CHAPTER 2

A Problem Worthy of Great Minds

2. This is the method of procedure when there are two players. If two players, playing in several throws, find themselves in such a state that the first lacks two points and the second three of gaining the stake, you say it is necessary to see in how many points the game will be absolutely decided.

It is convenient to suppose that this will be in four points, from which you conclude that it is necessary to see how many ways the four points may be distributed between the two players and to see how many combinations there are to make the first win and how many to make the second win, and to divide the stake according to that proportion. I could scarcely understand this reasoning if I had not known it myself before; but you also have written it in your discussion.

Pascal continues his letter, following his practice of numbering each section for subsequent reference. Since the two mathematicians have already exchanged several letters on the subject, in his second section, he can dive straight into the details.*

Probability

Today, we would use the word *probability* to refer to the focus of Pascal and Fermat's discussion, but that term was not introduced until nearly a century after the mathematicians' deaths. Instead, they spoke of "hazards," or number of chances. Much of their difficulty was that they did not yet have the notion of mathematical probability—because they were in the process of inventing it.

From our perspective, it is hard to understand just why they found it so difficult. But that reflects the massive change in human thinking that their work led to. Today, it is part of our very worldview that we see things in terms of probabilities.

Yet, for all its familiarity, probability remains a tricky notion to deal with. For one thing, there are actually several notions of probability. Two in particular are common: frequentist probability and subjective probability.**

*The entire correspondence took place in 1654. It is not known how many letters were involved. Seven are extant today: three from Pascal and four from Fermat, spanning the period July to October. They can all be found in the volume *Oeuvres de Fermat,* vol. 2 (1894). English translations are provided in D. E. Smith, *A Source Book in Mathematics* (New York: Dover, 1959).

**The distinction between different forms of probability was first made in 1713 by the Swiss mathematician Jakob Bernoulli, who is largely responsible for taking Pascal and Fermat's ideas and expressing them in what became the terminology of modern probability theory.

Frequentist probability is the one most people are familiar with today. It's the notion that arises in the classic games of chance such as cards, dice, or roulette, where you can use a theoretical mathematical argument to calculate the probability of a certain event, such as rolling boxcars (double 6's) when you throw two dice. (The probability in this case is 1/36.) It also occurs when you have some large population of people, animals, or objects, or some action that is repeated many times, where you can count the relative frequency of a particular outcome—say, the probability that a man chosen at random from a large population will die before he reaches sixty. The frequentist probability of an event *E* occurring in some population (or repeated action) *A* is the number of different ways E can occur (or the number of times E does occur) divided by the total number of different outcomes. For example, if you roll a die, it can land six different ways. There are three ways it can land with an even number face up (2, 4, or 6), so the probability of rolling an even number with an honest die is 3/6, or 1/2.

Subjective probability, also called *epistemic probability,* refers to a numerical estimate of the veracity of our knowledge of some event, where that knowledge is not based (entirely) on statistical data—for example, when a woman tells you she is 95 percent certain she knows the way to the post office.

This seemingly clear distinction becomes blurred when a subjective probability is based on data, but not in an entirely deterministic way, or when techniques such as Bayes' formula (which we'll get to later) are used to refine subjective probabilities in the light of concrete data. Although my primary focus in this book is on the development of frequentist probability, much of its impact on our world has come when its precise mathematical calculus has been used to base or to buttress subjective decision making.

But that was all in the future when Pascal and Fermat, with no idea of the consequences of their undertaking, embarked on their quest to solve the problem of the points.

Struggling Toward the Solution

Today, anyone who has had just a few hours of instruction in probability theory can solve the problem of the points with ease, either the simple version I gave in the last chapter or the one Pacioli considered or even, for someone comfortable with elementary algebra, the general case of "*N* rounds."* But the original solution came only after much effort, during which time different mathematicians proposed quite different answers, of which only one could possibly be correct.

Pacioli, the man who first wrote about the problem, considered a version in which the game is played until one player has won six rounds, but play is abandoned when the score is 5 to 2. He suggested that the solution is to divide the pot according to the current state of play, namely, 5 to 2, but this reasoning is incorrect. The flaw in Pacioli's reasoning

*I shall present a complete solution to the simple case later in this chapter.

was demonstrated in 1539 by the next person to try to solve the problem, his countryman Girolamo Cardano.

Cardano noted, correctly, that the apportionment of the pot depended not on how many rounds each player had already won (as Pacioli thought) but on how many each player must still win in order to win the contest. Although this insight helped pave the way to the eventual solution, Cardano, too, failed to find the right answer. He did, however, make several key observations that established the beginning of what, following Pascal and Fermat's work, became probability theory.

A mathematician and a physician, Cardano published books on both disciplines during a highly colorful life. Because of his illegitimate birth and sharp tongue, he was initially denied admission to the College of Physicians in Milan, only to be admitted later in life because of his tremendous achievements. He worked as a country doctor, a lecturer in mathematics in Milan (in which city he grew to be the most prominent and most sought-after physician), and then as a professor of medicine in Bologna—a position from which he was subsequently dismissed after being arrested and accused of heresy. In addition to books on mathematics and medicine, Cardano also wrote on astronomy, physics, chess, death, the immortality of the soul, wisdom, and games of chance, the last reflecting another prominent side of his multifaceted life, his compulsive gambling.

Cardano's lasting contribution to the creation of probability theory came in a manuscript he wrote in 1564, titled

Liber de ludo aleae (The Book on Games of Chance) and published in 1663, almost a century after his death. He wrote it not as a mathematics treatise but as a practical guide for gamblers. Buried among many pages of advice for the gaming table is some important mathematics, in particular the rule for when odds may be added (namely, when a game splits into separate cases) and the derivation of the hugely important multiplication rule for combining odds when a game is repeated several times. (The first rule tells you that if you throw a die, the odds of getting a 1 or an even number are $1/6 + 1/2 = 2/3$; the second rule tells you that when you throw a die twice, the odds of getting a 1 followed by an even number are $1/6 \times 1/2 = 1/12$.)

The extent to which our modern approach to probability calculations is a recent innovation is made clear by the fact that not only would it be some time before Cardano's ideas were generally accepted, but another leading Italian mathematician, Niccolò Tartaglia, in joining Cardano in deriding Pacioli's solution in 1556, added that he believed the problem could not be solved at all! In 1603, another Italian mathematician, Lorenzo Forestani, reached essentially the same conclusion. In his book *Practica d'arithmetica e geometria (The Practice of Arithmetic and Geometry)*, he suggested that the portion of the stake should be divided based on the number each had won in relation to the number of games played, with the remainder divided equally between them, because the remainder of the game favors neither player. This is a hopeless analysis by today's standards, but, as with Tartaglia, it reflected the widely held belief that the future was a matter of pure chance, with every possibility equally likely.

The last major figure to enter the picture prior to Pascal and Fermat was no less than Galileo Galilei, the father of modern science, who wrote a paper sometime between 1613 and 1623 titled *Sopra le scoperte dei dadi (On a Discovery Concerning Dice),* in which he analyzed (completely and correctly) all the ways three dice will give totals of 9 and 10 points. (Discounting order, each total can be obtained six different ways.) His motivation for doing this, it seems, was to justify the belief among gamblers that a total of 10 is a slightly better bet than 9. Galileo demonstrated that this is indeed the case, by calculating that a total of 10 can be obtained by way of twenty-seven different dice-throw outcomes, whereas 9 can be obtained by only twenty-five. This problem had also been considered and solved by Cardano. The significance of Galileo's contribution was that, whereas Cardano had reasoned theoretically, Galileo approached the problem scientifically, beginning with an empirical observation—that 10 occurs more often than 9 (the difference is sufficiently small that it takes a lot of plays and a very keen eye to spot this pattern)—and then seeking a mathematical explanation. This very much set the stage for what came next.

Pascal and Fermat

Blaise Pascal (1623–1662) was a child prodigy. He was born on June 19 in Clermont, France, today's Clermont-Ferrand.

His mother died when he was three, and a few years later, Pascal's father, Étienne, a wealthy tax official and a keen amateur mathematician, moved the family from Clermont to Paris, where he personally oversaw his son's education at home.

Étienne maintained some odd views. He decided that his son should not study mathematics before the age of fifteen and, accordingly, removed all mathematics texts from their house. This prohibition only raised young Blaise's curiosity about the banned subject, and he started to work on geometry in secret at the age of twelve. He discovered on his own that the sum of the angles of a triangle is two right angles; when Étienne found out, he was so impressed that he removed the ban and allowed his son to read mathematics texts, starting with Euclid's classic work *Elements.* He also started to take the obviously gifted Blaise to meetings of Mersenne's Academy, one of several semiformal groups of mathematicians and scientists in Paris that eventually gave birth to the Académie Royale des Sciences in 1666. As I noted in Chapter 1, when Pascal was sixteen, he wrote his first paper, on conic sections, and presented it to Mersenne's Academy.

To assist his father's tax-collecting work, the teenage Pascal also invented the calculating machine I mentioned earlier, and oversaw its manufacture and sale. The device, the Pascaline, looked much like the mechanical calculators that were sold throughout the world in the 1940s and 1950s. Pascal worked on developing his calculator for three years, between 1642 and 1645. Perhaps because his interests lay elsewhere, he was not a success as a calculator entrepreneur: the device sold only in small numbers and eventually went out of production.

