 2R, \quad \text{and Post. 5} \quad \Longrightarrow \text{not } g \parallel h.$$

(If you want to see a contradiction like the one in Euclid's proof, continue: but $g \parallel h$ by hypothesis....)

Propositions 27 and 29 together give us the fundamental property of parallels (*PP*), again with the notation of figure 13(b):

$$(PP) \qquad g \parallel h \Longleftrightarrow \alpha = \beta \qquad \text{(for alternating angles).}$$

$g \parallel h$ has the intuitive meaning "g and h have no common point," a property that has to be checked from here to infinity. On the other side, $\alpha = \beta$ can be verified locally and is a most useful practical device that connects the intuitive notion with the other concepts of congruence geometry. This is a very common feature of mathematics: Define a concept by an intuitive meaning (if possible), and then prove that this is equivalent to a technically useful other statement. Another striking example of this is the definition of a tangent to a circle in Book III. In modern mathematics, unfortunately, the technical devices are often put in the foreground at the expense of the intuitive meaning.

Proposition 30 shows the transitivity of parallelism, and Propostion 31 exploits (*PP*) for the construction of parallels by alternate angles.

Prop. I.32.
In any triangle, if one of the sides is produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.

Again, we will take the convenient notation from Fig. 4.14 for Euclid's proof. Prolong the line BC to the point D. Let EC be parallel to AB. Then, by Prop. 29, we have

$$\alpha = \alpha' \qquad \text{and} \qquad \beta = \beta'.$$

Hence the exterior angle $\alpha' + \beta'$ is equal to $\alpha + \beta$, and because of $\gamma + \alpha' + \beta' = 2R$, we have

$$\alpha + \beta + \gamma = 2R.$$

Euclid emphasizes the exterior angle $\alpha' + \beta'$ because he will use it on several later occasions.

The sum of the angles of a triangle is the most important and fundamental invariant in elementary geometry. No matter what the

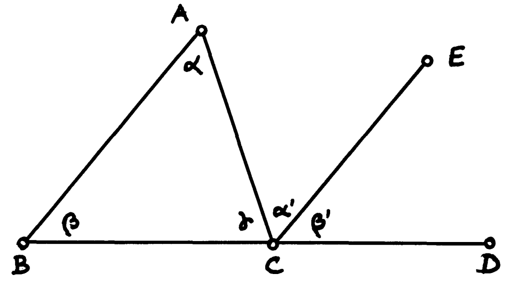

FIGURE 4.14

shape of the triangle is, its angles will invariably add up to two right angles (or 180 degrees, or π). This is used so often that one is prone to forget its significance. One first immediate consequence is the formula for the sum of the (interior) angles of a convex polygon. If it has n vertices, it can be dissected into $n-2$ triangles and has $(n-2)2R$ (or, expressed in another way, $(n-2)\pi$) as the sum of its angles. Proclus proves this in his comment on I.32. (Proclus–Morrow p. 301.) Proclus proceeds (p. 302) to state that "the property of having its interior angles equal to two right angles is an essential property of the triangle as such." He refers to Aristotle for the meaning of "essential property." In the words of today this means that the triangle is characterized (among convex polygons) by the sum of its angles: A convex polygon is a triangle if and only if the sum of its interior angles equals two right angles. Theorem I.32 has played its role in philosophy later on as well. For Immanuel Kant it is the quintessential example of what he calls "a synthetic a priori judgment," that is, a statement of absolute certainty (not depending on experience) that adds to our knowledge (*Critique of Pure Reason* B 744–746).

One of the far-reaching consequences of I.32 was found by Jacob Steiner (1796–1863). He used the formula $(n-2)\pi$ for the sum of the interior angles of a polygon for a simple proof of Euler's formula for convex polyhedra: If such a polyhedron has v vertices, e edges, and f faces, then "invariably"

$$v - e + f = 2.$$

Thus the simple invariant of triangles goes as far as proving one of the most important invariants of modern algebraic topology, the Euler characteristic, in its first significant case of convex polyhedra.

