9 CHAPTER Euclid Book III

About the Circle

9.1 The Overall Composition of Book III

Definitions 1–11	
1	How to find the center of a circle
2–15	A: Chords in circles, circles intersecting or touching each other
16–19	B: Tangents
20–22	C_1: Angles in segments of circles and quadrilaterals in circles
23–29	C_2: Chords, arcs, and angles
30	How to bisect an arc
31–34	C_3: More about angles in circles
35–37	D: Intersecting chords, secants, and tangents

B. Artmann, *Euclid—The Creation of Mathematics*

9.2 The Definitions of Book III

Def. 1.
Equal circles are those the diameters of which are equal, or the radii of which are equal.

In the first two Books, Euclid has used the notion of equality for rectilinear figures in the sense of "of equal content." For circles, now, "equal" has a different meaning. (For a more detailed discussion of this and the other definitions, see the Notes.)

Def. 2
[of a tangent] *A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle.*

Def. 6.
A segment of a circle is the figure contained by a straight line and an arc of a circle.

Def. 7.
An **angle of a segment** *is that contained by a straight line and an arc of a circle. [Fig. 9.1(a)].*

Def. 8.
An **angle in a segment** *[Fig. 9.1(b)] is the angle contained by the connected straight lines when some point is taken on the arc of the segment and straight lines are connected from it to the extremities of the straight line which is the base of the segment.*

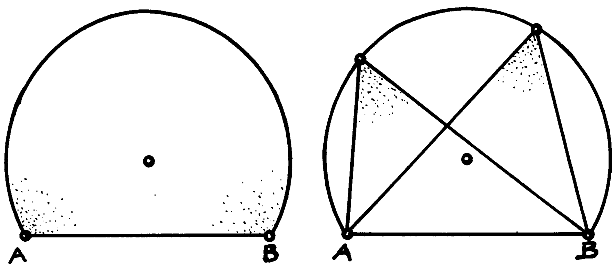

FIGURE 9.1 (a) angle *of* a segment (b) angle *in* a segment

In Def. 8 Euclid seems to presuppose the validity of Prop. III.21 about the equality of all the angles *in* a segment.

Def. 11.
Similar segments of circles are those which admit equal angles or in which the angles are equal to one another.

Here, as in Def. 7, mixed angles are used. Moreover, as in Def. 8, the equality of the angles α, α' *of* a segment is tacitly assumed. Third, the notion of similarity occurs for the first time. Equiangular triangles are treated likewise by Euclid in Props. IV. 2, 3 without using the word "similar".

In the discussion of Prop. I.5 we have seen how Aristotle used the (mixed) angles of a segment. It seems that Euclid used some older material in Book III that he did not rework thoroughly.

9.3 Book III, Part A: Chords in Circles, Circles Intersecting or Touching Each Other

Before starting with part A, Euclid poses the curious-looking problem III.1: how to find the center of a circle. The existence of a center is guaranteed by the very definition of a circle in I. Defs. 15, 16. But Euclid often enough starts from a circle and then finds its midpoint. One can only speculate that this is a remnant of an old drawing practice. Drawing a circle with a pair of compasses, which were in use in antiquity, automatically produces the center. But if an object fabricated on, say, a potters wheel, like a plate or a cup, was used to draw a circle in the sand, then indeed the midpoint had to be constructed afterwards.

The material of part A is not very coherent logically. We quote two typical examples.

Prop. III.4.
If in a circle two chords cut one another which are not through the center, they do not bisect each other.

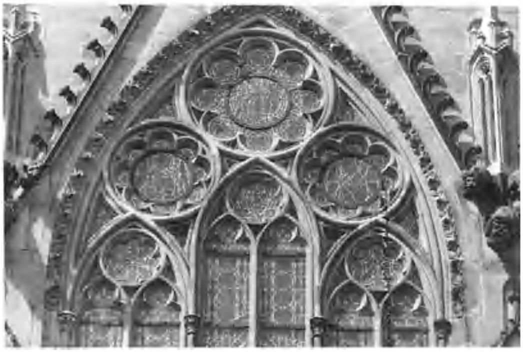

FIGURE 9.2 Ritterstiftskirche in Wimpfen, Germany, 1280

Prop. III.11, 12.
If two circles touch each other (internally or externally) and their centers are taken, the straight line joining their centers, when produced, will fall on the point of contact of the circles.

Props. III.11, 12 have found innumerably many applications in the art of the Middle Ages. They are basic for the construction of Gothic traceries (see Fig. 9.2).