The adult Pascal devoted his life to the study of mathematics, natural science, and religion, all privately supported by his family's fortune (he never took a university position). One of his better-known mathematical studies was what we call Pascal's triangle. To generate the triangle, you start with a 1, and then immediately below it, you put two 1's, one to either side. Then for each successive row, you put a new 1 at either end and complete the row between them by adding together each adjacent pair of entries in the row above and putting their sum halfway between them. Here are the first few rows:

```
                1
            1       1
        1       2       1
    1       3       3       1
1       4       6       4       1
1     5      10      10      5      1
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From today's perspective, Pascal's triangle does not appear mathematically deep; nor do the many interesting properties Pascal discovered that relate the various entries. But it turned out to be particularly important in elementary algebra and in probability theory, since the entries in each row are the so-called binomial coefficients that occur in the expansions of the expression $(a + b)^n$. For example

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad [1\text{–}2\text{–}1]$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad [1\text{–}3\text{–}3\text{–}1]$$

In his twenties, Pascal became ill and never fully regained his health (he died at age thirty-nine). In his later years, he lost interest in mathematics and focused his attention on writing religious treatises.

Pierre de Fermat (1601–1665) was born to a family of wealthy merchants. He studied law at the universities of Toulouse and Orléans, and mathematics at Bordeaux. His law studies prepared him for his subsequent career as a lawyer and jurist; his mathematical training prepared him for his lifelong interest in the subject. Although Fermat is often described today as an amateur mathematician, this is true only in the sense that he did not get paid for his research. He devoted immense amounts of time to it and was famed in his day as one of the greatest mathematicians in Europe. With no need to earn a living doing mathematics, he published almost nothing. Instead, he carried out his work and dispersed his ideas and results through ongoing correspondences with the leading mathematicians of the time.

Fermat worked in several areas of mathematics, including geometry, where he developed algebraic coordinate geometry independently of René Descartes, after whom it is often named. He also made important contributions to the early development of calculus. But he is known best for his research into number theory, the branch of mathematics that looks at properties of the positive whole numbers. With a distinguished history going back to the ancient Greeks, number theory was in Fermat's time regarded as one of the pinnacles of mathematics. (It remains so today.) Fermat made many profound discoveries in the field, though his most widespread fame came from an observation that (to the best of our knowledge) he never actually proved. He claimed that the equation

$$x^n + y^n = z^n$$

has no solution in which x, y, and z are all positive whole numbers and n is a whole number greater than 2. (The solutions when n is 2 are, of course, familiar to anyone who's taken high school geometry.) Fermat made a brief note in the margin of a book—Diophantus's *Arithmetica*—saying he had found a "truly marvelous proof" of this conjecture but that it was too big to fit in the margin. Discovered after his death, the conjecture eventually became known as Fermat's last theorem, the name reflecting the fact that of all the results he claimed during his mathematical career, this was the only one that no one was able to prove or disprove, even hundreds of years after his death. It was finally proved in 1994 by the English mathematician Andrew Wiles, by means of a long, complicated argument using techniques not available in Fermat's time. Most mathematicians believe Fermat was mistaken, and what he thought was a proof turned out to be flawed—which may be why he never revealed it.

The Strategy

In 1654, the gambler Antoine Gombaud, whose noble title was the Chevalier de Méré,* approached his friend Pascal with some questions about games of chance, including the problem of the unfinished game. After some thought, Pascal found a possible solution but was not completely sure his reasoning was correct. Accordingly, he sent his ideas to Fermat to see if his countryman agreed with the argument. The brief exchange of letters that ensued—and one letter in particular—represented one of the most profound advancements in the history of mathematical thought.

Before we take a look at their exchange and the methods it contains, let's look at a present-day solution of the simple version of the problem. In this version, the players, Blaise and Pierre, place equal bets on who will win the best of five tosses of a fair coin. We'll suppose that on each round, Blaise chooses heads, Pierre tails. Now suppose they have to abandon the game after three tosses, with Blaise ahead 2 to 1. How do they divide the pot?

The idea is to look at all possible ways the game might have turned out had they played all five rounds. Since Blaise

**Chevalier* was the lowest rank of nobility in France at the time, equivalent to a knight in England. The word is related to the English *chivalry.*

is ahead 2 to 1 after round three, the first three rounds must have yielded two heads and one tail.

The remaining two throws can yield

H H   H T   T H   T T

Each of these four is equally likely. In the first (H H), the final outcome is four heads and one tail, so Blaise wins; in the second and the third (H T and T H), the final outcome is three heads and two tails, so again Blaise wins; in the fourth (T T), the final outcome is two heads and three tails, so Pierre wins. This means that in three of the four possible ways the game could have ended, Blaise wins, and in only one possible play does Pierre win. Blaise has a 3-to-1 advantage over Pierre when they abandon the game; therefore, the pot should be divided 3/4 for Blaise and 1/4 for Pierre.

Many people, on seeing this solution, object, saying that the first two possible endings (H H and H T) are in reality the same one. They argue that if the fourth throw gives a head, then at that point, Blaise has his three heads and has won, so there would be no fifth throw. Accordingly, they argue, the correct way to think about the end of the game is that there are actually only three possibilities, namely

H   T H   T T

in which case, Blaise has a 2-to-1 advantage and the pot should be divided 2/3 for Blaise and 1/3 for Pierre, not 3/4 and

1/4. This reasoning is incorrect, but it took Pascal and Fermat some time to resolve this issue. Their colleagues, whom they consulted as they wrestled with the matter, had differing opinions. So if you are one of those people who finds this alternative argument appealing (or even compelling), take heart; you are in good company (though still wrong).

The issue behind the dilemma here is complex and lies at the heart of probability theory. The question is, What is the right way to think about the future (more accurately, the range of possible futures) and model it mathematically?

Interestingly, as far as we know, neither Pascal, Fermat, nor anyone else sought to resolve the issue empirically.* If you were actually to play out the completion of the game many times—that is, imagine the game had been halted after three tosses, with Blaise ahead 2 to 1, and then toss actual coins to complete the game—you would find that Blaise wins roughly 3/4 of the time. This would not constitute a mathematical proof, but it would indicate which solution is the right one. Many people today, when faced with such a puzzle about probabilities, do resort to a simulation (either in real life or on a computer) to help clarify their thoughts.

From today's perspective, we can explain exactly where the difficulty lies. You can indeed, if you wish, take the game's possible endings to be these:

H　　　T H　　　T T

*I'll come back to this interesting point later.

But if you do, you have to account for the frequencies of occurrence of each case. They are not all the same. If you do the math correctly, you find that the first outcome, H, occurs twice as often as either of the other two; the relative frequencies are 2 to 1 to 1, respectively. When you take account of those relative frequencies, you arrive at the same answer as you get with the previous approach: Blaise wins three out of four times.

Still confused? So was Pascal when he tried to understand the explanation of that very issue that Fermat had laid out in his previous letter. Let's take a closer look at how Pascal starts the second section of his letter:

> *2. This is the method of procedure when there are two players. If two players, playing in several throws, find themselves in such a state that the first lacks two points and the second three of gaining the stake, you say it is necessary to see in how many points the game will be absolutely decided.*
>
> *It is convenient to suppose that this will be in four points, from which you conclude that it is necessary to see how many ways the four points may be distributed between the two players and to see how many combinations there are to make the first win and how many to make the second win, and to divide the stake according to that proportion. I could scarcely understand this reasoning if I had not known it myself before; but you also have written it in your discussion.*

As I observed earlier in this chapter, Cardano had already realized that the key was to look at the number of points each player would need in order to win, not the points they had already accumulated. In the second section of his letter to Fermat, Pascal acknowledged the tricky point we just encountered ourselves, that you have to look at all possible ways the game could have played out, ignoring the fact that the players would normally stop once one person had clearly won. But Pascal's words make clear that he found this hard to grasp, and he accepted it only because the great Fermat had explained it in his previous letter.

When Pascal continued his letter, it was to make the same simplifying assumption I did in formulating my version of the problem, reducing it from regular dice, which have six possible outcomes, to dice that have just two values, making them equivalent to tossing coins. When you toss four coins, there are $2 \times 2 \times 2 \times 2 = 16$ possible outcomes. Pascal enumerated them all in a table. The table has sixteen columns, each column listing the outcomes of the four throws. Beneath each column of the table he wrote a "1" if that outcome gives a win for player 1 and a "2" if it gives a win for player 2. Here is how he described this in his letter:

> *Then to see how many ways four points may be distributed between two players, it is necessary to imagine that they play with dice with two faces (since there are but two players), as heads and tails, and that they throw*

four of these dice (because they play in four throws). Now it is necessary to see how many ways these dice may fall. That is easy to calculate. There can be sixteen, *which is the second power of* four; *that is to say, the square. Now imagine that one of the faces is marked* a, *favorable to the first player. And suppose the other is marked* b, *favorable to the second. Then these four dice can fall according to one of these sixteen arrangements.*

a	*a*	*a*	*a*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*b*	*b*	*b*	*b*
a	*a*	*a*	*a*	*b*	*b*	*b*	*b*	*a*	*a*	*a*	*a*	*b*	*b*	*b*	*b*
a	*a*	*b*	*b*	*a*	*a*	*b*	*b*	*a*	*a*	*b*	*b*	*a*	*a*	*b*	*b*
a	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*	*a*	*b*
1	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2

