From:

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3 The nature of abstract reasoning: philosophical aspects of Descartes' work in algebra

No one contributed more to the early development of algebra than Descartes. In particular, he was able to unify arithmetic and geometry to a significant extent, by showing their mutual connections in terms of an algebraic notation. This was an achievement that eclipsed his other scientific work, and Descartes believed that algebra could serve as a model for his other enterprises. The connection between algebra and his other scientific work was explored, via a consideration of the question of method, in Descartes' first published work, the *Discourse on the Method of rightly conducting one's reason and seeking the truth in the sciences, together with the Optics, the Meteorology and the Geometry which are essays in this method* (1637). What we are ostensibly presented with here is a general treatise on method, to which are appended three examples of the method. And three very successful examples they are, for in each case we are provided with at least one new fundamental result: the sine law of refraction in the *Optics*, the calculation and experimental confirmation of the angles of the bows of the rainbow in the *Meteorology*, and the solution of Pappus' locus problem for four or more lines in the *Geometry*. But it would be a grave mistake to see the *Geometry* as merely an exemplification of method. Descartes effectively treats the algebraic approach that he develops in the *Geometry* as a source of, rather than simply an exemplification of, correct method. Moreover, the methodological aspects of algebra do not in any way exhaust its interest, and although I shall touch on them, the focus of this paper will lie elsewhere.

The three principal themes that I want to take up are: what Descartes' algebraic work actually amounts to, what its originality consists in, and how the application of algebra to the physical world is possible. But underlying these themes is a deeper issue, namely the question of the abstract nature of algebra. One thing that I shall try to clarify is what this abstractness consists in for Descartes.

3.1. THE NATURE OF DESCARTES' ALGEBRA

Algebra, arithmetic, and geometry

The Greeks classified geometrical problems as being either plane, solid, or linear, depending on whether their solution required straight lines and circles, or conic sections, or more complex curves. Euclid had restricted himself to the two postulates that a straight line can be drawn between any two points, and that a circle can be drawn with any given point as center to pass through another given point. But the range of problems that can be solved purely on the basis of these postulates is very restricted, and a third postulate was added by later mathematicians; namely, that a given cone could be cut by a given plane. The geometry of conic sections that resulted was treated in antiquity as an abstruse branch of mathematics of little practical relevance. Aristotle had convincingly shown that the natural motion of bodies was either rectilinear (in the case of terrestrial bodies) or circular (in the case of celestial bodies), so from the physical point of view it appeared that we could get by without the more complex curves: these apparently had no basis in nature and were of purely mathematical interest. But by the seventeenth century the need to give some account of curves beyond the straight line and circle became pressing. The parabola, being the path taken by projectiles, was studied in ballistics, and astronomers were well aware of the fact that planets and comets described elliptical, parabolic, and hyperbolic paths. And in optics, which was one of the most intensely studied areas in natural science in the seventeenth century, a knowledge of at least conic sections was required for the construction of lenses and mirrors. The work of the Alexandrian mathematicians on conic sections left much to be desired, and many of their results were more often than not the result of ingenious one–off solutions of problems rather than being due to the application of some general procedure.

It is precisely such a general procedure that Descartes develops and puts to use in the *Geometry,* a treatise which had a revolutionary effect on the development of mathematics. The *Geometry* comprises three books, the first dealing with "problems that can be constructed using only circles and straight lines," the second dealing with "the nature of curves," and the third with the construction of "solid and supersolid problems." The first book is the most important as far as the fundamentals of algebra are concerned, and consequently I shall focus on this.[1]

From its title, which indicates that it concerns only those problems that utilize straight lines and curves in their construction, one might expect the first book to contain the traditional material, and the others to contain the new material. After all, Euclid had given a reasonably exhaustive account of problems which can be constructed using only straight lines and a circle. But in fact the purpose of the first book is, above anything else, to present a new algebraic means of solving geometrical problems by making use of arithmetical procedures and vice versa. In other words, the aim is to show how, if we think of them in algebraic terms, we can combine the resources of the two fields.

