25 CHAPTER Euclid Book XI

Solid Geometry

25.1 The Overall Composition of Book XI

Definitions 1–28	Angles and parallelism in space, sphere, cone, and cylinder, various polyhedra
1–19	A: Basic facts of solid geometry
20–23	B: Solid vertices
24/25	C_1: Parallelepipedal solids
26/27	D_1: Parallelepipeds and proportion
28–34	C_2: Part C_1 continued
35–37	D_2: Part D_1 continued
38/39	Two lemmas for Book XII

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25.2 The Definitions of Book XI

The definitions at the beginning of Book XI are intended to serve for all of the three Books XI–XIII. Accordingly, we find three groups of definitions: Defs. 1–8 determine angles between planes and similar objects. Next it is fixed what is to be understood by similar solid angles (9–11). Pyramids and prisms (12,13) are needed in Book XII, which culminates in the study of cylinders, cones, and spheres.

Def. XI.14.
When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere.

This sounds more like an artisan's than a mathematician's definition. If a potter were asked to make a spherical vase or a sculptor to carve a solid ball, they would proceed as in this definition. Plato has a similar description of a sphere in his description of the cosmos, but combines it with the definition that a mathematician (of today) would expect:

> And he [the creator] gave to the world the figure which was suitable and also natural ... that figure would be suitable which comprehends within itself all other figures. Wherefore he made the world in the form of a globe (sphairoeides), *round as from a lathe, having its extremes in every direction equidistant from the center*, the most perfect and the most like itself of all figures. [*Timaeus* 33a, my italics]

The definitions of cones (18) and cylinders (21) follow the same pattern as Def. 14. Whatever Euclid's motives may have been, these definitions once more convince us that the roots of mathematics lie in the traditions of sculptors, potters, architects, and other artisans.

There is another intrinsic reason for this unusual definition of a sphere. We, being three-dimensional, can look "from above" at a flat circular plate and see its center or reconstruct it as in Prop. III. 1 when it is lost. We would have to be four-dimensional in order to do and see the same things in a solid sphere. On a flat section of the cylindrical part of a column one can recover the center and hence the axis of the column, but the same is impossible for a stone ball.

The remaining, third, group, Defs. 25–28, are about the regular solids and will be quoted in the discussion of Book XIII.

25.3 Foundations of Solid Geometry

There are no foundations of solid geometry in Book XI, at least not in the sense of the axiomatic base of Book I. We may recapitulate from Book I the definition of a plane:

Def. I.7.
A plane surface is a surface which lies evenly with the straight lines on itself.

This is more a description similar to the definitions of points and lines. For solids we have the following analogue:

Def. XI.1.
A solid is that which has length, breadth, and depth.

This, together with Def. XI. 2 about the extremity of a solid, fixes the dimension of solids. The subsequent definitions are technical, like the next one.

Def. XI.3.
A straight line is at right angles to a plane, when it makes right angles with all the straight lines which meet it and are in the plane.

There has been much criticism as to how Euclid tries to derive from these definitions his first propositions, like

Prop. XI.3.
If two planes cut one another, their common section is a straight line.

Modern axiomatic analyses like the one by Hilbert [1899] have shown that there is no alternative to taking this proposition and the preceding one (three noncollinear points are in a unique plane) as axioms for solid geometry. We may conclude that here, as in his arithmetical books, Euclid has preserved a historically earlier stage of mathematical theories. Hence we were right when we said that there is no axiomatic foundation of solid geometry in the proper

sense in Book XI. In a broader sense, however, we may speak about foundations because Euclid collects many useful and basic statements about planes, lines, and their angles of inclination in part A of Book XI. As an example take

Prop. XI.14.
Planes to which the same straight line is at right angles will be parallel.

After XI.19 the subject changes to the treatment of solid vertices, or solid angles, as Euclid called them. Propositions 20–23 appear to be a unit, looking like preparatory material for Book XIII in a narrower sense. Neuenschwander [1975, 127] has conjectured that these propositions were stated by Theaetetus as a preparation for his theory of the regular solids, which is preserved in Book XIII.