and, because the first player lacks two points, all the arrangements that have two a*'s make him win. There are therefore 11 of these for him. And because the second lacks three points, all the arrangements that have three* b*'s make him win. There are 5 of these. Therefore it is necessary that they divide the wager as 11 is to 5.*

There is your method, when there are two players, whereupon you say that if there are more players, it will not be difficult to make the division by this method.

Having written out the table and noted which plays lead to a win for which player, Pascal had only to count the number of ways each can win—that is, the number of 1's and the number of 2's in his bottom row. His conclusion: for the game they were considering, the stake should be divided 11 to 5.

So far so good. But Pascal was by no means satisfied.

CHAPTER 3

On the Shoulders of a Giant

3. On this point, Monsieur, I tell you that this division for the two players founded on combinations is very equitable and good, but that if there are more than two players, it is not always just and I shall tell you the reason for this difference. I communicated your method to [some of] our gentlemen, on which M. de Roberval made me this objection:

That it is wrong to base the method of division on the supposition that they are playing in four *throws seeing that when one lacks* two *points and the other* three, *there is no necessity that they play* four *throws since it may happen that they play but* two *or* three, *or in truth perhaps* four.

Since he does not see why one should pretend to make a just division on the assumed condition that one plays four throws, in view of the fact that the natural terms of the game are that they do not throw the dice after one of the players has won; and that at least if this is not false, it *should he proved. Consequently he suspects that we have committed a paralogism.**

As Pascal makes clear at the beginning of the third section of his letter to Fermat, others who have seen Fermat's proposed solution are also having trouble understanding why he ignored the fact that the two players would surely stop as soon as they recognized that one of them had already won.

Number Was the Key

When a single document, such as the August 24, 1654, letter from Pascal to Fermat, turns out to have a pivotal effect, the document itself can tell only part of the story. Prior circumstances must have prepared the ground. As Isaac Newton once wrote to the British physicist Robert Hooke, "If I have seen further [than certain other men] it is by standing upon the shoulders of giants."** What Newton did not say is that getting up onto those shoulders often involves clambering over many other figures who have labored out of the limelight, making many small steps that together yield a ramp.

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*The word *paralogism* is rarely used nowadays. It means a fallacious or illogical argument or conclusion. It comes from the late Latin *paralogismus,* from the Greek *paralogismos,* itself from *paralogos,* meaning "unreasonable" or "beyond (= *para*) logic."

**He was referring to his dependency on Galileo's and Kepler's work in physics and astronomy.

The development of probability took place during an extraordinary time in human history. The seventeenth century saw the birth not only of calculus and probability theory but also of modern science. These were not coincidences; all were part of a major shift in the way humans understand our world. Key to everything was number.

Though numbers themselves were first introduced around ten thousand years ago in Sumeria, their use was restricted to those who could both master cumbersome notations and become skilled in using the mechanical devices (such as the abacus) for carrying out computations. The modern, so-called Hindu-Arabic number system, developed in India between 200 and 700 A.D., was the first truly efficient way to record and compute with numbers, though this system was not available in the West until Leonardo of Pisa (whom later historians dubbed "Fibonacci") learned it from North African traders and described it in his book *Liber abaci (The Book of Calculating)*, which first appeared in 1202.

The new number system made it possible for anyone to master and use basic arithmetic. The first group to take advantage of this powerful new tool was the Italian merchants, for whom Leonardo primarily wrote his book. (His hometown, Pisa, was the European capital of what then constituted global trade.) But as a mathematician, Leonardo was interested in the theoretical aspects of the newly learned number system as well as its use, and as a result, his book also provided a source for scholars to study the new methods and to make use of this powerful new tool in their own researches.

By the time Luca Pacioli was born in 1445 in Sansepolcro, a small town more or less in the center of Italy, the Hindu-Arabic number system was widely known and used by both businesspeople and scholars. Like Leonardo before him, Pacioli was an excellent mathematician who is largely remembered not so much for his original contributions as for his book *Summa de arithmetica, geometrica, proportioni et proportionalita* (which I mentioned earlier), which cataloged, in a systematic way, all of what was known at the time. Published in Venice in 1494, the book draws heavily on both *Elements* and *Liber abaci* but also contains much that had been discovered since those two earlier great works appeared, particularly in algebra, where great strides had been made in the solution of polynomial equations.

Pacioli became a good friend of Leonardo da Vinci, whose own interests in mathematics and science are well known. It was Leonardo who drew the illustrations for Pacioli's book *Divina proportione (The Divine Proportion).* Published in 1509, it is a study of what a nineteenth-century writer would rename the golden ratio, a mathematical constant (approximately equal to 1.618) first mentioned in *Elements.* This association is almost certainly the origin of the belief, which lacks the slightest supporting evidence, that Leonardo based many of his art works, including the *Mona*

Lisa, on that particular number. This belief is almost certainly false, but that has not prevented it from achieving the status of an urban legend in the world of art and, in due course, in popular culture.*

But I digress. Pacioli's *Summa* is significant in our story on two counts, first because it was so comprehensive. Occupying some six hundred densely printed folio pages, it treats arithmetic from both a theoretical and a practical standpoint, contains multiplication tables up to 60 × 60, provides a table of moneys, discusses weights and measures used in the various Italian states, and even provides one of the earliest accounts of double-entry bookkeeping, the technique that is so essential to all modern business and commerce. The second significance of *Summa* is that it includes the problem of the points—accompanied by Pacioli's incorrect solution—and thus provided a solid platform for the individual who set the stage for the Pascal-Fermat correspondence: Girolamo Cardano.

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*Two other beliefs about this particular number are often mentioned in magazines and books: that the ancient Greeks believed it was the proportion of the rectangle the eye finds most pleasing and that they accordingly incorporated the rectangle in many of their buildings, including the famous Parthenon. These two equally persistent beliefs are likewise assuredly false and, in any case, are completely without any evidence. For one thing, tests have shown that human beings who claim to have a preference at all vary in the rectangle they find most pleasing, both from person to person and often the same person in different circumstances. Also, since the golden ratio is actually *not* a ratio of two whole numbers, it is impossible to construct (by measurement) a rectangle having that proportion, even in theory.

The Remarkable Man from Milan

Fazio Cardano, Girolamo's father, was a successful lawyer based in Milan. He was also an accomplished mathematician who lectured on geometry both at the University of Pavia and at the Piatti Foundation in Milan. Leonardo da Vinci consulted him on questions of geometry. When Fazio was in his fifties, he had an affair with Chiara Micheria, a young widow in her thirties who was struggling to raise her three children. Chiara became pregnant, but before she was due to give birth, the plague hit Milan, and Fazio persuaded her to leave the city to have her child in the relative safety of nearby Pavia, where she stayed with wealthy friends of his. There, on September 24, 1501, she gave birth, naming her son Girolamo Cardano after his father. Her joy was short-lived, however. Shortly after the birth, she learned that her first three children, whom she had left behind in Milan, had all died of the plague.

Fazio and Chiara lived apart for many years but eventually married. When Girolamo grew up, he became his father's assistant. Fazio taught his son mathematics, leading the young man to contemplate an academic career. Fazio, however, wanted his son to study law, and the two at first argued until the father relented and allowed Girolamo to enter Pavia University (where he himself had studied) to read medicine.

When war broke out, the university was forced to close, and the young Cardano transferred to the University of

Padua to complete his studies. There he campaigned to become rector of the university. But for all his brilliance as a student, he was an outspoken and somewhat obnoxious individual who was not well liked. In the end, he beat his rival by only a single vote.

Cardano was one of the first major figures in mathematics to write an autobiography *(De vita propria liber [The Book of My Life])*, and as a result, we know far more about him than about most of his predecessors. Some might say he tells us far more than we care to know, since he was not sparing in the personal details.

Of his appearance, we know by his own words that he was skinny, had a long neck, a heavy lower lip, a wart over one eye, and a voice so loud that even his friends complained about it. We know, too, that he was constantly in ill health, suffering from, among other ailments, diarrhea, ruptures, kidney trouble, and palpitations. He was by his own account "ever hot-tempered, single-minded, and given to women." Elsewhere, he describes himself as "cunning, crafty, sarcastic, diligent, impertinent, sad, treacherous, magician and sorcerer, miserable, hateful, lascivious, obscene, lying," and "obsequious."

Of the characteristics that almost lost him the campaign for rector, he writes:

> This I recognize as unique and outstanding amongst my faults—the habit, which I persist in, of preferring to say above all things what I know to be displeasing to the ears of

my hearers. I am aware of this, yet I keep it up willfully, in no way ignorant of how many enemies it makes for me.

None of those character flaws seems to have prevented Cardano from giving the world its first systematic understanding of how to compute probabilities. Given his intellectual curiosity and abilities, his primary interest in the mathematics of games of chance may well have been scientific, but it is hard to discount another (perhaps even primary) motive. Throughout his life, Cardano was a compulsive gambler who needed every bit of help he could find at the gaming tables, from mathematics or any other source. (And he did find other sources of help. Once, when he suspected he was being cheated at cards, he took out the knife he always carried with him and slashed his opponent's face.)

Cardano's gambling began at an early age. Shortly after he moved to Padua, his father died, leaving him a small inheritance. It did not take Girolamo long to squander it all, and he turned to the gaming tables to maintain his lifestyle. From then on, he was addicted. "I have played not off and on but, as I am ashamed to say, every day," he wrote. Though his mathematical ability often helped him get the better of his opponents, things did not always go his way, and like any gambler, he lost more money than he won, sometimes causing significant personal hardship for him and those close to him.