4.5 Book I, Part C: Parallelograms and Their Areas

In part C we find a systematic study of the interrelations between the concepts of "parallelism" and "of equal content."

Euclid defines various types of "quadrilaterial figures" in Def. 22 at the beginning of Book I, but not the parallelograms that figure so prominently in this section C. Instead, he introduces them together with their basic symmetry properties in Propositions 33 and 34.

Prop. I.33.
The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves equal and parallel.

Prop. I.34.
In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas.

In Prop. 34 Euclid speaks about halving the "area" of a parallelogram, but he does not use this word in the subsequent propositions, which are—in our understanding—equally statements about areas. In daily life, the Greeks measured their properties, and in fact the very word "geometry" means "measuring the fields." Measuring a field means attaching a number to it; it measures so and so many square feet. In mathematical language this amounts to a function that associates numbers to certain (polygonic) plane surfaces. But the concept of a function is alien to the *Elements*. Euclid does not use it, and moreover, he does not use any formulas that in effect would define functions. For a modern description of what Euclid does, we quote Hartshorne [2000], Section 3 (but see also Section 22) about Euclid's notion of "equal figures":

> So what did Euclid have in mind? Since he does not define it, we will consider this new equality as an undefined notion, just as the notions of congruence for line segments and angles were undefined. We will call this new notion equal content, to avoid confusion with other notions of equality or congruence. We do not want to use the word area, because this notion is quite different from our common understanding of area as a function associating a number to each figure.
>
> From the way Euclid treats this notion, it is clear that he regards it as an equivalence relation, satisfying the common notions. In particular
>
> (a) Congruent figures have equal content.

(b) If two figures each have equal content with a third, they have equal content.

(c) If pairs of figures with equal content are added in the sense of being joined without overlap to make bigger figures, then these added figures have equal content.

(d) Ditto for subtraction, noting that equality of content of the difference does not depend on where the equal pieces were removed.

(e) Halves of figures of equal content have equal content (used in the proof of I.37).

(f) The whole is greater than the part, which in this case means if one figure is properly contained in another, then the two figures cannot have equal content (used in the proof of I.39).

In terms of the axiomatic development of the subject, at this point Euclid is introducing a new undefined relation, and taking all the properties just listed as new axioms governing this new relation.

In the next propositions, 35–41, Euclid achieves more flexibility in handling the concept of equal content, or equality, as he says.

Prop. I.35.
Parallelograms which are on the same base and in the same parallels are equal to one another.

Prop. I.36.
Parallelograms which are on equal bases and in the same parallels are equal to one another [Fig. 4.15].

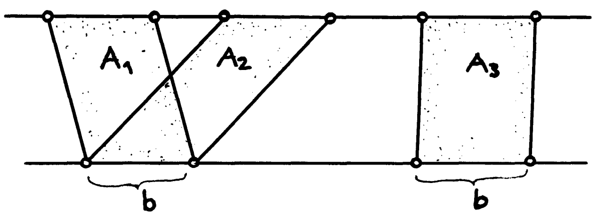

FIGURE 4.15

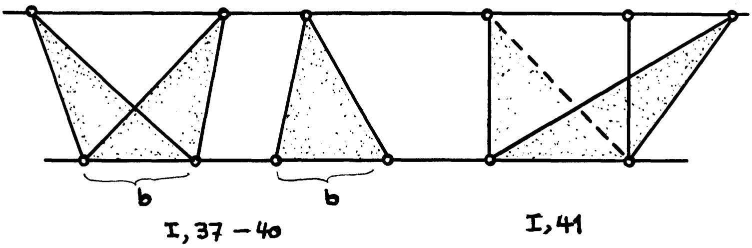

FIGURE 4.16

Propositions 37–40 say similar things for triangles, and Proposition 41 associates parallelograms and triangles. (The parallelogram situated as in Fig. 4.16 is the double of the triangle.)