25.4 The Affinities of Books I and XI

Plato, in his *Republic*, 528 a–d, discusses the state of solid geometry. On the one hand, he praises the "extraordinary attractiveness and charm" (528 d 2) of the subject and observes progress of the investigators in spite of obstacles because of the "inherent charm" of their results. On the other hand, he deplores the state of affairs and speaks of an "absurd neglect" (528 d 9/10) of solid geometry, the reasons for this being (a) the state gives no money for the research in this area and (b) there is no director coordinating the efforts of the investigators (528 b 6–c 4). How would this make sense in the historical situation? We may propose the following solution:

(i) Theaetetus had created the concept of a regular polyhedron and shown the existence of exactly five such polyhedra, certainly a result of "inherent charm" and "extraordinary attractiveness,"

(ii) There did not yet exist a book on the "Elements of solid geometry", so that the results of Theaetetus did not have an elementary foundation consisting of certain simple propositions about the properties of planes etc. comparable to the

foundations of plane geometry in section (Aa) of Euclid's Book I.

Under these circumstances, Plato's seemingly contradictory remarks would make perfect sense, and moreover:

(iii) They could be understood as a challenge to write a book on the foundations of solid geometry.

Following the authors quoted below, I assume that such a book was indeed written, that it has, essentially, survived as Euclid's book XI.1–23, 26, and that it was modeled after an existing earlier version of Euclid's Book I. The similarity of Euclid's Books I and XI has been observed by Neuenschwander [1975, p. 118] and Mueller [1981, p. 207], and has been worked out in detail in the paper Artmann [1988]. We direct our attention to some specific details, which show the state of affairs before the introduction of the parallel postulate. The solid geometry in Book XI has no axiomatic foundation like the plane geometry in Book I. This is not what Plato criticizes. On the contrary, it confirms the age of Book XI. In Plato's time mathematics was based on definitions, not on axioms. (See Mueller [1991] for a substantiation.) Parallel straight lines are defined in the very last definition 18 of Book I as lines in the same plane that do not meet. Definition XI. 8 says that parallel planes are those that do not meet. In order to get a better synopsis, we list the propositions of Books I and XI side by side (see Table 25.1).

The most conspicuous analogies between the two books are certainly the respective constructions (11/12) of perpendiculars and (20) the triangular inequality. A closer look at some other propositions is helpful. After 11/12, Book XI goes on to show the uniqueness of the perpendiculars constructed in 11 and 12. This is missing in Book I, but given by Proclus in his commentary on I.17 [Proclus–Morrow p. 244]. Then the two books deviate:

Prop. XI.14:
Planes to which the same straight line is orthogonal will be parallel.

This should be seen in connection with Props. XI. 6 and 8:

Prop. XI.6:
If two straight lines are at right angles to the same plane, the straight lines will be parallel.

TABLE 25.1

Book I		Book XI	
(Aa)		(Aa)	
1–10	foundations	1–5	foundations
		6–10	parallelism and angles for lines and planes
11/12	how to draw lines perpendicular to a given line	11/12	how to draw lines perpendicular to a given plane
		14	a line orthogonal to two planes makes the planes parallel
(Ab)		(Ab)	
13–15	supplementary, vertical angles	15/16,	orthogonality and
		18/19	parallelism for lines and planes
16/17		17	solid analogue to VI.2 (isolated theorem)
(Ac)		(Ac)	
18/19	greater side and angle		
20	triangle inequality for sides of a triangle	20	triangle inequality for angles at a vertex
		21	sum of angles at a vertex
21/22	construction of a triangle from its sides	22/23	construction of a solid vertex from plane angles
23	copying of angles	26	copying of solid angles
24/25			
(Ad)			
26			
(B)			
27/28	equal angles make parallels		
29	parallels make equal angles		
31	construction of parallels		
32	sum of angles in a triangle		

Prop. XI.8:
If two straight lines are parallel, and one of them is at right angles to any plane, the remaining one will also be at right angles to the same plane.