Despite the gambling, Cardano's studies went well enough that in 1525, he was awarded his doctorate in medicine. He applied to join the College of Physicians in Milan, where his mother still lived. But by then the college was well aware of his difficult nature, and in spite of his exceptional performance as a student, college officials were reluctant to admit him. When they learned of his illegitimate birth, they had the excuse they wanted to reject his application.

On the advice of a friend, Cardano moved to Sacco, a small village about ten miles from Padua. Despite not being a member of the College of Physicians, he was able to set up a small, not very successful medical practice. It was there that he met Lucia Bandarini, whom he married in 1531. Since his small medical practice in Sacco did not provide enough income to support a wife, the couple moved to Gallarate, near Milan, the following year.

He applied once more to the College of Physicians in Milan, but again without success. Unable to practice medicine and losing money steadily at the gaming tables, he was eventually forced to pawn his wife's jewelry and some of his furniture. When a move into Milan itself brought no improvement, the couple had to enter the poorhouse.

Cardano's situation changed a little when he managed to secure Fazio's former post of lecturer in mathematics at the Piatti Foundation. When not occupied with teaching, he supplemented his salary by treating some patients. Strictly speaking, since he was not a member of the College of Physicians, this was not allowed, but he achieved some remarkable cures and his reputation as a doctor grew rapidly. With a client list that soon included wealthy people of influence in Milan—including some members of the college—it was surely only a matter of time before the college would be forced to admit him. But then, in 1536, still fuming at his continuing exclusion, he killed his chances by publishing a book attacking not only the college members' medical ability but their character as well:

> The things which give most reputation to a physician nowadays are his manners, servants, carriage, clothes, smartness and caginess, all displayed in a sort of artificial and insipid way.*

Not surprisingly, Cardano's application to join the college the following year was again rejected. Two years later, however, after the heat caused by his book had died down, the continued pressure from his many supporters finally persuaded the college to relent. They modified the rule regarding legitimate birth and admitted Cardano into their ranks.

In the same year, Cardano's first two mathematical books were published, beginning a prolific literary career that saw him writing on mathematics, medicine, philosophy, astronomy, and theology. In 1540, he resigned his mathematics post at the Piatti Foundation and spent the next two years doing nothing but gamble. From 1543 until 1552, he lectured on medicine at the universities of Milan and Pavia.

In 1545, he published his greatest mathematical work, *Ars magna,* a hugely influential book on algebra that pre-

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*Girolamo Cardano, *Autobiography* (New York, 1930).

sented, for the first time, a method to solve a cubic equation. The result became known as Cardano's formula, even though he clearly stated that the method was discovered by Scipione del Ferro around 1500 and independently rediscovered by Niccolò Tartaglia in 1535. Cardano's publication of the method led to a heated dispute with Tartaglia, who had shown his formula to Cardano around 1539 on the strict condition that it be kept secret. Cardano's response was that he published the method only after he learned that del Ferro had obtained it much earlier, making his agreement with Tartaglia no longer binding.

In between his mathematical work, Cardano also found the time to engage in some engineering projects. His name was attached to a number of inventions, among them Cardano's suspension, the Cardano joint, and the Cardano shaft.

Cardano's wife, Lucia, died in 1546. By then, he was enjoying his reputation as the greatest physician in the world and basking in the fame his books and inventions had brought him. He became rector of the College of Physicians and later was appointed professor of medicine at Pavia University. With so many wealthy patients, he soon became rich.

But then disaster struck. His older son, Giambatista, who had qualified as a doctor in 1557, secretly married a young woman called Brandonia di Seroni. Cardano described her in his autobiography as "a worthless, shameless woman," a description that is richly supported by the evidence. With Cardano's financial support, though not his blessing, the young couple moved in with Brandonia's parents. However, the di

Seronis were interested only in what they could extort from Giambatista and his wealthy father, and Brandonia publicly mocked her husband for not being the father of their three children. These taunts eventually drove Giambatista to poison his wife, and after his arrest, he confessed to the crime. Giambatista's father hired the best lawyers, but at the trial, the judge decreed that to save his son's life, Cardano must come to terms with the di Seronis. They demanded a sum which Cardano could never have raised, and his son's fate was sealed. Giambatista was tortured in jail, his left hand was cut off, and on April 13, 1560, he was executed.

Cardano never forgave himself for failing to save his son. Moreover, as the father of a convicted murderer, his reputation was damaged beyond repair. Realizing he had to move, he secured a professorship of medicine at Bologna.

His time in Bologna was full of controversy. His reputation and his arrogant manner combined to create many enemies, and at one point, he humiliated a fellow medical professor in front of his students by pointing out errors in his lectures. After a few years, his colleagues tried to get the senate to dismiss him. They began spreading rumors that his lectures attracted few students.

Cardano had further problems with his children. His remaining son, Aldo, was a gambler and associated with individuals of dubious character. In 1569, the young man gambled away all his own clothes and other possessions in addition to a considerable sum of his father's money. In an attempt to pay off the debt, Aldo broke into his father's house and stole a large amount of cash and jewelry. Cardano sadly reported his son to the authorities, and Aldo was banished from Bologna.

In 1570, in what was probably a deliberate attempt to gain notoriety by offending the Church, Cardano cast the horoscope of Jesus Christ and wrote a book in praise of Nero, tormentor of Christian martyrs. Cardano was jailed on charges of heresy, but because of his fame, he was treated leniently and spent just a few months in prison. On his release, however, he was forbidden to hold a university post and was barred from further publication of his work. For a man such as Cardano, this could have been a more painful punishment than imprisonment, but once free, he went to Rome, and there he received an unexpectedly warm reception. He was granted immediate membership to the College of Physicians, and the pope, who had now apparently forgiven him, granted him a pension. It was in this period that he wrote his autobiography, although it was not published until 1643, long after his death. Cardano is said to have correctly predicted the exact date of his own death, September 21, 1576, but some historians have surmised that he achieved this triumph by committing suicide.

Book of Games of Chance

Cardano's *Liber de ludo aleae (Book of Games of Chance)* was published in 1663, long after his death, but according to his autobiography, he completed it in 1525, while still a young man, and rewrote it in 1565.* It is the first scientific study of dice rolling, based on the premise that there are fundamental principles governing the likelihood of particular outcomes. The book is partly observational (he had a lot of opportunity for observation) and partly a theoretical analysis of how chance events, such as particular outcomes of rolls of dice, aggregate when repeated many times. In modern parlance, it was the first study of frequentist probability.

Cardano did not use the word *probability*; rather, he talked of "chances." The word *probability,* which came later, derives from the Latin *probare* (to prove or test) and *ilis* (to be able) and may thus be understood as "able to be verified," where the verification is empirical—though it has also been suggested that "able to be believed" is a closer approximation to the original meaning.**

Not only did he not use the word *probability,* but Cardano did not even conceive of his work in a way that we would now classify as frequentist probability (i.e., counting relative frequencies). Rather, he saw his enterprise as very much "predicting the future" in the sense of formulating practical rules that would increase the likelihood of winning bets. (A more accurate description would be that the rules decrease the likelihood of losing. Cardano clearly realized

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*The word *aleae* refers to dice games. *Aleatorius,* from the same Latin root, refers to games of chance in general.

**Attempts to trace origins and interpret earlier meanings of words are often fraught with difficulty, and in this case particularly so, since our present-day conception of probability, which has the two aspects of cataloging the past and predicting the future, was not developed until much more recently.

this himself, for he wrote, "The greatest advantage from gambling comes from not playing at all." A present-day echo of that observation can be found in the gambler's joke "I hope I break even tonight; I need the money.") The book is in part a how-to book for the gambler, peppered with asides about Cardano's own gambling experiences and beliefs.

But it is also part mathematics. Cardano defined, for the first time, what we now call *the probability of an event* as a fraction: the number of ways the event can occur divided by the total number of possible outcomes.* (He referred to the latter as the "circuit.") For example, if the event is getting an even number when a die is rolled (one-half the number of faces), the probability is 3/6, that is, 1/2. Here is how Cardano expressed this observation in his book:

> One-half the total number of faces always represents equality [of chance]; thus the chances are equal that a given point will turn up in three throws for the total circuit is completed in six, or again that one of three given points will turn up in one throw. For example, I can as easily throw one, three or five as two, four or six. The wagers there are laid in accordance with this equality if the die is honest.

He likewise observed that the probability of getting, say, a 3 or a 5 on a single roll is 2/6, or 1/3.

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*For ease of understanding, I shall use the present-day term *probability* to describe Cardano's observations.

More penetrating, he observed that to obtain the probability of getting a certain outcome on two successive throws, you square the probability of getting it on a single throw. For example, the probability of getting a 6 twice in succession is $1/6 \times 1/6 = 1/36$. Similarly, the probability of getting an even number twice in succession is $1/2 \times 1/2 = 1/4$. Extending the reasoning, the probability of getting a 6 three times in a row will be $1/6 \times 1/6 \times 1/6 = 1/216$, and the probability of rolling three even numbers is $1/2 \times 1/2 \times 1/2 = 1/8$.

