7 CHAPTER
Euclid Book II

The Geometry of Rectangles

Book II is short and homogeneous, with only 14 propositions and two definitions at the beginning. For the most part it is about various combinations of rectangles and squares of equal content. At the end we find the generalization of the theorem of Pythagoras to what today is called the law of cosines and, as the last proposition, the squaring of a rectangle. There are reasons to assume a Pythagorean origin for the main part of Book II, but some historians have different opinions.

We will follow Euclid and speak of "equal" rectangles where we could use "of equal content" in the modern way.

II. Def. 1. *Any rectangle is said to be contained by the straight lines containing the right angle.*

II. Def. 2. *(For rectangles) Let any one of the rectangles around the diameter of any rectangular area (together) with the two complements be called a gnomon.*

B. Artmann, *Euclid—The Creation of Mathematics*

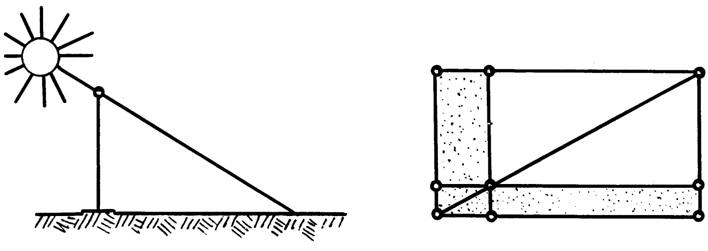

FIGURE 7.1 Gnomons

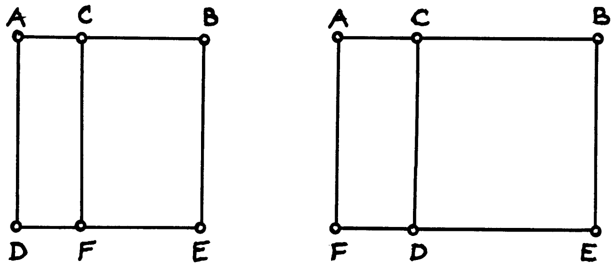

FIGURE 7.2

A gnomon was originally a sort of primitive sundial, a pole perpendicular to the horizon whose shadow was used for the measurement of time. The mathematicians apparently transferred the name to the similar-looking geometric figure. Figure 7.1 shows both of them.

We omit Prop. II.1, which is a generalization of II.2/3 and has probably been interpolated after Euclid. For II.2/3, a figure (Fig. 7.2) and a statement in shorthand suffice.

II.2: □AE = ▭DC + ▭FB
II.3: ▭AE = ▭AD + □BD

In the first case, we might say that we see a square with defect, namely ▭DC, and in the second one we have a square with excess ▭AD. The technical terms "defect" and "excess" will reappear later.

Prop. II.4.
If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

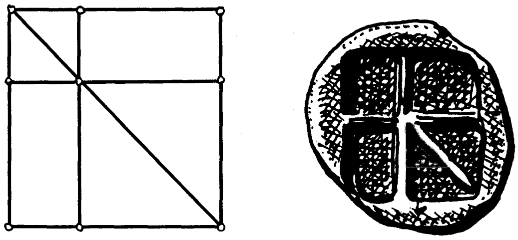

FIGURE 7.3

For the modern reader, this, together with the figure, is the familiar geometric interpretation of the binomial formula for $(a+b)^2$. For Euclid, however, it is a purely geometrical statement. (He does, however, apply the equally geometric Prop. II.6 to numbers in the lemma to Prop. X.28.) The proof is not difficult, but straightforward and somewhat lengthy.

Curiously enough, Prop. II.4 has found its way to a Greek coin (Fig. 7.3). From the year 404 B.C.E. onwards, the reverse of the coins of the city (and island) of Aigina, not far from Athens, shows the design of Euclid's Prop. II.4. We do not know why the people of Aigina selected this particular design for their money. But in any case the coins indicate the familiarity of Prop. II.4 some hundred years before Euclid's time.

Prop. II.5.

If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segments of the whole together with the square on the straight line between the points of section is equal to the square on the half.