At this point the theory of equal content branches out in two directions. The first branch leads directly to the theorem of Pythagoras (I.46–48), which in any case is a goal in its own right; and the second one leads via I.42–45 and the theorem of Pythagoras to the important result II.14: *It is possible to construct a square of content equal to that of any rectilinear figure.* Or shorter: Any rectilinear figure can be squared.

We stick to Euclid's sequence and discuss I.42–45, which will find their sequels in Book II.

Prop. I.42.
To construct, in a given rectilineal angle, a parallelogram equal to a given triangle.

The construction is easy enough, compare Fig. 4.17, where $\triangle ABC$ and the angle δ are given and E is the midpoint of BC.

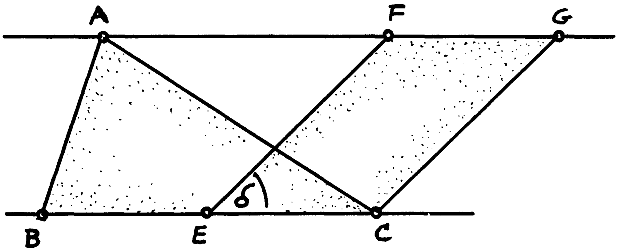

FIGURE 4.17

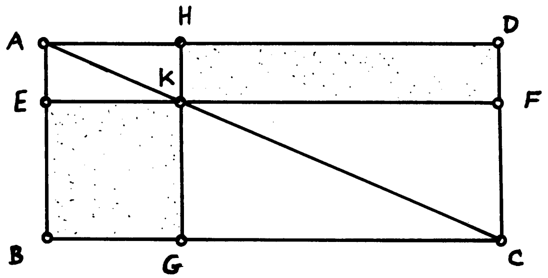

FIGURE 4.18

The "given angle" will be a right angle in Euclid's subsequent applications. So we might as well specialize it to this case in the next propositions. (The generalization from rectangles to parallelograms is easy enough because shears preserve areas.) We replace "parallelogram" by "rectangle" and "given angle" by "right angle" in Euclid's Props. I.43–45. Book II is about rectangles throughout.

Prop. I.43.
In any rectangle the complements of the rectangles about the diagonal are equal to one another.

Figure 4.18 shows a diagram that is used over and over again in the *Elements*. In several contexts Euclid simply calls it "the schema." The point K is on the diagonal of the rectangle $\square ABCD$, and the lines EF, GH are parallel to the sides. Euclid denotes $\square BGKE$ by BK, and $\square KFDH$ by KD. These latter rectangles are the "so-called complements." (The reader may want to look ahead at Props. VI.16, 24, 26.)

We have to prove:

$$K \text{ is on } AC \Longrightarrow \square BK \text{ and } \square KD \text{ are of equal content (are equal).}$$

By I.34, the triangle ΔABC is equal to ΔDAC. For the same reason, ΔGCK and ΔFKC as well as ΔAEK and ΔKHA are equal. Subtracting the two smaller triangles from the large one on each side of the diagonal gives the result.

Simple as it is, Prop. 43 has very many useful consequences. The next proposition is the first one. By "applying" a figure C to a line (segment) AB Euclid means to construct a rectangle with one side AB of equal content with figure C.

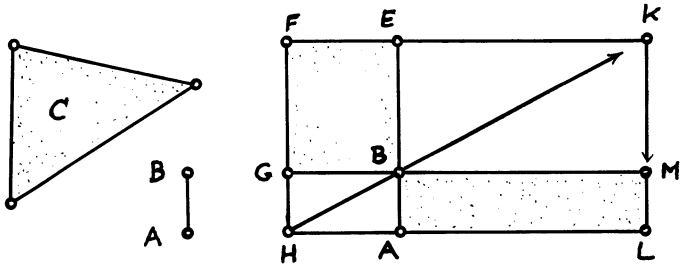

FIGURE 4.19

Prop. I.44.
To a given straight line to apply a rectangle equal to a given triangle.

Construction. Let ΔC and line AB be the given figures (Fig. 4.19). Construct *a* rectangle $\square BF$ of equal content with ΔC via I.42. Place AB so that it prolongs side EB and construct $\square BH$. Prolong FE and HB until they meet in K. (Euclid shows that they will meet by means of Post. 5.) Complete the figure as shown in Fig. 4.19. $\square BL$ has one side AB and is of equal content with $\square BF$ by Prop. I.43.