The more general version of this is that if an action occurs twice, and if the probability of an event or outcome E occurring the first time is p_E and the probability of an event or outcome F occurring the second time is p_F, then the probability that both will occur (in that order) is $p_E \times p_F$. This assumes that the first action does not influence the next—in modern parlance, the two occurrences of the action have to be *independent.* (Things will work out differently—in fact, you may not be able to calculate an answer—if the action does not satisfy this requirement.) For example, the probability of rolling a 6 on the first throw and any even number on the second is $1/6 \times 1/2 = 1/12$.

(If you remove the restriction on the order in which the 6 and an even number occur, there are more possibilities and, accordingly, the probability is bigger. The easiest way to calculate the probability in this case is to do what Fermat did in considering the problem of the points, and enumerate all the favorable outcomes. They are 2–6, 6–2, 4–6, 6–4, 6–6. Since there are 36 possible outcomes altogether, this gives the answer 5/36.)

More tricky is Cardano's next result: he calculated the probability of throwing a 1 or a 2 with not one die but a pair of dice. The probability of throwing a 1 or a 2 with a single die is 1/3, so the naive intuition is that with two dice, you double your chances—that is, to a probability of 2/3. But as Cardano observed, this is incorrect. The problem is that a 1 or a 2 could come up on both throws, and by adding the two individual probabilities, you would be counting this possibility twice. To allow for this, you have to subtract from the figure 2/3 the 1/9 probability of getting a 1 or a 2 on both throws. Hence, the correct answer is $2/3 - 1/9 = 5/9$.

(Using Fermat's enumeration method, if you list all the thirty-six possible outcomes, you find that twenty of them include 1 or 2 at least once. Still another way to calculate the answer—more efficient when the number of possibilities grows very large—is to compute the probability that the event does *not* occur, since this avoids the problem of double counting. The probability of not rolling a 1 or a 2 with a single die is 4/6, or 2/3, so the probability of not rolling a 1 or a 2 with two dice is $2/3 \times 2/3 = 4/9$. Hence, since the probability of rolling *something* is 1, the probability of rolling a 1 or a 2 must be $1 - 4/9 = 5/9$.)

With Cardano's analysis, the stage was set for Pascal and Fermat to make their breakthrough.

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CHAPTER 4

A Man of Slight Build

. . . I replied to him [M. de Roberval] that I did not found my reasoning so much on this method of combinations, which in truth is not in place on this occasion, as on my universal method from which nothing escapes and which carries its proof with itself. This finds precisely the same division as does the method of combinations. Furthermore, I showed him the truth of the divisions between two players by combinations in this way. Is it not true that if two gamblers finding according to the conditions of the hypothesis that one lacks two *points and the other* three, *mutually agree that they shall play four complete plays, that is to say, that they shall throw four two-faced dice all at once,—is it not true, I say, that if they are prevented from playing the four throws, the division should be as we have said according to the combinations favorable to each? He agreed with this and this is indeed proved. But he denied that the same thing follows when they are not obliged to play the four throws.*

Advances in science are often delayed because the common wisdom holds that something is impossible. From the time of the ancient Greeks, it was believed that the future was in the hands of the gods—a matter of pure fate. Quantifying the way a pair of dice may fall, as Cardano had done, was one thing; predicting what future throws of the dice might bring (as in the unfinished game) was quite another.

As recently as 1756, over a hundred years after Pascal and Fermat had their famous exchange, the great French mathematician Abraham de Moivre (whom we shall meet later), the man who discovered the normal distribution that forms the basis of contemporary predictive statistics, firmly believed the future was determined by God. He wrote (in his book *Doctrine of Chances*):

> Again, as it is thus demonstrable that there are, in the constitution of things, certain Laws according to which Events happen, it is no less evident from Observation, that those Laws serve to wise, useful and beneficent purposes to preserve the steadfast Order of the Universe, to propagate the several Species of Beings, and furnish to the sentient Kind such degrees of happiness as are suited to their State. But such Laws, as well as the original Design and Purpose of their Establishment, must all be from *without;* the *Inertia* of matter, and the nature of all created Beings, rendering it impossible that any thing should modify its own essence, or give to itself, or to anything else, an original determination or propensity. And hence, if we blind not ourselves with

> metaphysical dust, we shall be led, by a short and obvious way, to the acknowledgement of the great MAKER and GOVERNOR of all; *Himself all-wise, all-powerful* and *good.*[*]

Today, we have no difficulty in accepting that an event of pure chance may have sufficient "predictability" that we may calculate precisely the likelihoods pertaining to its outcome without needing to assume divine determination—the result really is a matter of pure chance. Indeed, today's religious person who sees the hand of God in all things simply accepts that the laws of probability are themselves a manifestation of God's ways. In short, we have taken the notions of undetermined random events and mathematically computed probabilities and folded them into our worldview. Quantum theory starts from the assumption that this is the nature of the very fabric of our universe.

But in the early seventeenth century, things looked very different. Pascal and Fermat not only had to figure out how to perform the calculation that resolves the problem of the unfinished game, but also had to do so within a worldview that considered what they were doing impossible. Prior to 1654, tomorrow was viewed as a matter of Fate, something over which a person had no control. In solving the problem, they were instrumental in changing that view, though as the passage from de Moivre indicates, it took a long time for the true import of their work to break through.

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[*]A. de Moivre, *The Doctrine of Chances* (London: Pearson, 1718).

With the solution to the problem of the points and the subsequent acceptance of probability theory, mortals did have (a measure of) control over the future. An individual might not know with certainty what would happen, but by calculating the probabilities of various likely events, one might choose one's actions so as to minimize dangers and maximize preferred outcomes, or one might purchase insurance to facilitate recovery if things went badly. In short, Pascal and Fermat gave us the ability to manage risk.

An Abundance of Talents

Unlike Cardano, Pascal did not write an autobiography, and accordingly, we must rely on the scholarship of others to discover what kind of a man he was. In *Blaise Pascal: Mathematician, Physicist and Thinker About God,* biographer Donald Adamson writes that his subject was

> a man of slight build with a loud voice and somewhat overbearing manner . . . he lived most of his adult life in great pain. He had always been in delicate health, suffering even in his youth from migraine . . . [and was] precocious, stubbornly persevering, a perfectionist, pugnacious to the point of bullying ruthlessness yet seeking to be meek and humble.*

*D. Adamson, *Blaise Pascal: Mathematician, Physicist and Thinker About God* (Basingstoke: Palgrave Macmillan, 1995).

Historian René Taton, in his biography of Pascal in the *Dictionary of Scientific Biography,* provides this assessment of Pascal's life:

> At once a physicist, a mathematician, an eloquent publicist in the Provinciales . . . Pascal was embarrassed by the very abundance of his talents. It has been suggested that it was his too concrete turn of mind that prevented his discovering the infinitesimal calculus, and in some of the Provinciales the mysterious relations of human beings with God are treated as if they were a geometrical problem. But these considerations are far outweighed by the profit that he drew from the multiplicity of his gifts, [and] his religious writings are rigorous because of his scientific training.*

In December 1639, six years before Pascal had the conversation with the Chevalier de Méré that led to the famous correspondence with Fermat, the Pascal family left Paris to live in Rouen. Blaise's father, Étienne, had been appointed a tax collector for Upper Normandy. In February the following year, the young Blaise published his first genuine mathematical paper, *Essay on Conic Sections.* (The earlier "paper" he had presented to Mersenne's Academy in Paris as a teenager was a single-page affair—little more than a student

*René Taton, *Dictionary of Scientific Biography* (New York: Scribner, 1970–1990).

exercise, though Blaise's young age at the time made it a portent of greater things to come.)

It was in Rouen, from 1642 to 1645, that he worked on the Pascaline, his mechanical calculator. The only previous attempt to construct such a device was in 1624 by a man called Schickard, so Pascal's machine was the second mechanical calculator ever built.

The year after Blaise finished work on his device, his father slipped on the ice and broke his hip. Two young men were hired to care for Étienne. The caretakers were brothers in a nearby religious community of Jansenists. Jansenism was a proselytizing Catholic sect that preached an extreme form of asceticism, sacrifice, and strict adherence to the Scriptures as the only path to salvation. Though the two brothers seemed to have little effect on Étienne Pascal, they were more successful with the young and impressionable son, who adopted their faith. From that point forward, there was to be no pleasurable social life for young Blaise. Intellectual pursuits such as mathematics or science were also distractions to be avoided.

For a short while, Pascal renounced mathematics as he fought to save his soul from eternal damnation. The terror of such a fate eventually became too great for the young man to bear, however, and he became ill, suffering severe headaches and a partial paralysis. His doctor advised him that for the sake of his health, he should abandon the Jansenist ways and lead a life more normal for a young man. Although he would remain strongly religious for the remainder of his all-too-short life, Blaise resumed normal activities. Indeed, he did so with vigor, adding regular visits to the gaming rooms to his earlier academic pursuits. It was at the gambling table that Pascal met the Chevalier de Méré, a keen gambler with sufficient mathematical ability to figure out for himself some of the more favorable odds.

Pascal's first focus, when he resumed his researches, was in physics. He carried out a series of experiments in which he observed that the pressure of the atmosphere decreases with height and so deduced that, at some height, it must thin out to a vacuum. In September 1647, he informed Descartes of his finding when the great philosopher came on a visit. Descartes refused to accept Pascal's reasoning, and the pair argued about it for two days. Reporting on the dispute afterward in a letter to the physicist Christiaan Huygens, Descartes noted rather haughtily that Pascal "has too much vacuum in his head." In October 1647, Pascal published his findings under the title *New Experiments Concerning Vacuums,* which led to disputes with a number of scientists who, like Descartes, did not believe in a vacuum.