If we, in these three propositions, replace "plane" by "line," then 14 and 6 will coincide and, together with 8, will give us a convenient description of parallels in the (one) plane of Book I. Similarly, the transitivity of parallelism for lines is stated in XI. 9 and I.30.

Proposition XI.15, 16 are specific to solid geometry and cannot have plane analogies. The same is true for XI.18, 19.

Proposition XI.17 is a singularity. It is the solid analogue of VI, 2–10 and is proved by means of VI.2 (proportional segments), which is the basis of all similarity geometry.

With the two triangular inequalities in I.20 and XI.20 we are back to the sequence of Book I. The sum of angles at a solid vertex (XI.21) may correspond to the sum of the angles in a triangle (I.32.) Proposition XI, 22/23 have their exact counterparts in I.21/22. The copying of a plane angle in I.23 corresponds to the copying of a solid angle in XI, 26. At XI.24, another section of Book XI about parallelepipedal solids starts, which has to be compared with section C on parallelograms of Book I. Again the similarities are compelling. In spite of a great number of definitions at the beginnings of both books, neither parallelograms nor parallelepipeds are defined. We leave out a more detailed investigation of the respective propositions because these parts are less important for our purposes. Besides, they seem to be somewhat younger than the preceding material.

Our structural comparison of Books I and XI works in two ways. First, we have seen how Book XI was composed after the pattern of Book I, but with some variations concerning parallels. It is generally assumed that the parallel postulate (5) and the corresponding treatment of parallels in the *Elements*, Props. I. 27–32, were fairly recent at Euclid's time. Hence we may assume that Book XI was modeled after an earlier version of Book I, where parallels were treated in a less sophisticated way. Our second line of investigation will be to recover information from Book XI to find out what that earlier treatment of parallels could have been. Because we have no explicit witnesses, the emerging picture can only be hypothetical.

Nevertheless, it seems to be convincing and may allow us a glimpse at the *Elements* of Leon, which was the textbook of Plato's Academy.

If we look at our schematic representation of the two books (see Table 25.1), we see parallelism in the solid case treated in subsection (Ab), Props. 15–19. The corresponding subsection in Book I has only two remarks about angles and Props. I. 16/17, which we have already recognized in our discussion of Book I as rather late. We conclude that before somebody rearranged the propositions now following I.12, there should have existed a section on parallels and angles and after that a section concerning the triangular inequality. From the plane versions of XI.6, 8, 14 we may even see how parallels were originally constructed. The construction of perpendiculars in Props. XI.11, 12 is the basis for the criterion of parallelism for planes in space, XI.14. In the plane case, Props. I.11, 12 would have served the same purpose. (The more general alternate angles in I.27–29 could be seen as a generalization of right angles.) The pre-Euclidean subsection (Ab) on angles and parallels might then have consisted of the following propositions (we use the numbering of Euclid's *Elements*):

I. 13/14: Angles "on a line" adding up to two right angles.
I. 15: Vertical angles are equal.
I. 27/28/29/31: Alternate angles are equal and make parallels, parallels make equal alternate angles, how to draw parallels.
I. 30: Transitivity of parallelism.
I. 32: The exterior angle of a triangle is equal to the sum of the two interior and opposite angles, and the three interior angles together make two right angles.

At this stage, I.16 would be superfluous, and I.32 could have been used as a basis for a subsection (Ac) on greater relations for lines and angles, consisting of Euclid's Props. I. 18–22 and 24/25. Together with an initial subsection (Aa) consisting of Euclid's Props. I. 1–12, this would have been an attractive (but hypothetical) introduction to plane geometry and fit with the structure of Book XI.

Unfortunately, we have no written evidence of an earlier definition of parallels. There are three key properties of parallel lines (in a fixed plane, as in Book I):

(i) Parallel lines do not meet.
(ii) They make equal (right) angles with (orthogonal) transversals.
(iii) They are equidistant.

All three properties were known by artisans, especially stone masons, from time immemorial. In every wall made from rectangular blocks or bricks one could see parallel lines. Plato acknowledges the strong interrelations of the building trade and geometry in his *Philebus* (56 b–57 d), even if he contrasts practical and scientific geometry in the same place. Maybe even all three properties were stated in the definition. This would be similar to the definition of the diameter of a circle in Def. I. 17, where it is defined as a line through the center that also bisects the circle.