For let a straight line AB be cut into equal segments at C and into unequal segments at D [Fig. 7.4];

I say that the rectangle contained by AD, DB together with the square on CD is equal to the square on CB.

The proposition is true because $\square AL$ and $\square DF$ are of equal content.

The next proposition is very similar. We quote it in the shorter version before we comment on both of them.

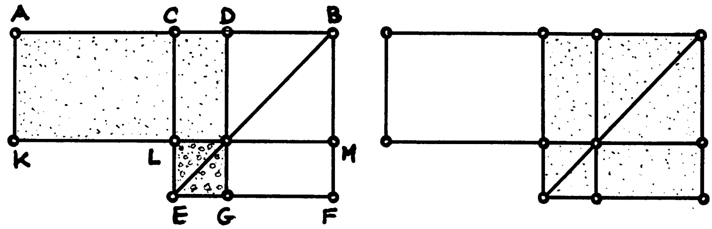

FIGURE 7.4

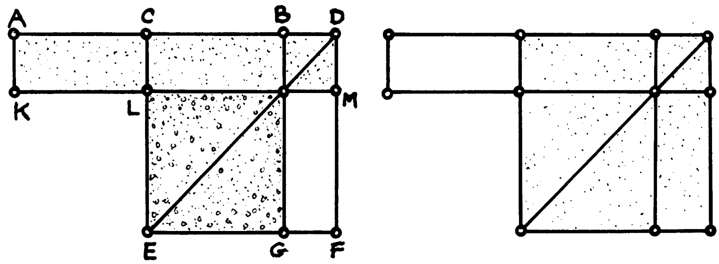

FIGURE 7.5

Prop. II.6.
Let a straight line AB be bisected at the point C, and let a straight line BD be added to it in a straight line. I say that the rectangle contained by AD, DB together with the square on CB is equal to the square on CD. (See Fig. 7.5.)

Again the truth of the proposition is easily seen from the figure.

Comments on II.5/6.

The last two propositions have often been associated with quadratic equations in the following way. In II.5, let $AB = b$ and $DB = x$. Then II.5 translates into the algebraic formula

$$\left(\frac{b}{2}\right)^2 = \left(\frac{b}{2} - x\right)^2 + x(b - x),$$

or equivalently,

$$x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2} - x\right)^2.$$

Similarly, for II.6 with $AB = b$ and $BD = x$,

$$(b+x)x + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}+x\right)^2.$$

That is, II.5/6 show us, in the literal sense, how to complete the square. Euclid had to consider these two cases separately because there are no negative segments or areas. Completing the square is the first step in solving a quadratic equation like

$$\begin{aligned} x^2 + bx + c &= 0, \\ x^2 + bx + \left(\frac{b}{2}\right)^2 &= \left(\frac{b}{2}\right)^2 - c, \\ \left(x + \frac{b}{2}\right)^2 &= \left(\frac{b}{2}\right)^2 - c. \end{aligned}$$

For the next step, one has to extract the square root out of $\left(\frac{b}{2}\right)^2 - c$. This depends on the size and sign of c and hence would again need special consideration for Euclid. He executes this second step in Book VI, somewhat hidden in VI.25, but not in Book II, even if it would be easy enough with the help of Pythagoras's theorem. Therefore one cannot speak of II.5/6 as "solutions to quadratic problems." They can be used only in order to verify known solutions and not in order to find unknown solutions. This interpretation is confirmed by the fact that Euclid introduces the condition $\left(\frac{b}{2}\right)^2 > c$, which is necessary for the existence of a solution, only in Props. VI.27/28.

Let us add another translation of II.5. Let $AC = a$ and $CD = b$. Then $AD = a + b$ and $BD = a - b$, and hence by II.5,

$$(a+b)(a-b) + b^2 = a^2, \tag{7.1}$$

or

$$(a+b)(a-b) = a^2 - b^2. \tag{7.2}$$

The next propositions, II.7–10, are variations on the themes of II.4–6. After these, Euclid turns to three different subjects. II.11 is basic for the construction of the regular pentagon in Book IV; Propositions II.12, 13 generalize the theorem of Pythagoras; and II.14 is the capstone of the sequence, dealing with the squaring of polygons.