Étienne died in September 1651. The event prompted Blaise to write to one of his sisters describing his Christian views on death and how they applied to their deceased father. The ideas he expressed in that letter were to form the basis for *Pensées,* Pascal's most famous work of philosophy, a collection of personal thoughts on human suffering and faith in God, which he began in late 1656 and continued to work on during 1657 and 1658. *Pensées* was finally published in

1670, eight years after Pascal's death. While it is a philosophical text, *Pensées* is not totally devoid of mathematical content. Among its essays is one that has come to be known as *Pascal's Wager*, an argument that belief in God is rational. Stripped of its numerical calculations, Pascal's case is essentially that if God does not exist, one will lose little by believing in the supreme being, while if God does exist, one will lose everything by not believing. We'll look at this argument in its full mathematical glory in Chapter 7.

In 1653, Pascal wrote *Treatise on the Equilibrium of Liquids,* in which he explains what is nowadays known as Pascal's law of pressure. At the same time, he resumed his earlier investigations of conic sections and proved some important theorems in projective geometry. In particular, he completed a work titled *The Generation of Conic Sections,* most of which he had written in 1648. This was meant to be the first part of a comprehensive treatise on conics, but he never got back to it.

His *Traité du triangle arithmétique (Treatise on the Arithmetical Triangle),* completed and printed in 1654 but not released until 1665, contained the numerical triangle that today bears his name. Although he was not the first to study this particular mathematical structure, his insights were far more illuminating than anything written beforehand. Pascal's results concerning binomial coefficients were to lead Isaac Newton to his discovery of a hugely important result known as the general binomial theorem (the binomial theorem for fractional and negative powers).

The full title of Pascal's work, rarely given in historical accounts, was *Traité du triangle arithmétique, avec quelques autres petits traités sur la même matière.* The additional clause translates as "with some other small treatises on the same topic," and the essay covers, among other things, Pascal's fairly comprehensive treatment of everything then known about probability theory (as we now call it). He sent a copy to Fermat, who referred to it in one of his letters to Pascal.

It was around the time he started his correspondence with Fermat about the problem of the points, in the summer of 1654, that Pascal first fell ill. Although Pascal did not know it at the time, he was suffering from the first stages of the stomach cancer that would eventually kill him. In one of his letters, written in July 1654, he told Fermat, "though I am still bedridden, I must tell you that yesterday evening I was given your letter." Before his correspondence with Fermat began, Pascal had been unsure if his own initial attempt at a solution was correct, and he first discussed the matter with his colleague Pierre de Carcavi, another member of Mersenne's Academy. Carcavi in turn suggested that Pascal contact Fermat, acknowledged to be greatest mathematician alive.

A few weeks after writing the August 24 letter that is the focus of our story, Pascal almost lost his life in an accident. The horses pulling his carriage bolted, and the carriage was left hanging over a bridge above the river Seine. Although he was rescued without injury, the experience seems to have affected him psychologically, and not long afterward, he underwent another religious experience. On November 23, 1654, he pledged his life to Christianity once again and, soon after, made the first of several visits to the Jansenist monastery Port-Royal des Champs, about twenty miles southwest of Paris. He began to publish anonymous works on religious topics, including eighteen *Provincial Letters* published during 1656 and early 1657. These were written in defense of his friend Antoine Arnauld, an opponent of the Jesuits and a defender of Jansenism, who was on trial before the faculty of theology in Paris for his controversial religious works.

As the pain from the malignancy in his stomach increased, Pascal lost interest in science. In 1658, he tried to do some mathematics, much of it in correspondence with others, but that was largely as a distraction from the pain and he soon gave up. He spent his last years giving to the poor and going from church to church in Paris, attending one religious service after another.

On August 19, 1662, at age thirty-nine, Blaise Pascal died in Paris, in intense pain after the cancer reached his brain. He left behind a legacy of ideas made all the greater by his brief correspondence with Fermat in 1654.

First Letters

The letter from Pascal to Fermat that began their collaboration has unfortunately been lost, and our knowledge of the two men's correspondence begins with Fermat's undated reply. After a brief discussion of the part of Pascal's letter that Fermat has found problematic—concerning the ever troublesome issue of how to handle games where one player wins in fewer than the maximum number of rounds—Fermat explains the difficulty he sees in Pascal's approach:

> *But you proposed in the last example in your letter (I quote your very terms) that if I undertake to find the six in eight throws and if I have thrown three times without getting it, and if my opponent proposes that I should not play the fourth time, and if he wishes me to be justly treated, it is proper that I have 125/1296 of the entire sum of our wagers.*
>
> *This, however, is not true by my theory. For in this case, the three first throws having gained nothing for the player who holds the die, the total sum thus remaining at stake, he who holds the die and who agrees to not play his fourth throw should take 1/6 as his reward.*
>
> *And if he has played four throws without finding the desired point and if they agree that he shall not play the fifth time, he will, nevertheless, have 1/6 of the total for his share. Since the whole sum stays in play it not only follows from the theory, but it is indeed common sense that each throw should be of equal value.*
>
> *I urge you therefore [to write me] that I may know whether we agree in the theory, as I believe [we do], or whether we differ only in its application.*
>
> *I am, most heartily, etc.,*
>
> *Fermat*

Pascal acknowledges the validity of Fermat's objection in his next letter, dated Wednesday, July 29, 1654, which begins:

> *1. Impatience has seized me as well as it has you, and although I am still abed, I cannot refrain from telling you that I received your letter in regard to the problem of the points yesterday evening from the hands of M. Carcavi, and that I admire it more than I can tell you. I do not have the leisure to write at length, but, in a word, you have found the two divisions of the points and of the dice with perfect justice. I am thoroughly satisfied as I can no longer doubt that I was wrong, seeing the admirable accord in which I find myself with you.*

Pascal was a mathematician of formidable powers. But as their respective mathematical records make plain, Fermat was far superior, and anyone who reads the entire correspondence between the two will see readily that Fermat was the dominant collaborator.

Pascal's Universal Method

Fermat's initial reply to Pascal and the famous August 24 letter from Pascal to Fermat make it clear that the two men had adopted different approaches to solving the problem of the points. While Fermat listed explicitly all the possible combinations of outcomes that could arise in the completion of the game, Pascal adopted a different strategy. Since both methods are correct, they will of course yield the same result, as Pascal made clear in his letter:

> *I did not found my reasoning so much on this method of combinations, which in truth is not in place on this occasion, as on my universal method from which nothing escapes and which carries its proof with itself. This finds precisely the same division as does the method of combinations.*

The "universal method" that Pascal refers to is an instance of what is generally known as a *recursive method.* Recursion is the process whereby a sequence of numbers is generated by a rule that gives the new number in terms of the last number produced (or sometimes the last two numbers). The most famous example of a recursion is the rule for generating the Fibonacci numbers. As an exercise in his book *Liber abaci,* Leonardo of Pisa (Fibonacci) posed a problem about the growth of a rabbit colony. To solve the problem, you have to generate the sequence of numbers that starts with two 1's and grows according to the rule that the new number at each stage is the sum of the previous two numbers. When you follow this rule, you get the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Pascal approaches the problem of the points by looking at the quantity $e(a, b)$ that gives the share of the stake that the first player should be given if player 1 requires a winning throws to win and player 2 requires b winning throws. Clearly, if the numbers a and b are equal, then $e(a, b) = 1/2$. The idea is to see how $e(a, b)$ changes when each player wins one more throw. This leads to an algebraic expression for $e(a, b)$ in terms of $e(a - 1, b)$ and $e(a, b - 1)$, and you can solve the problem of the points by using recursion to calculate $e(2, 3)$, the desired share in the particular game they considered.

The main drawback to Pascal's approach was that it requires some complicated algebra dependent upon the theory of combinations he worked out in connection with his famous triangle. Nevertheless, when carried out correctly (not an easy matter), it does lead to the correct result, and the same approach was subsequently used by Abraham de Moivre, Joseph Louis Lagrange, Pierre-Simon Laplace, and other mathematicians to find general methods for solving a variety of other problems.

A detailed description of Pascal's recursive solution to the problem of the points quickly becomes too technical for this book. But my main reason for not including it is that Fermat's solution is simply much better.

Outside observers often assume that the more complicated a piece of mathematics is, the more mathematicians admire it. Nothing could be further from the truth. Mathematicians admire elegance and simplicity above all else, and the ultimate goal in solving a problem is to find the method that does the job in the most efficient manner. Though the major accolades are given to the individual who solves a particular problem first, credit (and gratitude) always goes to those who subsequently find a simpler solution.

Pascal's solution to the problem of the points was far more complicated than Fermat's and required some sophisticated—and daunting!—algebra. It is hard to follow even for a professional mathematician. But for all its complicated appearance, following it (or even producing it in the first place) is, again from the perspective of the professional mathematician, routine. Fermat's approach is undoubtedly more pedestrian, requiring only that you list all the possibilities and then simply count them. But by penetrating to the heart of the problem and doing *just what is required* to get the answer, Fermat's approach shows true genius.

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CHAPTER 5

The Great Amateur

. . . I therefore replied as follows:

It is not clear that the same gamblers, not being constrained to play the four throws, but wishing to quit the game before one of them has attained his score, can without loss or gain be obliged to play the whole four plays, and that this agreement in no way changes their condition? For if the first gains the two first points of four, will he who has won refuse to play two throws more, seeing that if he wins he will not win more and if he loses he will not win less? For the two points which the other wins are not sufficient for him since he lacks three, and there are not enough [points] in four throws for each to make the number which he lacks.