The *Geometry* opens with a direct comparison between arithmetic and geometry (AT VI 369). Just as in arithmetic the operations we use are addition, subtraction, multiplication, division, and finding roots, so too in geometry we can reduce any problem to one that requires nothing more than a knowledge of the lengths of straight lines, and in this form the problem can be solved using nothing more than the five arithmetical operations. Descartes therefore introduces arithmetical terms directly into geometry. Multiplication, for example, is an operation that can be performed using only straight lines (i.e. using only a ruler):

> Let AB be taken as one unit, and let it be required to multiply BD by BC. I have only to join the
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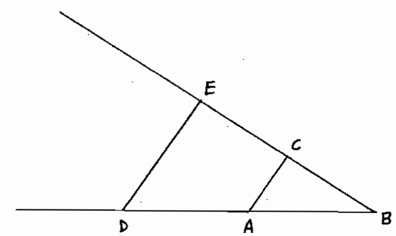
> 
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>
> points A and C, and draw DE parallel to CA; then BE to the product of this multiplication. (AT 370)

If we wish to find a square root, on the other hand, we require straight lines and circles (i.e. ruler and compass):

> In order to find the square root of GH, I add, along the straight line, FG equal to one unit; then, divid-
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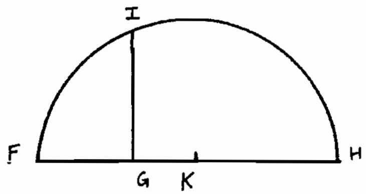
> 
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> ing FH into two equal parts at K, I describe the circle FIH about K as a center, and draw from the point G a straight line at right angles to G extended to I, and GI is the required root. (AT VI 370–1)

Note that, given FG as the arbitrarily chosen unit, GI may well turn out to be irrational: this is not relevant in the geometrical construction.

Descartes next points out that we do not actually need to draw the lines, but can designate them by letters. He instructs us to label all lines in this way, those whose length we seek to determine as well as those whose length is known, and then, proceeding as if we had already solved the problem, we combine the lines so that every quantity can be expressed in two ways. This constitutes an equation, and the object is to find such an equation for every unknown line. In cases where this is not possible, we choose lines of known length arbitrarily for each unknown line for which we have no equation, and:

> if there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines; and we must combine them until there remains a single unknown line which is equal to some known line, whose square, cube, fourth, fifth or sixth power etc. is equal to the sum or difference of two or more quantities, one of which is known, while the other consists of mean proportionals between the unit and this square, or cube, or fourth power etc., multiplied by other known lines. I may express this as follows:
>
> $z = b$
>
> or $z^2 = -az + b^2$
>
> or $z^3 = az^2 + b^2z - c^3$
>
> or $z^4 = az^2 - c^3z + d^4$ etc.

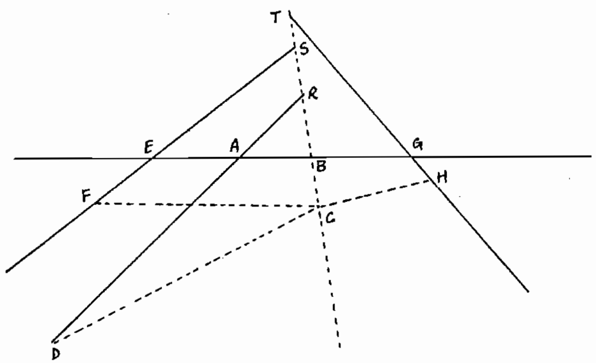
> That is, z, which I take for the unknown quantity, is equal to b; or the square of z is equal to the square of b minus a multiplied by z . . . Thus all the unknown quantities can be expressed in terms of a single quantity, whenever the problem can be constructed by means of circles and straight lines, or by conic sections, or by a curve only one or two degrees greater. (AT VI 373–4)

This is a novel approach to the question. Algebraic equations in two unknowns, $F(x,y) = O$, were traditionally considered indeterminate since the two unknowns could not be determined from such an equation. All one could do was to substitute arbitrarily chosen values for x and then solve the equation for y for each of these values, something that was not considered to be in any way a general solution of the equation. But Descartes' approach allows this procedure to be transformed into a general solution. What he effectively does is to take x as the abscissa of a point and the corresponding y as its ordinate, and then one can vary the unknown x so that to every value of x there corresponds a value of y which can be computed from the equation. We thereby end up with a set of points that form a completely determined curve satisfying the equation.