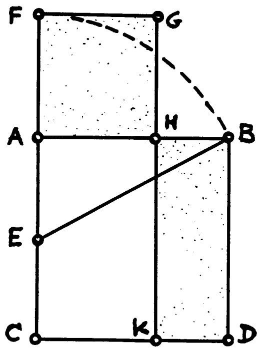

FIGURE 7.6

Prop. II.11.
To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.

Let the given line be AB. We are looking for a point H on AB such that the rectangle contained by AB and HB is equal to the square on AH.

Euclid first describes the construction of the accompanying figure (Fig. 7.6):

Let the square $ABDC$ be described on AB;
let AC be bisected at the point E, and let BE be joined;
let CA be drawn through to F, and let EF be made equal to BE;
let the square FH be described on AF, and let GH be drawn through to K.

With the help of II.6 and the theorem of Pythagoras I.47 he shows that the point H has the desired properties:

On the line CF we have the situation of II.6: AC has been bisected in E, and AF has been added to it; hence

$$\boxed{}(CF, FA) + \square AE = \square EF.$$

By construction, $\square EF = \square EB$, and by I.47, $\square EB = \square AE + \square AB$, hence

$$\boxed{}(CF, FA) + \square AE = \square AE + \square AB.$$

This implies

$$\square\!\!\square(CF, FA) = \square AB.$$

Subtraction of $\square\!\!\square(AC, AH)$ on both sides gives us the desired result:

$$\square AH = \square AF = \square\!\!\square(BD, HB) = \square\!\!\square(AB, BH).$$

Comment.

Typically, Euclid does not tell us how he arrived at his construction, but only verifies his solution. This is in accordance with the remarks about II.5/6. Later on, in VI.30, he unveils the secret. Let us again use modern notation as a convenient shorthand. Let b be the given segment and x the larger part of it. We have the condition

$$b(b-x) = x^2, \tag{7.3}$$
$$b^2 = x^2 + bx. \tag{7.4}$$

With II.6 this is transformed into

$$b^2 + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2,$$

and the theorem of Pythagoras provides us with the segment $EB = d$ such that

$$d^2 = b^2 + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2;$$

hence

$$x = d - \frac{b}{2}, \text{ as constructed.}$$

In VI.30 this derivation is embedded into the general procedure for solving quadratic problems, VI.29. This last proposition explains how to find the point D (here F) under the circumstances of II.6. Essentially, it is done as above with the help of I.47 or, in more general situations, its generalization, VI.25.

We are still left with the question why the problem of II.11 should be interesting at all. A short glance into Book XIII, Prop. 8 tells us what is going on. XIII.8 is about the regular pentagon with side x and diagonal b (in our present notation). It is proved that

$$b : x = x : (b - x),$$

which is, after VI.16, equivalent to

$$b(b - x) = x^2.$$

Knowing this, we understand the essential meaning of II.11: to determine the side x of a regular pentagon when the diagonal b is given. But about this Euclid keeps quiet in Book II, and even in Book IV, where he constructs the pentagon.

THE CHAIR

This chair was once a student of Euclid.

The book of his laws lay on its seat.
The schoolhouse windows were open,
So the wind turned the pages
Whispering the glorious proofs.

The sun set over the golden roofs.
Everywhere the shadows lengthened,
But Euclid kept quiet about that.

(Charles Simic: Hotel Insomnia, New York: Harcourt, Brace 1992. With kind permission of the publisher.)

Remark

One thing should be said very clearly: Mathematical truth is not dependent on motivation. Euclid has taught this to many generations of mathematicians. Teaching a class is another thing. Students have every right to understand how particular steps are directed to a specific goal.