25.5 The Duplication of the Cube

From the rest of Book XI we will pick out just the one theorem Prop. XI. 33, which is closely related to the famous problem of the duplication of the cube.

Prop. XI.33.
Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

This proposition is the extension to the solid case of Prop. VI. 20 for similar polygons. In the plane a similarity factor k for segments gives k^2 for areas; now we will have the factor k^3 for the corresponding volumes of similar parallelepipeds.

There is a precisely parallel statement with essentially the same proof in Book VIII about similar solid numbers. We will not discuss similar solid numbers and similar parallelepipedal solids but restrict our attention to cubes in both cases. Except for some very easy modifications, the proofs are the same for the special and the more general situations.

Prop. VIII.19.
Two mean proportional numbers fall between two similar solid numbers; and the solid has to the similar solid the triplicate ratio of that which the corresponding side has to the corresponding side.

Let us first repeat the statements in modern terminology. Let the cubes have sides a and b, so that the respective cube numbers

(or volumes) are a^3 and b^3. Then it is said that there are *two* mean proportionals, which is the definition of "in triplicate ratio." This means

$$a^3 : a^2b = a^2b : ab^2 = ab^2 : b^3,$$

the numbers a^2b and ab^2 being the said mean proportionals. In the problem of duplicating a cube, the two numbers $a^3 = 2$ and $b^3 = 1$ are given, and a is sought. Clearly, this amounts to $a = \sqrt[3]{2}$. By transforming the original problem into one about two mean proportionals, it was brought into a version that could be handled by the mainstream methods of Greek mathematics. From Eratosthenes of Cyrene (ca. 250 B.C.E.) we learn that the reduction of the cube duplication to the finding of two mean proportionals was the discovery of Hippocrates of Chios. (For this and many more details of the story see Knorr [1986, esp. p. 23] and Saito [1995].)

For the proof of Props. XI. 33 and VIII. 19 Euclid uses what one might call a solid gnomon (see Fig. 25.1). We have the two cubes A and B with their respective sides a and b. Let R be the square with side a, and S the rectangle contained by a and b. Then, using Props. VI. 1 and XI. 25/32, we get

$$\begin{aligned} a : b &= R : S = A : C, \\ a : b &= C : D, \\ a : b &= D : B. \end{aligned}$$

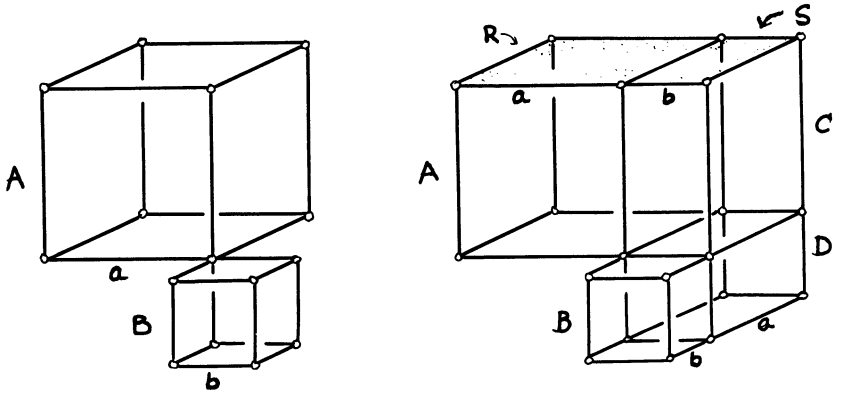

FIGURE 25.1

Hence we have

$$A : C = C : D = D : B,$$

and the two mean proportionals for A and B have been found.

The Greeks could not know that it was hopeless to look for a ruler-and-compass construction of $\sqrt[3]{2}$. Instead of using these "plane" methods, solutions have been found by "solid" methods, that is, spatial curves (Archytas) or conic sections (Menaechmus). For the details see Knorr [1986].