An example: Descartes' treatment of Pappus's locus-problem[2]

This procedure is exemplified in Descartes' resolution of one of the great unsolved mathematical problems bequeathed by antiquity, Pappus's locus problem for four or more lines. The problem had been posed by Pappus in terms of a three- or four-line locus problem. Essentially, what is at issue is this. In the case of the three-line problem, we are given three lines with their positions, and the task is to find the locus of points from which three lines can be drawn to the given lines, each making a given angle with each given line, such that the product of the lengths of two of the lines bears a constant proportion to the square of the third. In the case of the four-line problem, we are given four lines with their positions, and we are required to find the locus of points from which four lines can be drawn to the given lines, such that the product of the length of two of the lines bears a constant proportion to the product of the other two.

It was known in antiquity that the locus in each case is a conic section passing through the intersections of the lines, but no general procedure for solving the problem was developed. Descartes' treatment of the question is algebraic and completely general, allowing us to express relations between the lines using only two variables. His approach is to show how the problem, explicitly solved for four lines but in a way which is theoretically generalizable to *n* lines, can, like all geometrical problems, be reduced to one in which all we need to know are the lengths of certain lines. These lines are the coordinate axes, and the lengths give us the abscissae and ordinates of points. The four-line problem is presented as follows (AT VI 382–7):



Here the full lines are the given lines and the broken lines those sought. Descartes takes AB and BC as the principal lines and proceeds to relate all the others to these. Their lengths are x and y respectively, and in fact AB is the x axis, and BC the y axis, a point obscured in Descartes' diagram by the fact that AB and BC are not drawn perpendicular to one another (since to do so would obscure the proportions). Now the angles of the triangle ABR are given, so the ratio AB:BR is known. If we let this ratio be $\frac{z}{b}$ then BR $= \frac{bx}{z}$, and CR $= y + \frac{bx}{z}$ (where B lies between C and R). The angles of the triangle DRC are also known, and if we represent the ratio CR:CD as $\frac{z}{c}$ then CR $= y + \frac{bx}{z}$ and CD $= \frac{cy}{z} + \frac{bcx}{z^2}$. Moreover, since the positions of AB, AD, and EF are fixed, the length k of AE is thereby given; therefore EB $= k + x$ (where A lies between E and B). The angles of

the triangle ESB are also given, and hence so is the ratio BE:BS. If we let this ratio be $\frac{z}{d}$, then we get BS $= \frac{dk + dx}{z}$ and CS $= \frac{zy + dk + dx}{z}$ (where B is between S and C). The angles of the triangle FSC are given, therefore the ratio CS:CF is known. If we let this ratio be $\frac{z}{e}$ then we obtain CF $= \frac{ezy + dek + dex}{z^2}$. Letting l denote the given length of AB, we have BG $= l - x$; and if we let the known ratio BG:BT in the triangle BGT be $\frac{z}{f}$, then BT $= \frac{fl - fx}{z}$ and CT $= \frac{zy + fl - x}{z}$, and if we let CT:CH in the triangle TCH be $\frac{z}{g}$ then CH $= \frac{gzy + fgl - fgx}{z^2}$.

No matter how many lines of given position we are dealing with, the length of a line through C making a given angle with these lines can always be expressed in three terms of the form $ax + by + c$. For three or four fixed lines, the equation will be a quadratic equation, and this means that, for any known value of y, the values of x can then be found using only ruler and compass, and a sufficiently large number of values will enable us to trace the curve on which C must lie. For five or six lines the equation is a cubic, for seven or eight a quartic, for nine or ten a quintic, and so on, rising one degree with the introduction of every two lines.

Descartes' advance beyond ancient mathematics

In solving Pappus's problem Descartes has solved one of the most difficult problems bequeathed by ancient mathematics, and he has solved it in a simple, elegant, and generalizable way. In doing so, he has developed a technique that goes well beyond those employed in antiquity.