It certainly is convenient to consider that it is absolutely equal and indifferent to each whether they play in the natural way of the game, which is to finish as soon as one has his score, or whether they play the entire four throws. Therefore, since these two conditions are equal and indifferent, the division should be alike for each. But since it is just when they are obliged to play the four *throws as I have shown, it is therefore just also in the other case.*

That is the way I prove it, and, as you recollect, this proof is based on the equality of the two conditions true and assumed in regard to the two gamblers, the division is the same in each of the methods, and if one gains or loses by one method, he will gain or lose by the other, and the two will always have the same accounting.

Pascal convinces himself that it is correct to consider the imagined completion of the game in such a way that the players complete all remaining throws, not stopping when one player has won, the same way that many people today resolve the issue. It might be a waste of time to play after one person has won, they say, but there is nothing *wrong* with doing that. So it cannot lead to an incorrect answer if you look at it that way. Today, however, accustomed as we are to the fact that probability theory can predict the future, this argument is easier to accept than it was in Pascal and Fermat's time, when it was not even clear that you *could* predict the future, let alone *how* to do it.

The Lawyer in Toulouse

Fermat was born on August 17, 1601, in the French city of Beaumont-de-Lomagne, where his father was a wealthy leather merchant and second consul of the city. Pierre had one brother and two sisters and was almost certainly brought up in the town of his birth. Although there is little evidence concerning his early education, most likely it took place at the local Franciscan monastery.

Fermat attended the University of Toulouse before moving to Bordeaux in the second half of the 1620s. In Bordeaux, he began his first serious mathematical researches, producing important work on maxima and minima.

From Bordeaux, Fermat went to Orléans, where he studied law at the university. He received a degree in civil law and purchased the office of counselor at the parliament in Toulouse. By 1631, he was a lawyer and government official in Toulouse, which allowed him to change his name from Pierre Fermat to Pierre de Fermat, signifying minor nobility. For the remainder of his life, he lived in Toulouse but regularly spent time in his home town of Beaumont-de-Lomagne and the nearby town of Castres.

Fermat's career as a lawyer and jurist progressed rapidly. Initially, he worked in the lower chamber of the parliament, but in 1638, he was appointed to the higher chamber, and in 1652, he was promoted to the highest level at the criminal court. Still further promotions followed, though his rapid rise does not necessarily mean that he put any unusually great effort into his duties. Promotion was done mostly on seniority, and when the plague struck the region in the early 1650s, many of the older men died. Fermat himself survived the plague in 1653.

The common description of Fermat as "the great amateur" reflects only that he did not pursue mathematics for a living and did not publish his work. In reality, he devoted enormous time and effort to the subject, corresponding on a regular basis with some of the best mathematicians in Europe. He had little interest in practical problems, focusing most of his attention on number theory, the branch of mathematics that most mathematicians regard as the "purest of the pure." As recently as 1944, the great English mathematician G. H. Hardy remarked* that number theory (his preferred subject, also) had absolutely no practical use, a claim that remained true until the 1970s, when the theory of prime numbers was used to construct a highly secure encryption system that is now used to protect most of the confidential traffic sent over the Internet.

In his 1994 book *The Mathematical Career of Pierre de Fermat (1601–1665),* Michael Sean Mahoney describes Fermat as "secretive and taciturn, he did not like to talk about himself and was loath to reveal too much about his thinking."** Fermat's secrecy was legendary. His standard method for informing others of his work was to send out letters in which he stated his results, giving little or no indication as to how he had obtained them, let alone complete proofs or detailed solutions. Though occasionally one

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*G. H. Hardy made the statement about number theory in his autobiographical book *A Mathematician's Apology*, canto ed. (Cambridge: Cambridge University Press, 2001).

**Michael Sean Mahoney, *The Mathematical Career of Pierre de Fermat (1601–1665),* 2nd rev. ed. (Princeton, NJ: Princeton University Press, 1994), xii.

of his claims turned out to be wrong, the vast majority were correct, and accordingly, the receipt of a letter from Fermat amounted to a direct challenge: "I, Fermat, can do this; can you?"

Fermat's avoidance of publication suggests he had no ambition for fame. In a letter dated April 26, 1636, in which he described to Marin Mersenne, the founder of Mersenne's Academy, some results on geometric spirals, he wrote, "I will share all of this with you whenever you wish and do so without any ambition, from which I am more exempt and more distant than any man in the world." On the other hand, he did regularly inform others of his results, so it would appear he desired recognition from those he regarded as his peers—a decidedly elite group.

Others have interpreted his behavior differently. Jean-Baptiste Colbert, a leading figure in France at the time, wrote that "Fermat, a man of great erudition, has contact with men of learning everywhere. But he is rather preoccupied, he does not report cases well and is confused."

Certainly, his letters to Pascal about the problem of the unfinished game were far shorter—though mathematically much more insightful and to the point—than those written by Pascal, although the entire correspondence was written in exquisitely polite language, and each writer displayed great respect for the other.

This mutual respect was clearly genuine. In a letter to Pierre de Carcavi, Fermat wrote: "I am delighted to have had opinions conforming to those of M. Pascal, for I have infinite esteem for his genius."*

Fermat appears to have had less regard for Descartes. When asked by Mersenne for his opinion of Descartes' *La dioptrique,* Fermat dismissed it as "groping about in the shadows" and suggested that Descartes had not correctly deduced his law of refraction.

Descartes, needless to say, was livid. His anger grew even worse when he recognized that Fermat's work on maxima, minima, and tangents could be seen as reducing the importance of his own work, *La géométrie.* In retaliation, Descartes attacked Fermat's work. Roberval and Étienne Pascal became involved in the ensuing argument, and eventually so did another prominent colleague, Desargues, whom Descartes asked to act as a referee. In the end, Fermat proved correct, and eventually Descartes admitted this, albeit somewhat churlishly: "seeing the last method that you use for finding tangents to curved lines, I can reply to it in no other way than to say that it is very good and that, if you had explained it in this manner at the outset, I would have not contradicted it at all."

For Descartes, however, this was not the end of the matter. Although he wrote to Fermat praising Fermat's work on determining the tangent to a cycloid (which was correct), he wrote to Mersenne claiming that it was incorrect and that Fermat was a second-rate mathematician and thinker. Descartes' importance was such that his words severely

*Ibid., p. 61.

damaged Fermat's reputation, but there is no indication that this bothered Fermat one iota.

For eleven years, from 1643 to 1654, Fermat largely lost touch with his scientific colleagues in Paris. In part this was due to the pressure of work, which kept him from devoting so much time to mathematics. Second, the Fronde, a civil war waged in France, affected Toulouse in 1648. Finally, there was the plague of 1651, with its near-terminal consequence for Fermat himself.

Fermat's correspondence with the Paris mathematicians resumed in 1654, when Blaise Pascal wrote him to ask his advice on the problem of the unfinished game. After his collaboration with Pascal, Fermat lived another eleven years, remaining active in mathematics and continuing to work as he had always done, by sending letters to others, describing, without proofs, his latest results. He died at age sixty-three, on January 12, 1665, in Castres, France.

Though their four months of correspondence in 1654 would change the world, Fermat and Pascal never met in person. On one occasion, Fermat tried to arrange a meeting. Hearing that his friend was in Clermont, not too far from his home in Toulouse, he wrote suggesting they try to get together. Sadly, Pascal wrote back immediately to say that his health was now so bad he could not manage even so short a journey by carriage and would be returning soon to Paris by river. It had, he said, taken him twenty-one days to get to Clermont, as he was unable to travel more than a very short distance each day.

Two years later, Pascal was dead.

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CHAPTER 6

Terrible Confusions

4. Let us follow the same argument for three *players and let us assume that the first lacks* one *point, the second* two, *and the third* two. *To make the division, following the same method of combinations, it is necessary to first discover in how many points the game may be decided as we did when there were two players. This will be in three points for they cannot play three throws without necessarily arriving at a decision.*

It is now necessary to see how many ways three throws may be combined among three players and how many are favorable to the first, how many to the second, and how many to the third, and to follow this proportion in distributing the wager as we did in the hypothesis of the two gamblers.

Although Pascal has solved the problem of the points to his satisfaction using his recursive method, he still struggles to understand Fermat's approach.* The root of his difficulty is, as we noted earlier, the vexing question of what difference it makes when, in the imagined continuation of the game, the players stop playing as soon as one of them has won. To try to grasp Fermat's solution, Pascal does what all mathematicians are trained to do: he makes the problem slightly more complicated, to see what happens. Suppose, he says, there are not two but three players, and they are rolling a die that has not two equally likely outcomes but three (say, two faces marked *a,* two marked *b,* and two marked *c*). He analyzes this modified game using Fermat's method, insofar as he understands it (and he makes it clear he is not at all sure that he does):

> *It is easy to see how many combinations there are in all. This is the third power of 3; that is to say, its cube, or 27. For if one throws three dice at a time (for it is necessary to throw three times), these dice having three faces each (since there are three players), one marked* a *favorable to the first, one marked* b *favorable to the second, and one marked* c *favorable to the third,—it is evident that these three dice thrown together can fall in 27 different ways as:*

*This chapter is a bit more technical than the remainder of the book, and many readers may prefer to skip through with merely a cursory glance at the details. There is no shame in that; after all, Pascal found it hard going as well. The chapter title refers to Pascal, not to my reader!

aaa	*aaa*	*aaa*	*bbb*	*bbb*	*bbb*	*ccc*	*ccc*	*ccc*
aaa	*bbb*	*ccc*	*aaa*	*bbb*	*ccc*	*aaa*	*bbb*	*ccc*