9.4 Book III, Part B: Tangents

The four propositions of this section are fundamental for elementary geometry. They transform the intuitive notion of the tangent as a line "touching the circle" into an easily usable tool by providing a simple construction for a tangent: The tangent is orthogonal to the radius drawn from the center to the point of contact.

Prop. III.16.
The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further, the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

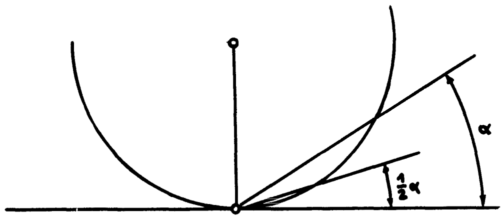

FIGURE 9.3

Before going on with tangents we need to have a look at the "hornlike" angles formed by straight lines and circles at the point of contact. Euclid states that the angle between the circle and the tangent is smaller than any rectilinear angle.

This has serious consequences for the ordering of angles according to their size. The rectilinear angle α may be bisected (and bisected again ...), but never will the resulting smaller rectilinear angle be smaller than the hornlike angle β (See Fig. 9.3). The ordering of angles, including mixed ones, is non-Archimedean in modern terms. Euclid was well aware of the situation, because on other occasions he in effect excludes non-Archimedean orderings. In Def.V.4 he speaks about magnitudes "which are capable, when multiplied, of exceeding one another," and in Prop.X.1 he has exactly the situation of III.16 and shows with the aid of Def.V.4 that the successive halves will eventually be smaller than the second magnitude. Non-Archimedean and Archimedean orderings were known some hundred years before Archimedes.

At the end of the proof of Prop. III.16 Euclid adds a corollary ("porism"):
From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremities touches the circle.

Props. III. 18/19 are (partial) converses of the corollary.

Prop. III.17 is of special interest because of its typical use of symmetry in a mathematical argument and because of the similarity of the figure with the one of Aristotle's old proof for Prop. I.5.

Prop. III.17.
From a given point to draw a straight line touching a given circle.

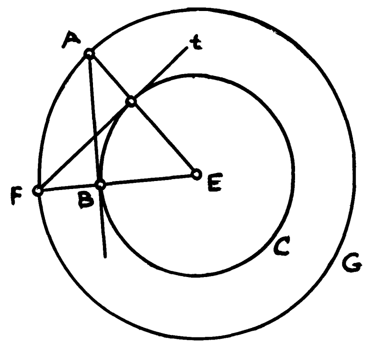

FIGURE 9.4

Let C be the given circle with center E and A be the given point (Fig. 9.4). Join A to E and let D be the intersection point of AE with the circle. Let t be the tangent to C at the point D, which is orthogonal to ED. Draw the circle G with center E and radius EA; let it intersect t in F. Find the intersection point B of FE and circle C. Claim: AB is the tangent sought. For the proof it has to be shown that AB is orthogonal to EF. The triangles ΔEDF and ΔEBA are congruent by *SAS*; and hence the angle at B is right because the angle at D is right. The unexpected way in which the tangent t, which at first sight has nothing to do with the problem, is used to produce the wanted tangent AB is a nice surprise and makes the construction very elegant.

9.5 Book III, Part C_1: Angles in Segments of Circles

The four most useful theorems in elementary geometry are Euclid's Prop. I.32 about the sum of the angles in a triangle, the theorem of Pythagoras I.47, the theorem about the invariance of angles in a segment Prop. III.21, and Prop. VI. 2 about the proportionality of sides in similar triangles. Prop. III.20 marks a fresh start in Book III.

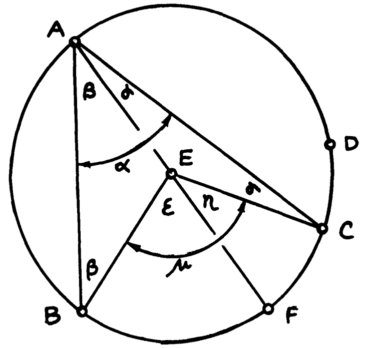

FIGURE 9.5

Prop. III.20.

In a circle the angle at the center is double the angle at the circumference when the angles have the same arc as base.

With the notation of Fig. 9.5, Euclid's proof proceeds as follows. Let $\alpha = \angle BAC$ with the arc BFC as its base. E is the center of the circle. For the first case, let E be in the interior of the angle α. The line AE is extended to F. Then ΔABE is isosceles; hence the exterior angle ϵ is equal to 2β. Similarly, $\eta = 2\gamma$. Hence $\mu = \epsilon + \eta$ is the double of $\alpha = \beta + \gamma$. (The case of the angle at D is to be treated in the same way, with subtraction instead of the addition of angles.)