In the second book of the *Geometry*, Descartes extends his treatment of the Pappus loci for three or four lines by distinguishing the curves corresponding to equations of the second degree, namely the ellipse, hyperbola and parabola. This treatment is fairly exhaustive, but he considers very few cases corresponding to cubics, maintaining (somewhat optimistically as it turns out)[3] that his method shows how these are to be dealt with. His general classification of curves, and in particular his dismissal of transcendental curves, has provoked much discussion,[4] but will not concern us here. It is perhaps worth mentioning, however, that his method of drawing a tangent to curves took on a new importance with the development of calculus (to which Descartes made no direct contribution) as it is effectively equivalent to finding the slope of a curve at any point, which is a form of differentiation. Finally, in the third book, solid and supersolid problems are examined. This marks an important advance beyond the Alexandrian mathematicians, who only recognized constructions making use of curves other than straight lines and circles with reluctance, and the category of solid problems was never systematically thought through. Here Descartes extends his algebraic analysis far beyond the concerns of mathematicians of antiquity. The most striking feature of his approach is that, in order to preserve the generality of his structural analysis of the equation, he is prepared to allow not only negative roots but also imaginary roots, despite the otherwise completely counterintuitive nature of these. To grasp just how radical this is, we need first to say a bit more about the nature of algebra and Descartes' place in its development.

3.2 THE ORIGINALITY OF DESCARTES' APPROACH

Geometrical algebra

The characteristic feature of algebra is its abstractness. It comprises mathematical structures defined purely in operational and relational terms, without any constraint on the nature of the *relata*. Strictly speaking, it has no content of its own, but acquires content only through interpretation. This is how we think of algebra now, but it has not always been seen in such abstract terms, and we can distinguish two crucial stages in its development: the freeing of number from spatial intuitions, and the freeing of algebra itself from an exclusively numerical interpretation. The first of these we owe largely to Descartes. It is not always appreciated, however, just how novel Descartes' algebraic approach is. Until relatively recently it has been thought that the Greeks possessed a "geometrical algebra," i.e. a procedure for dealing with genuinely algebraic problems which, because of the crisis brought about by the Pythagorean discovery of linear incommensurability, resulted in the geometrical formulation and resolution of these problems. This geometrical algebra, it was argued, was subsequently rediscovered, stripped of its geometrical language, and hence made more general, in the work of Descartes and others. What is at issue here is whether the geometrical formulation and resolution of certain classes of mathematical problems by the Greeks can be construed as algebra in geometrical dress. It cannot be denied that there are many propositions in Euclid, for example, for which we can easily find algebraic results to which they are equivalent. Moreover, many of the propositions of the second book of Euclid's *Elements* can be given a very straightforward algebraic interpretation, whereas there has often been perceived to be problems in providing purely geometrical interpretations for these. Finally, it seems that geometrical algebra was exactly what was required as a response to the crisis in mathematics occasioned by the discovery of linear incommensurability, a discovery with which the available arithmetical procedures were unable to cope.

Challenges to this kind of interpretation have in fact existed since the 1930s, but it is only more recently that it has been widely appreciated that there is something wrong with the geometrical algebra view. Jacob Klein, in his pioneering work on the early development of algebra, for example, showed that very radical changes in the concept of number were required before algebra became possible, and that these were not effected until the work of Vieta at the end of the sixteenth century.[5] Secondly, it is now clear that the Pythagorean geometry of areas, far from being a geometrical algebra designed to solve the problem of incommensurability, was in fact designed to eliminate what was effectively regarded as an insoluble problem.[6] Third, all the propositions of Euclid's *Elements* do in fact have geometrical interpretations[7] and in a number of cases their algebraic presentation simply trivializes them.[8] The conclusion that we must draw from this is, I believe, that there is simply no evidence to support the traditional contention that Greek mathematicians operated with any genuinely algebraic ideas, consciously or otherwise.

However, to say that the Greeks did not operate with a geometrical algebra is not to say that geometry did not play a significant role in Greek arithmetic. It in fact played a very significant role indeed, but one quite contrary to the traditional interpretation, for it diminished rather than increased the abstractness of arithmetic. An understanding of this role is important if we are to appreciate fully the novelty of Descartes' algebra, and his approach is best contrasted with the very influential account of number that Aristotle had offered in his *Metaphysics*.[9]

The Aristotelian conception: number, matter, and space

For Aristotle, mathematical objects have matter and this matter is what he calls "noetic matter." Now mathematics is distinguished for Aristotle by the fact that its objects do not change and do not have independent existence. These objects are noetic, as opposed to sensible, and we come by them through abstraction from "sensible" numbers and shapes, i.e. the numbers and shapes of sensible objects. Sensible objects are made up of sensible matter, and Aristotle thinks that mathematical objects must be made up of noetic matter. He adopts this doctrine because he believes that numbers and shapes are properties, and that properties must always be instantiated in something. Sensible numbers and shapes are instantiated in sensible matter, but noetic numbers and shapes cannot be for these are only objects of thought; since they are properties, however, they must be instantiated in something, so Aristotle invents a new kind of purely abstract matter for them to be instantiated in.