Propositions II.12, 13 are very important as seen from today, because they translate into the law of cosines. In the *Elements*, however, they are just stated and never again mentioned or applied. Let us first look for a motivation of the statements (which, clearly, Euclid does not supply). By the congruence theorem SAS, a triangle—and

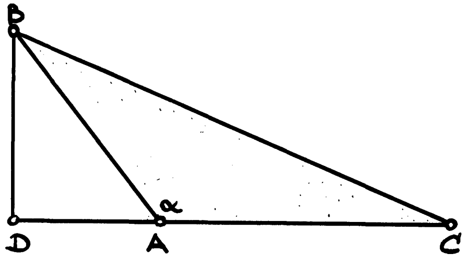

FIGURE 7.7

hence its third side—is determined by the two sides, say a, b, and the enclosed angle γ. By the theorem of Pythagoras we know the third side c if γ is a right angle. Is it possible to determine c for a general angle γ? This question is answered in II.12 for an obtuse and in II.13 for an acute angle γ. Let the first case stand for both of them.

Prop. II.12.

Let $\triangle ABC$ be an obtuse-angled triangle having $\angle BAC$ obtuse, and let BD be drawn from the point B perpendicular to CA produced [Fig. 7.7]. I say that the square on BC is greater than the squares on BA, AC together by twice the rectangle contained by CA, AD.

Proof.

By the theorem of Pythagoras I.47:

$$\square BC = \square BD + \square DC,$$
$$\square AB = \square BD + \square DA.$$

By the binomial theorem II.4,

$$\square DC = \square DA + \square AC + 2\sqsubset\!\!\sqsupset(DA, AC); \tag{7.5}$$

hence

$$\square BC = \square BD + \square DA + \square AC + 2\sqsubset\!\!\sqsupset(DA, AC) \tag{7.6}$$
$$= \square AB + \square AC + 2\sqsubset\!\!\sqsupset(DA, AC) \tag{7.7}$$

as asserted.

Modern translation. Let $AB = c$, $BC = a$, $CA = b$, and $\angle BAC = \alpha$. Then $DA = c\cos\alpha$. Because α is obtuse, $\cos\alpha$ is negative, and we get

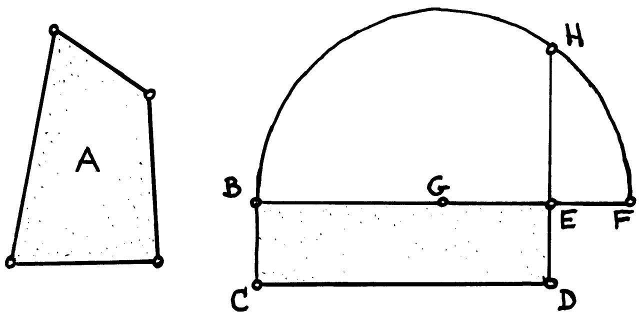

FIGURE 7.8

the result

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

the modern version. The sign of the cosine function will take care of both cases II.12,13. The law of cosines could develop its full power only a long time after Euclid.

Prop. II.14.
To construct a square equal to a given rectilinear figure.

Solution:
Let A be the given figure [Fig. 7.8]. Construct a rectangle $\boxed{}(BE, ED)$ of equal content with A via I.45. If this rectangle is a square, we are ready. Otherwise let BE be greater than ED. Prolong BE to F and make EF equal to ED. Bisect BF in G, describe the semicircle BHF with center G. Produce DE to H.

Assertion:
$\boxed{}(BE, ED) = \square EH$.

By II.5 for the line BF we get

$$\boxed{}(BE, EF) + \square EG = \square GF.$$

GF is equal to GH by construction, hence

$$\begin{aligned} \boxed{}(BE, EF) + \square EG &= \square GH \\ &= \square EG + \square EH \qquad \text{by I.47.} \end{aligned}$$

Subtracting the square $\square EG$ on both sides and observing that $EF = ED$ completes the proof.

This theorem is a goal for its own sake. It is the objective of a theory developed from I.35–45 via I.47 and II.5 to its culminating point II.14. It serves no other purpose than to transmit a certain knowledge or answer a question that is important in theory and, we may add, in practice as well.