abc	*abc*	*abc*	*abc*	*abc*	*abc*	*abc*	*abc*	*abc*
111	111	111	111	1	1	111	1	1
	2		2	222	2		2	
		3			3	3	3	333

By analogy with the way Fermat proceeded with the original game, Pascal charts out all the possible combinations of the three-die outcomes (of which there are $3 \times 3 \times 3 = 27$ in all); notes which player wins when each comes up (he numbers the players 1, 2, and 3); and then adds up the number of ways each can win. He sees that things are quite a bit more complicated this time around, since some die-rolls result in two players winning. (Pascal's three-person game is best three of seven, and the seven rolls can end with the score 3 to 3 to 1.) He continues:

> *Since the first lacks but one point, then all the ways in which there is one* a *are favorable to him. There are 19 of these. The second lacks two points. Thus all the arrangements in which there are two* b*'s are in his favor. There are 7 of them. The third lacks two points. Thus all the arrangements in which there are two* c*'s are favorable to him. There are 7 of these. If we conclude from this that it is necessary to give each according to the proportion 19, 7, 7, we are making a serious mistake and I would hesitate to believe that you would do this. There are several cases favorable to both the first and the second, as* abb *has the* a

> *which the first needs, and the two* b*'s which the second needs. So too, the* acc *is favorable to the first and third.*

By this reasoning, Pascal finds himself faced with outcomes that will give two players a win, which he handles by giving each half a point:

> *It therefore is not desirable to count the arrangements which are common to the two as being worth the whole wager to each, but only as being half a point. For if the arrangement* acc *occurs, the first and third will have the same right to the wager, each making their score. They should therefore divide the wager in half. If the arrangement* aab *occurs, the first alone wins. It is necessary to make this assumption.*

With the possibility of using halves in dividing the pot, he continues his calculation:*

> *There are 13 arrangements which give the entire wager to the first, and 6 which give him half and 8 which are worth nothing to him. Therefore if the entire sum is one pistole, there are 13 arrangements which are each worth one pistole to him, there are 6 that are each worth ½ a pistole, and 8 that are worth nothing.*

---

*He refers to the stake in units of a *pistole*. *Pistole* is the French name given to a Spanish gold coin used in the sixteenth century. The name was also used to refer to other European gold coins of roughly the same value as the Spanish coin.

Then in this case of division, it is necessary to multiply

	13	*by one pistole which makes*		13
	6	*by one half which makes*		3
	8	*by zero which makes*		0
Total	27		*Total*	16

and to divide the sum of the values 16 by the sum of the arrangements 27, which makes the fraction 16/27 and it is this amount which belongs to the first gambler in the event of a division; that is to say, 16 pistoles out of 27.

The shares of the second and the third gamblers will be the same:

There are	4	*arrangements which are worth 1 pistole; multiplying,*		4
There are	3	*arrangements which are worth 3/2 pistole; multiplying,*		1½
And	20	*arrangements which are worth nothing*		0
Total	27		*Total*	5½

Therefore 5 pistoles belong to the second player out of 27, and the same to the third. The sum of the 5½, 5½, and 16 makes 27.

This answer is wrong, and he knows it. He has arrived at an incorrect conclusion because he has not really grasped

Fermat's method. His certainty about the source of his difficulty becomes evident as he continues:

> *5. It seems to me that this is the way in which it is necessary to make the division by combinations according to your method, unless you have something else on the subject which I do not know. But if I am not mistaken, this division is unjust.*

Pascal is still hung up on the issue of when, in practice, the players in the imagined continuation would stop playing because someone had won. He struggles to clarify his thoughts on the matter:

> *The reason is that we are making a false supposition,—that is, that they are playing three throws without exception, instead of the natural condition of this game which is that they shall not play except up to the time when one of the players has attained the number of points which he lacks, in which case the game ceases.*
>
> *It is not that it may not happen that they will play three times, but it may happen that they will play once or twice and not need to play again.*

He hopes that by comparing the original, two-person game with his three-person variation, he will be able to pin down the source of his difficulty in understanding Fermat's approach.

But, you will say, why is it possible to make the same assumption in this case as was made in the case of the two players? Here is the reason: In the true condition [of the game] between three players, only one can win, for by the terms of the game it will terminate when one [of the players] has won. But under the assumed conditions, two may attain the number of their points, since the first may gain the one point he lacks and one of the others may gain the two points which he lacks, since they will have played only three throws. When there are only two players, the assumed conditions and the true conditions concur to the advantage of both. It is this that makes the greatest difference between the assumed conditions and the true ones.

If the players, finding themselves in the state given in the hypothesis,—that is to say, if the first lacks one *point, the second* two, *and the third* two*; and if they now mutually agree and concur in the stipulation that they will play* three *complete throws; and if he who makes the points which he lacks will take the entire sum if he is the only one who attains the points; or if two should attain them that they shall share equally, in this case, the division should be made as I give it here. The first shall have 16, the second 5½, and the third 5½ out of 27 pistoles, and this carries with it its own proof on the assumption of the above condition.*

But if they play simply on the condition that they will not necessarily play three throws, but that they will only play until one of them shall have attained his points, and *that then the play shall cease without giving another the opportunity of reaching his score, then 17 pistoles should belong to the first, 5 to the second, and 5 to the third, out of 27. And this is found by my general method which also determines that, under the proceeding condition, the first should have 16, the second 5½, and the third without making use of combinations,—for this works in all cases and without any obstacle.*

Fermat Stumbles

When Fermat receives Pascal's long letter (there is a little more of it to come as we continue our story, but we now have the hard part behind us), he sees at once where his correspondent has gone wrong, and writes back immediately. Perhaps out of kindness to Pascal, or perhaps because, having solved the problem of the points to his own satisfaction, his interests have now moved on to other things, he buries his assessment of Pascal's confused analysis of the three-person game in the second section of his letter and starts instead with something that will make his correspondent feel good, namely, his work on the arithmetic triangle.

Saturday, August 29, 1654

Monsieur,

1. Our interchange of blows still continues, and I am well pleased that our thoughts are in such complete adjustment as it seems since they have taken the same di-

rection and followed the same road. Your recent Traité du triangle arithmétique *and its applications are an authentic proof and if my computations do me no wrong, your eleventh consequence went by post from Paris to Toulouse while my theorem, on figurate numbers, which is virtually the same, was going from Toulouse to Paris. I have not been on watch for failure while I have been at work on the problem and I am persuaded that the true way to escape failure is by concurring with you. But if I should say more, it would be of the nature of a Compliment and we have banished that enemy of sweet and easy conversation.*

It is now my turn to give you some of my numerical discoveries, but the end of the parliament augments my duties and I hope that out of your goodness you will allow me due and almost necessary respite.

Now Fermat cuts to the chase:

2. I will reply however to your question of the three players who play in two throws. When the first has one [point] and the others none, your first solution is the true one and the division of the wager should be 17, 5, and 5. The reason for this is self-evident and it always takes the same principle, the combinations making it clear that the first has 17 chances while each of the others has but five.

3. For the rest, there is nothing that I will not write you in the future with all frankness.

In other words, Pascal can solve the three-person game correctly when he uses his own (considerably more complicated) recursive method, but he still has not grasped Fermat's enumeration approach. Not for the first time has a mathematician—even one as accomplished as Pascal—found that it is possible to get the right solution by correctly applying an appropriate method while not really understanding the subtleties of the problem.*

At this point, again perhaps out of kindness to Pascal, Fermat changes the topic:

> *Meditate however, if you find it convenient, on this theorem: The squared powers of 2 augmented by unity are always prime numbers. [That is,]*
>
> *The square of 2 augmented by unity makes 5 which is a prime number;*
>
> *The square of the square makes 16 which, when unity is added makes 17, a prime number;*
>
> *The square of 16 makes 256 which, when unity is added, makes 257, a prime number;*
>
> *The square of 256 makes 65536 which, when unity is added, makes 65537, a prime number;*
>
> *and so to infinity.*

---

*For example, few university undergraduates fully understand differential calculus, but that does not prevent them from using it to solve problems correctly. That was certainly true for me when I was a student. I came to understand the method only many years later, when I was faced with teaching it to students of my own. I gather that my experience is by no means unique among professors of mathematics.

This is a property whose truth I will answer to you. The proof of it is very difficult and I assure you that I have not yet been able to find it fully. I shall not set it for you to find unless I come to the end of it.

This theorem serves in the discovery of numbers which are in a given ratio to their aliquot parts, concerning which I have made many discoveries. We will talk of that another time.

I am, Monsieur, yours etc.

Fermat

At Toulouse, the twenty ninth of August, 1654

By a remarkable good fortune in terms of social equality, in the very letter in which Fermat corrects Pascal's faulty reasoning, he makes a major blunder of his own—one of the few in his entire career. His claim is that each of the numbers

$$2^{2^n} + 1$$

is prime. Nowadays, numbers of this form are called *Fermat numbers,* generally denoted by F_n.

Fermat lists the first few cases: $F_1 = 2^2 + 1 = 5$; $F_2 = 2^4 + 1 = 17$; $F_3 = 2^8 + 1 = 257$; $F_4 = 2^{16} + 1 = 65{,}537$, and observes, correctly, that each is prime. He then claims that all numbers of this form are prime. This is not so. The great Swiss mathematician Leonhard Euler discovered in 1732 that the very next Fermat number, $F_5 = 4{,}294{,}967{,}297$, is not prime, being the product of the primes 641 and 6,700,417.

In fact, no Fermat number beyond F_4 has been shown to be prime, and all Fermat numbers from F_5 through F_{32} have been shown, through the use of computers, to be composite. Some mathematicians have suggested that perhaps no Fermat number is prime other than the four he looked at. Even Fermat made a mistake occasionally.