26 CHAPTER The Origin of Mathematics 13

The Role of Definitions

We have seen how Euclid formulates a definition of the sphere that goes back to the actual process of making one. Plato, in this case more advanced than Euclid, seems to waver between the artisan's and the mathematician's definition when he adds the condition of equidistance from a center. The example shows how a particular mathematical concept changes from a practical and realistic origin toward a more abstract notion that fits better into a developed theory. Usually in the *Elements* we find this mathematically advanced kind of definition. Many of them are explicit descriptions of geometrical objects like "equilateral triangle" or "gnomon," which simply determine certain notions for later use. However, not everything can be defined. Every mathematical theory starts with some undefined concepts. Today these may be in principle no more than "set" and "function" or one of these two, but one cannot start from nothing. In the *Elements* a typical example of this sort is "to measure." We are never told what "measuring" is, in spite of its fundamental importance in Book V for magnitudes and in Book VII for numbers. Somehow one knows what is meant, and it works. We have strong

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connotations of an everyday procedure that help us to make sense of the concept.

As soon as mathematics develops a little further, another type of definition emerges. When mathematicians discovered that certain segments have no common measure, the notions of commensurability and incommensurability entered the field. This again led to another problem of definition. How should "proportion" be defined, given the new phenomenon? In this way the search for the "right" definition becomes a research problem.

This may sound curious to a philosophically trained reader. After Blaise Pascal (1623–1662) it became firmly established that in mathematics definitions are what are called nominal definitions. The notion "square" is nothing but an abbreviation of the phrase "equilateral and equiangular quadrilateral," and the same is true for any other mathematical concept. In principle one should always be able to replace the shorthand notion by the terms that were used in its definition. In fact, many routine proofs do just this. And more than that: Definitions are never right or wrong, but can only be more or less useful.

What mathematicians mean by a "right" or "good" definition is one that on the one hand fits closely to the more intuitive notion that is under discussion, and on the other hand allows for the fruitful development of the corresponding theory. A paradigmatic example for such a definition is Def.V.5 about the proportionality of magnitudes. A definition of this type has a strong creative component. One cannot know beforehand what will work, and many different approaches may be necessary. Aristotle has a clear understanding of this situation when he speaks about the proof of Prop.VI.1, which is the basis of similarity geometry:

> In mathematics, too, some things would seem to be not easily proved for want of a definition, e.g., that the straight line parallel to the side, which cuts a plane figure [rectangle] divides similarly both the line and the area. But once the definition is stated, the said property is immediately manifest.... [*Topics* 158 b 29–32]

Euclid's definition of similar plane numbers may fall into this same category. A modern example could be the definition of quaternions by Hamilton (1843), who looked for a multiplication in $\mathbb{R}^3$, but

found one in $\mathbb{R}^4$ and so created a whole new theory. In this case we are on the borderline of yet another type of definition that mathematicians hold in highest esteem. We might call them creations out of nothing, *creatio ex nihilo*. In the *Elements*, we meet this type in Book XIII about the regular polyhedra. We will see how Theaetetus, knowing three of them from the Pythagoreans, created the concept and classified all examples. This subject is as fascinating today as it was in the time of Plato. Great advances in mathematics are based on definitions of this kind. Newton and Leibniz defined derivatives and integrals; Cantor created infinite cardinal numbers, and Poincaré the fundamental group.

And yet all these definitions are nominal in the strict logical sense. Logical theories are valuable and useful for foundational questions, but in some way they miss what is going on in real-life mathematics. Look, for instance, at theorems that *characterize* a mathematical object. In principle, a characterization should be nothing but the replacement of one definition by another one. Example: A quadrilateral is cyclic if and only if its opposite angles add up to two right angles (cf. Prop. III. 22). Any such characterization will expand our knowledge and reveal, in the important cases, hidden properties and open up new paths of investigations.