In the case of geometry, Aristotle employs two different kinds of abstraction. The first involves disregarding the matter of sensible objects so that we are left with properties like "being triangular" and "being round." Geometry investigates "being round" in very general terms as the form of whatever, most generally speaking, is round. And whatever, most generally speaking, is round is something we arrive at by a complementary kind of abstraction, in which we disregard the properties of sensible objects so that what has these properties becomes the object of investigation. What we are left with is a substratum of indeterminate extension characterized solely in terms of its spatial dimensions: length, breadth, and depth. This abstraction can then be carried further yielding planes, and finally lines and points, each of these substrata having different dimensions. Now these substrata can neither be sensible, since they have been deprived of the properties that would render them sensible, nor can they have an independent existence, since they are merely abstractions, and they are what Aristotle calls noetic matter.

Aristotle makes the same claim about numbers, however, and this is more problematic. We can imagine geometrical noetic matter as spaces of one, two, and three dimensions, but how are we to imagine the noetic matter of number? The answer is: in much the same way – provided we bear in mind that, in Greek mathematics, whereas geometry operates with lines, arithmetic operates with line *lengths* (or areas or volumes). The distinction is of the upmost importance, as Aristotle is well aware. A line length, insofar as it is a determinate length, can be seen to be potentially divisible into discontinuous parts, i.e. into a determinate plurality of unit lengths. It is by treating the foot length as being indivisible, for example, that we can treat it as being a unit length, as being the measure of other lengths (cf. Book I of Aristotle's *Metaphysics*). And in this case the line length becomes effectively the same as number, which Aristotle defines as a plurality measured by a unit or a "one." The central distinction between arithmetic and geometry lies in the fact that the former deals with discontinuous and the latter with continuous magnitudes. The line considered simply as a line comes within the subject matter of geometry because it is infinitely divisible and hence a continuous magnitude, but considered either as a unit length or as a sum of unit lengths it comes within the subject matter of arithmetic.

In terms of this distinction, we can grasp clearly what arithmetic amounts to on Aristotle's conception: it is metrical geometry. Although he never explicitly mentions metrical geometry, his arithmetical terminology – *linear, plane,* and *solid* numbers, numbers being *measured,* factors *measuring* products in multiplication – consistently suggests that this is the conception of arithmetic that he is taking for granted. Indeed, metrical geometry is an essentially arithmetical discipline, common to the whole of ancient mathematics from the old-Babylonian period to the Alexandrians.[10] In the present context, its importance lies in the fact that, although it deals with lines, planes, etc., it deals with them not *qua* lines and planes but *qua* unit lengths and unit areas, or sums or products of such unit lengths and areas. Aristotle talks throughout his work of numbers in one dimension, plane numbers, and solid numbers and he never introduces the idea of the geometrical *representation* of numbers. Nor, indeed, does any Greek or Alexandrian author talk of numbers being represented geometrically. It is instructive here that the arithmetical propositions of Euclid's *Elements,* those taking up books VII to IX, are explicitly stated in terms of line lengths, as if numbers were line lengths: And this is exactly what they are.