Prop. I.45.
To construct a rectangle equal to a given rectilinear figure.

For the given figure Euclid takes a quadrangle, dissects it into two triangles, and transforms these by Prop. I.44 into two rectangles with a common side. By combining the two rectangles with the common side he gets one rectangle as desired. The proof is done meticulously by justifying every single step.

In spite of the general assertion Euclid, selects a quadrangle for the proof. But the procedure is quite transparent, and it is obvious how to proceed in the general case. This way of handling proofs, which today might be done by mathematical induction, is quite typical for Euclid. We will see it now and then on other occasions.

4.5.1 Comment on Props. I.44/45

We will for a moment use modern formulas. The area A of a rectangle with sides (of length) a, b is given by $A = ab$. In I.44, let R be the given rectangle and a be the given side. In terms of these formulas, the problem of I.45 amounts to the solution of the linear equation

$$R = ax,$$

where x is the second side of the desired new rectangle. Seen this way, I.45 is algebra in geometric disguise, and hence it has been interpreted as "geometric algebra." Historians have said that this interpretation is not justified and an anachronism. Mathematicians have replied that the formulas represent an isomorphic image of the geometric situation and hence are the correct modern way of describing Euclid's procedures. The same problem arises again in Book VI, where the geometric equivalent of quadratic equations is treated. Because the geometric version is quite sufficient for the understanding of Euclid's text, we will leave the formulas aside. Occasionally we will use them in order to facilitate understanding for the modern reader.

4.6 Book I, Part D: The Theorem of Pythagoras

In proposition I.46 Euclid shows how to construct a square on a given line; I.47 is the famous theorem of Pythagoras, and I.48 is its converse. We quote I.47 and its proof verbatim.

Prop. I.47.
In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Let ABC be a right angled triangle having the angle BAC right;
I say that the square on BC is equal to the squares on BA, AC.
For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC; through A let AL be drawn parallel to either BD or CE, and let AD, FC be joined [Fig. 4.20].
Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

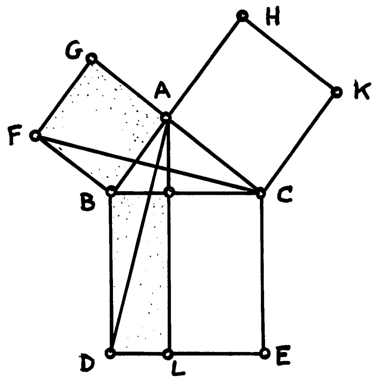

FIGURE 4.20

therefore *CA* is in a straight line with *AG*.
For the same reason

BA is also in a straight line with *AH*.

And, since the angle *DBC* is equal to the angle *FBA*: for each is right: let the angle *ABC* be added to each;
therefore the whole angle *DBA* is equal to the whole angle *FBC*.
And, since *DB* is equal to *BC*, and *FB* to *BA*, the two sides *AB*, *BD* are equal to the two sides *FB*, *BC* respectively, and the angle *ABD* is equal to the angle *FBC*;
therefore the base *AD* is equal to the base *FC*, and the triangle *ABD* is equal to the triangle *FBC*. [I.4]
Now the parallelogram *BL* is double of the triangle *ABD*, for they have the same base *BD* and are in the same parallel *BD*, *AL*. [I.4I]
And the square *GB* is double of the triangle FBC, for they again have the same base *FB* and are in the same parallels *FB*, *GC*. [I. 4I]
Therefore the parallelogram *BL* is also equal to the square *GB*.
Similarly, if *AE*, *BK* are joined, the parallelogram *CL* can also be proved equal to the square *HC*;
therefore the whole square *BDEC* is equal to the two squares *GB*, *HC*.
And the square *BDEC* is described on *BC*, and the squares *GB*, *HC* on *BA*, *AC*.
Therefore the square on the side *BC* is equal to the squares on the sides *BA*, *AC*.