Let us add a few words about definitions in the world outside mathematics. Once a mathematical concept is defined, it is and remains fixed in the context of the mathematical theory. For professions other than that of a mathematician's, sometimes exactly the opposite seems to be true. In many cases it is the business of a lawyer or a politician to argue about the meaning of definitions, to expand and twist them until they match one's purposes. The mathematician excludes secondary connotations; the lawyer exploits them. And there is a third person, the poet, who creates them. Writers and poets take care of the living language by creating ever new metaphors and adding fresh connotations to words that have become stereotypes or have worn down over many years of careless usage. The writer Robert Musil (1880–1942), who held a degree in mathematics, says in his great novel "The Man Without Qualities" exactly this: "And so every word demands to be taken literally, or else it would decay to a lie; and yet one cannot take it verbatim, for that would render the world a madhouse" (Musil [1981] p. 749).

27 CHAPTER Euclid Book XII

Volumes by Limits

27.1 The Overall Composition of Book XII

1/2	A: The area of a circle
3–9	B: The volume of a pyramid
10/15	C: Cylinders and cones
16/18	D: Spheres

27.2 The Circle

As a preparation for his main theorem on circles Euclid needs some information on polygons.

Prop. XII.1.
Similar polygons inscribed in circles are to one another as the squares on the diameters.

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This is easily reduced to Prop. VI, 20 about the areas of similar polygons by a little geometric manipulation that replaces corresponding sides by the diameters of the circumcircles.

By inscribing successively larger polygons into a circle he approximates the area. This is called the method of exhaustion.

Prop. XII.2.
Circles are to one another as the squares on the diameters.

(In the same way as with polygons in this context he means the area when he says "circle.")

Before sketching Euclid's proof we describe the transition to modern formulas. If two circles have areas A_1, A_2 and diameters d_1, d_2 or (to use the modern convention) radii r_1, r_2, then

$$A_1 : A_2 = d_1^2 : d_2^2 = r_1^2 : r_2^2.$$

Consequently,

$$A_1 : r_1^2 = A_2 : r_2^2,$$

so that the ratio of the area A of a circle to the square r^2 on its radius is constant. This constant is now called π, and we write

$$A : r^2 = \pi, \quad \text{or} \quad A = \pi r^2.$$

The second (and in this context secondary) question is to determine the constant π as precisely as possible. In his *Measurement of a Circle* Archimedes showed that

$$3\frac{10}{71} < \pi < 3\frac{10}{70}.$$

(For details and sources about π the interested reader will find complete information in Berggren et al. [1997].)

The final section of Book XII deals with spheres and establishes

Prop. XII.18.
Spheres are to one another in the triplicate ratio of their respective diameters.

Again we may express this result by saying that if V_1, V_2 are the volumes of two spheres with diameters d_1, d_2 and radii r_1, r_2, then

$$V_1 : V_2 = d_1^3 : d_2^3 = r_1^3 : r_2^3,$$

or, as before, for some constant k,

$$V : r^3 = k.$$

One of the major achievements of Archimedes (*On the Sphere and the Cylinder*, Prop. 34) establishes the following relation between this constant k and π:

$$k = \frac{4}{3}\pi.$$

The Proof of Prop. XII.2

The proof of Prop. XII. 2 is by what has later been called the method of exhaustion. We reproduce Euclid's arguments in modern notation. Let the two circles $\mathcal{K}_1$ and $\mathcal{K}_2$ have areas A_1 and A_2 and diameters d_1 and d_2. There exists a certain area B such that

$$d_1^2 : d_2^2 = A_1 : B,$$

and we have to show that $B = A_2$. Euclid achieves this by excluding $B < A_2$ and $A_2 < B$.