Prop. III.21.

In a circle the angles in the same segment are equal to one another.

This is an obvious corollary to Prop. III.20. It is, however, in a strict sense proved only for segments greater than half circles. We will not dwell on this point, which can be corrected easily. The next theorem is a first substantial application of Prop. 21.

Prop. III.22.

The opposite angles of quadrilaterals in circles are equal to two right angles.

Proof

Notation as in Fig. 9.6. In the triangle ΔABC, we have $\alpha+\beta+\gamma = 2R$. The angles α, α' are in the same segment defined by BC; hence $\alpha = \alpha'$ by Prop. 21. Similarly, $\gamma = \gamma'$, and because of $\delta = \alpha' + \gamma'$ we have $\beta + \delta = \beta + \gamma + \alpha = 2R$.

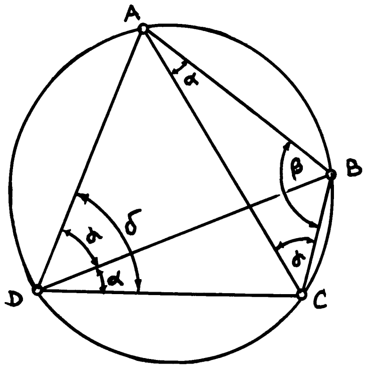

FIGURE 9.6

Doesn't this clever combination of the two invariants from I.32 and III.21 constitute a jewel of a proof? We will return to it in the following section on problems and theories.

9.6 Book III, Part C_2: Chords, Arcs, and Angles

In this section of Book III the results of the preceding one are extended in a technical way to more general situations resembling the propositions on the areas of parallelograms and triangles in Book. I.36–38. The equality of angles is transferred from "in the same segment" to "in equal segments of equal circles" and similarly for chords and arcs: Equal arcs determine equal chords, and vice versa; and the same for angles and arcs. The statements have important applications in Book IV.

9.7 Book III, Part C_3: More About Angles in Circles

In section C_1, two extreme cases have not been dealt with. The first one is the angle in a semicircle. The diameter of a circle is a straight

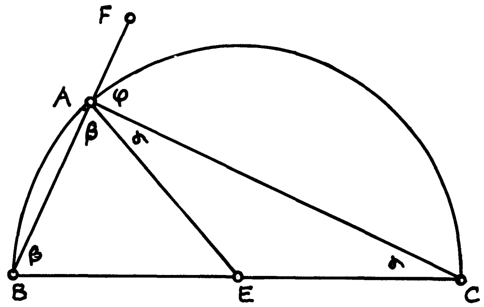

FIGURE 9.7

line, and for Euclid this does not define an angle (of 180° or 2 right angles) at the center.

Prop. III.31.
In a circle the angle in a semicircle is right.

(The proposition goes on with statements about the angles in and of segments greater or smaller than semicircles.) Prop. III.31 is ascribed to Thales by some antique sources and is called "the theorem of Thales" in Germany. (In France "the theorem of Thales" is VI.2!)

The proof again uses isosceles triangles; see Fig. 9.7. Euclid concludes that the exterior angle φ is equal to the interior angle $\beta + \gamma$; hence both must be right angles. This (and the proof of III.20 and other occasions) shows that Euclid was well aware of the later applications of theorem I.32 about the sum of the angles in a triangle and chose a formulation of I, 32 suitable to his purposes.

Prop. III.32.
If some straight line touches a circle and some straight line cutting the circle is drawn from the contact into the circle, the angles which it makes with the tangent will be equal to the angles in the alternate segment of the circle.

We will present Euclid's essential steps while abbreviating his proof and using the modern notation of Fig. 9.8. The assertion is $\alpha = \varphi$. Because of the invariance of the angle α in the segment BAC we may place A' so that BA' is a diameter of the circle. Then, by the last proposition, $\angle A'CB$ will be right; hence by I.32 we have $\alpha + \beta = R$. But because $A'B$ is a diameter and BF a tangent, also

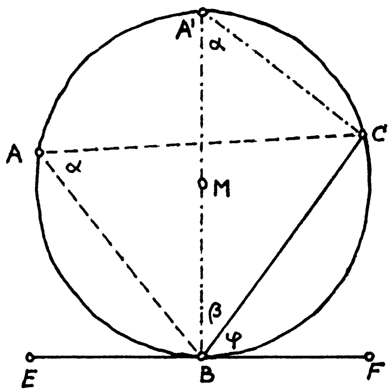

FIGURE 9.8

$\beta + \varphi = R$, from which it follows that $\varphi = \alpha$, as asserted. Notice how many previous theorems play essential roles in this one proof: I.32, III.18, 21, 31.