Let us recapitulate the main points of the proof. The right angle at A guarantees that G, A, C are in a straight line parallel to FB. This is the decisive point. For the equality of ▭GB and ▭BL Euclid has to resort to their respective halves, $\triangle FBA$ (which is not shown) and $\triangle BDL$ (also not shown). By I.41 these are equal to (i.e., are of equal content with) $\triangle FBC$ and $\triangle BDA$. These two triangles are congruent by *SAS*, and we are done.

Comment There are many dozens of different proofs of Pythagoras's theorem. Proclus credits Euclid personally with this one. It is a marvellous piece of mathematics, and I personally like it better than any other proof. There is no special trick or need of a formula, one sees in such a clear way how the square □GB is transformed into the rectangle ▭BL, and in spite of its simplicity the argument is in no way trivial.

The theorem of Pythagoras is as fundamental for mathematics today as it was in Euclid's time. It is the progenitor of all the different kinds of metrics and of quadratic forms, and of theorems like $\sin^2\alpha + \cos^2\alpha = 1$. Via its generalization, the law of cosines, and the corresponding scalar product in vector spaces, it pervades mathematics as far as the eye can see.

Prop. I.48

If in a triangle the square on one of the sides is equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.

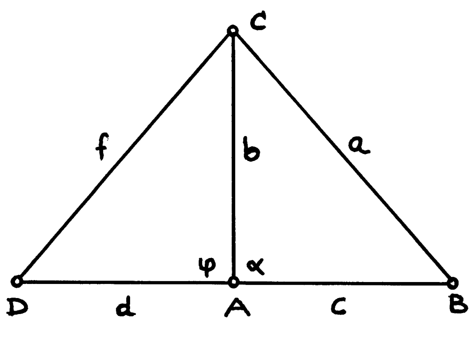

FIGURE 4.21

Proof
Let ΔABC be the given triangle. We take the notation from Fig. 4.21 and abbreviate. Let AD be equal to AB and φ be a right angle, then by I.47 we have $f^2 = d^2 + b^2 = c^2 + b^2$, which by assumption is equal to a^2. Hence f is equal to a. (Here is a little gap. This implication has not been proved before.) Now, by the congruence theorem SSS the two triangles ΔABC and ΔADC are congruent, and hence $\alpha = \varphi$ is a right angle.

Propositions I.47 and 48 combined are the full theorem of Pythagoras.

We conclude this chapter by quoting a fine sonnet by the German poet Adelbert von Chamisso, translated by Max Delbrück, together with a nice remark by C. L. Dodgson. According to a legend from antiquity, Pythagoras sacrificed a hundred (a hecatomb, or, in another version, only one) oxen to the gods after he had discovered his theorem. (The German original of the poem can be found in the notes.)

Adelbert von Chamisso: The Truth

(Translated by Max Delbrück)

The TRUTH: her hallmark is ETERNITY
As soon as stupid world has seen her light
PYTHAGORAS' theorem today is just as right
As when it first was shown to the FRATERNITY.

The GODS who sent to him this ray of light
to them PYTHAGORAS a token sacrificed:
One hundred oxen, roasted, cut, and sliced
Expressed his thank to them, to their delight.

The oxen, since that day, when they surmise
That a new truth may be unveiling
Forthwith burst forth in fiendish railing.

PYTHAGORAS forever gives them jitters –
Quite powerless to stem the thrust of such emitters
of LIGHT, they tremble and they close their eyes.

But neither thirty years, not thirty centuries, affect the clearness, or the charm, of Geometrical truths. Such a theorem as "the square of the hypotenuse of a rightangled triangle is equal to the sum of the squares of the sides" is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it, and celebrated its advent, it is said, by sacrificing a hecatomb of oxen – a method of doing honor to Science that has always seemed to me *slightly* exaggerated and uncalled-for. One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant scientific discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But a *hecatomb* of oxen! It would produce a quite inconvenient supply of beef.

C. L. Dodgson (Lewis Carroll)