First, assume $B < A_2$. Inscribe a square $EFGH$ in $\mathcal{K}_2$ and circumscribe another one as indicated in Fig. 27.1. This shows that $\square EFGH > \frac{1}{2}A_2$. In the next step bisect the arcs EF etc. and inscribe an octagon in $\mathcal{K}_2$. Each one of the triangles like $\triangle EFK$ takes away more than half of the area of the remaining segments. At this point

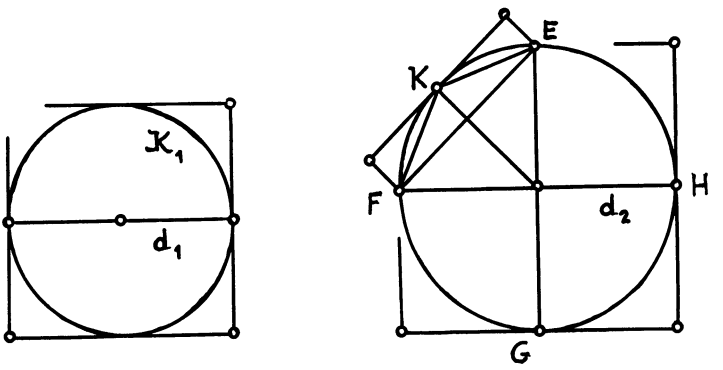

FIGURE 27.1

Euclid quotes Prop. X. 1 and says, "Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle $\mathcal{K}_2$ exceeds the area B." Now let $\mathcal{P}_n$ be a polygon such that $B < P_n$. Inscribe in $\mathcal{K}_1$ a polygon $\mathcal{Q}_n$ similar to $\mathcal{P}_n$. Then, by Prop. XII. 1,

$$\begin{aligned} \mathcal{Q}_n : \mathcal{P}_n &= d_1^2 : d_2^2. \text{ Hence} \\ \mathcal{Q}_n : \mathcal{P}_n &= A_1 : B. \end{aligned}$$

By alternation he derives

$$\mathcal{Q}_n : A_1 = \mathcal{P}_n : B,$$

and this is a contradiction because $\mathcal{Q}_n < A_1$, and $\mathcal{P}_n > B$.

By reducing the case $B > A_2$ to the preceding one the proof is complete.

27.3 The Pyramid

Euclid has defined a pyramid at the beginning of Book XI:

Def. XI.12.
A pyramid is a solid figure, contained by planes, which is constructed from one plane to one point.

Clearly, the Greeks knew the great pyramids of Egypt. Just imagine that for them these pyramids were as old as for us a building constructed by the Roman emperor Augustus some 2000 years ago. One might think of an Egyptian origin of the word "pyramid" itself, but the Greek dictionary gives a more plausible explanation by the way of *pyra* = funeral pile (associated to *pyr* = fire) with the extended meaning of a holy burying place. This is exactly what the Egyptian pyramids were.

From elementary geometry we know the formula for the volume P of a pyramid with base b and altitude h to be $P = \frac{1}{3}bh$. (See, for instance, the high-school text by Jacobs [1987], who states it without proof.) Euclid proves this result and expresses it in his way without a formula. We will sketch his theory and see how he realizes various

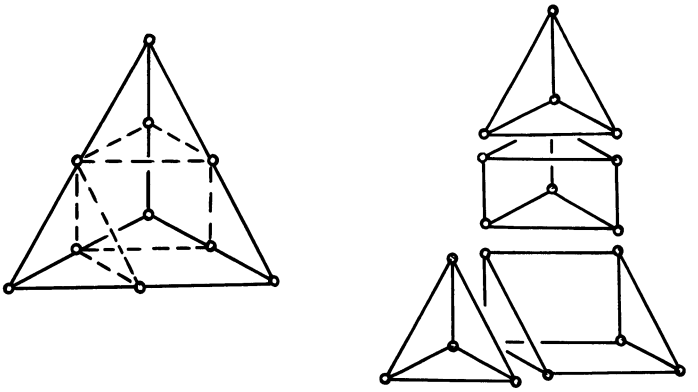

FIGURE 27.2

aspects of $P = \frac{1}{3}bh$. (For the proofs see Heaths's text.) His general procedure is, even if more complicated, similar to his treatment of the circle. The first step is to find a solid replacement for the polygons inscribed in a circle (see Fig. 27.2).

Prop. XII.3.
Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

The second step provides us with an analogy to Prop. XII. 1 by looking at the proportions of the approximating solids. Let P_1, P_2 be two triangular pyramids with bases b_1, b_2 and equal altitudes, and let W_i be the combined volumes of the two prisms in P_i.