The next two propositions present variants of III.32 that are useful because they create situations suitable for the application of III.21.

Prop. III.33.
To describe on a given straight line a segment of a circle admitting an angle equal to a given rectilinear angle.

Prop. III.34.
To cut off from a given circle a segment admitting an angle equal to a given rectilinear angle.

9.8 Book III, Part D: Intersecting Chords, Secants, and Tangents

The last three propositions of Book III are placed in the context of similarity geometry in today's high-school geometry, where they

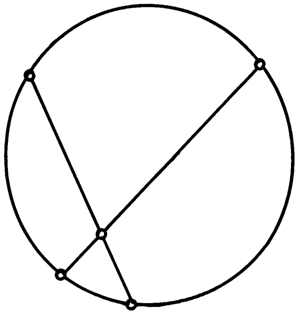

FIGURE 9.9

have easy proofs. At this stage, Euclid cannot use proportions, and he supplies complicated proofs. This is an important feature of the architecture of the *Elements*.

Implicit in III. 35/36 is the construction of an invariant, which is today called the power of a point with respect to a circle, and which has played an important role in the history of geometry. Because Prop. 35 is not used in the later books, we present the two proofs for Prop. 36/37 only.

Prop. III.35.
If in a circle two straight lines cut one another, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other (See Fig.9.9).

Prop. III.36.
If some point (D) is taken outside a circle and from it there fall on the circle two straight lines, and if one of them cuts the circle (in A and C), the other touches it (in B), the rectangle contained by DC and DA will be equal to the square on the tangent DB.

We will restate this in a somewhat modernized notation so as to include its converse III.37. Compare Fig. 9.10.

In the situation of Fig. 9.10:

$$DB \text{ is a tangent} \quad \Leftrightarrow \quad \square DB = \square(DC, DA).$$

(i) Let DB be a tangent.

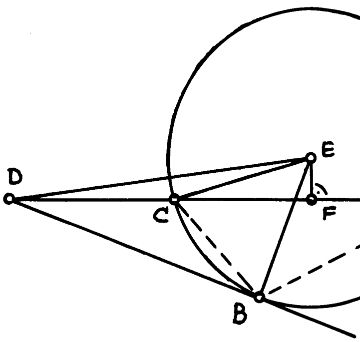

FIGURE 9.10

Similarity proof:

Consider the triangles $\triangle DBC$ and $\triangle DBA$. They have $\angle ADB$ in common. If DB is a tangent, then $\angle DBC = \angle BAC$ by III.32. By I.32 the remaining angles are equal and the triangles $\triangle DBC$ and $\triangle DAB$ are equiangular.

Using VI.4 we get

$$DC : DB = DB : DA,$$

and by VI.16,

$$\boxed{}(DC, DA) = \square DB.$$

Euclid's proof.

Euclid first considers the case where DA passes through the center of the circles. We ignore this because the methods are the same as in the other case, where the center E of the circle is not on DA. Let EF be orthogonal to DA; join E to D and B.

On the line DA, we have the situation of II.6, resulting in

$$\boxed{}(AD, DC) + \square CF = \square FD.$$

Now three applications of Pythagoras's theorem will help. By construction,

$$\square EC = \square CF + \square FE,$$
$$\square ED = \square DF + \square FE.$$

Hence

$$\sqsubset\!\!\sqsupset(AD, DC) + \square EC = \square ED,$$

and because $EC = EB$,

$$\sqsubset\!\!\sqsupset(AD, DC) + \square EB = \square ED.$$

Now, because DB is a tangent,

$$\square ED = \square DB + \square EB.$$

Substituting and subtracting $\square EB$ gives

$$\sqsubset\!\!\sqsupset(AC, DC) = \square DB.$$

At first sight, Euclid's proof looks much more complicated than the similarity proof. The latter, however, uses VI.4, 16, which need conceptually difficult foundations in Book V. Altogether, Euclid's proof is still simpler, in spite of being technically complicated. However, it does not present itself in a natural way. Seen from today, it is an artificial product. As long as the concepts of similarity and proportion were used in a naive way, the first proof was obvious.

(ii) The converse direction of the proof (Prop. 37) uses part (i). It is of no particular interest for us.