Prop. XII.4.
[abbreviated] $W_1 : W_2 = b_1 : b_2$.

Step 3 is the proof by exhaustion of the main result.

Prop. XII.5.
Pyramids which are of the same height and have triangular bases are to one another as the bases.

The proof of XII.5 follows the pattern of XII.2. From XII.3 we have the subdivision of a pyramid P_0 into two prisms (more than its

half) and two smaller pyramids P_1, P_2. Subdivide these as before and iterate, and so on.

What follows is a technical extension of the preceding result for pyramids with triangular bases.

Prop. XII.6.
Pyramids which are of the same height and have polygonal bases are to one another as the bases.

Let us see how close we are to the formula. For pyramids with equal heights we know that

$$P_1 : P_2 = b_1 : b_2.$$

If we ignore dimensions (something that Euclid could not do) and regard P_i, b_i as real numbers, we could alternate to

$$P_1 : b_1 = P_2 : b_2 = k, \text{ a constant.}$$

Even if Euclid could not calculate as we did, he determines k in his next step. (Fig. 27.3).

Prop. XII.7.
Any prism which has a triangular base is divided into three pyramids equal to one another which have triangular bases.

This gives us $k = \frac{1}{3}$. We still do not yet know how the volume changes with different heights. If one cuts a pyramid by a plane parallel to the base, the problem is that the size of the base changes together with the size of the height. (Fig. 27.4) In his next proposi-

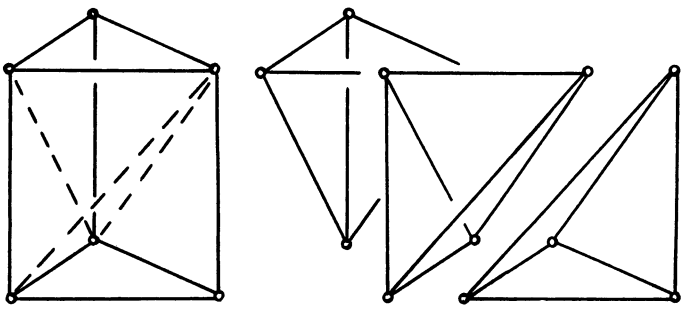

FIGURE 27.3

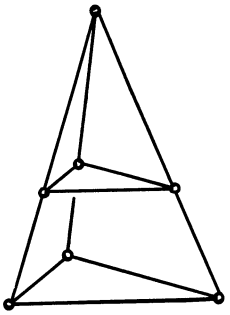

FIGURE 27.4

tion, Euclid solves this problem in a roundabout way. He extends Prop. XI. 33 about similar parallelepipeds to pyramids.

Prop. XII.8.
Similar pyramids which have triangular bases are in the triplicate ratio of their corresponding sides.

This together with the last proposition solves the problem of the volume of pyramids completely.

Prop. XII.9.
Let P_1, P_2 be equal pyramids with triangular bases b_1, b_2 and altitudes h_1, h_2. Then

$$P_1 = P_2 \quad \Leftrightarrow \quad b_1 : b_2 = h_2 : h_1.$$

As with triangles, Euclid never measures volumes. He thinks geometrically and does not measure volumes in our sense, that is, he does not assign a number to a solid as its volume. On the other hand, one could, given the above information and starting from a unit cube, derive the formula $P = \frac{1}{3}bh$.

27.4 Cylinders, Cones, and Spheres

Euclid's treatment of cylinders, cones, and spheres follows the pattern established for the circle and the pyramid. The geometry gets more complicated, but the method of exhaustion is basically the

same. Only in Props. XII, 13/14 does a new element enter the proofs. In Prop. XII. 14 he treats cylinders cut by planes parallel to the base (that is, of different heights) and employs the definition Def. V. 5 of proportionality in a way similar to Prop. VI. 1. Thus he gets

Prop. XII.14.
Cones and cylinders on equal bases are to one another as their heights.

One wonders why he does not state a similar result for prisms and pyramids. At this point one could speculate about the contributions of various pre-Euclidean authors to the text, but we will leave this to the specialists.