Aristotle was not an innovator in mathematics. He was not attempting to develop a new form of mathematics but to provide a proper philosophical basis for the mathematics of his day. What he is providing a basis for in the case of arithmetic is not a form of arithmetic which, because of its grounding in geometrical algebra, is particularly abstract and general, but rather a form of arithmetic that, being construed in terms of metrical geometry, is dependent on spatial intuitions and as a result is severely limited. Consider, for example, the arithmetical operation of multiplication and, in particular, the dimensional change involved in this operation, which results in the product always being of a higher dimension. This is not a notational constraint, it is inherently connected with the idea that numbers, for Greek mathematicians, are always numbers *of* something. A consequence of this is that when we multiply, we must multiply numbers of something: we cannot multiply two by three, for example, we must always multiply two somethings by three somethings. It is in this sense that Klein has called numbers "determinate" for the Greeks. They do not symbolize general magnitudes, but always a determinate plurality of objects.[11] Moreover, not only are the dimensional aspects of geometry retained in arithmetical operations, so too is the physical and intuitive nature of these dimensions, so that, for example, no more than three line lengths can be multiplied together since the product here is a solid, which exhausts the number of available dimensions.[12]

Cartesian algebra and abstraction

Descartes explicitly opposes this spatial conception. At the beginning of the *Geometry*, after having shown us the geometrical procedures for multiplication and finding square roots, he introduces single letters to designate line lengths. But his interpretation of these letters is significantly different from the traditional interpretation. On the traditional interpretation, if a is a line length, a^2 is a square having sides of length a, ab is a rectangle having sides of length a and b, and a^3 is a cube having sides of length a. On Descartes' interpretation, however, these quantities are all dimensionally homogeneous:

> It should be noted that all the parts of a single line should always be expressed by the same number of dimensions as one another, provided that the unit is not determined in the condition of the problem. Thus, a^3 contains as many dimensions as ab^2 or b^3, these being the component parts of the line that I have called $\sqrt[3]{a^3 - b^3 + ab^2}$ (AT VI 371).

Here the shift between arithmetic and geometry is something that furthers the abstraction of the operations, not something that constrains their abstraction, as on ancient conceptions. The question of level of abstraction is crucial. For mathematicians of antiquity, it was only if a determinate figure or number could be constructed or computed that one could be said to have solved a mathematical problem. Moreover, the only numbers allowable as solutions were natural numbers: negative numbers in particular were "impossible" numbers. It is true that toward the end of the Alexandrian period, most notably in Diophantus's *Arithmetica*, we begin to find the search for problems and solutions concerned with general magnitudes, but these procedures never make up anything more than auxiliary techniques forming a stage preliminary to the final one, where a determinate number is computed. Descartes is explicitly opposed to this, and in Rule XVI of the *Rules for the Direction of Our Native Intelligence* he spells out the contrast between his approach and the traditional one in very clear terms:

> It must be pointed out that while arithmeticians have usually designated each magnitude by several units, i.e. by a number, we on the contrary abstract from numbers themselves here just as we did above [Rule XIV] from geometrical figures, or from anything else. Our reason for doing this is partly to avoid the tedium of a long and unnecessary calculation, but mainly to see that those parts of the problem which are the essential ones always remain distinct and are not obscured by useless numbers. If for example we are trying to find the hypotenuse of a right-angled triangle whose given sides are 9 and 12, the arithmetician will say that it is $\sqrt{225}$, i.e. 15. We on the other hand will write a and b for 9 and 12, and find that the hypotenuse is $\sqrt{a^2 + b^2}$ leaving the two parts of the expression a^2 and b^2 distinct, whereas in the number they are run together . . . We who seek to develop a clear and distinct knowledge of these things insist on these distinctions. Arithmeticians, on the other hand, are satisfied if the required result turns up, even if they do not see how it depends on what has been given, but in fact it is in knowledge of this kind alone that science consists.
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> (AT X 455–6, 458: CSM I 67–8, 69)

For Descartes, concern with general magnitudes is constitutive of the mathematical enterprise. He recognizes no numbers or figures to be "impossible" on intuitive grounds. Indeed, he readily accedes to purely algebraic constraints requiring that "number" be extended to include not just integers, but fractions and irrationals as well. And his structural analysis of the equation leads him to accept negative and imaginary roots. Here our intuitions about what numbers are are effectively sacrificed to the structural definition of number provided by algebra.

In this respect, Descartes inaugurates a development in which the range of items coming under the category of "number" is expanded and consolidated as the generality of algebra is increased and its rules of operation define new kinds of entity as numbers. As Kneale has pointed out,[13] up to and including the introduction of complex numbers, mathematicians took an unreflective attitude to their extension of the idea of number. Retaining the general rules of algebra required them to introduce novel kinds of entities which they were forced to adopt to solve problems posed at an earlier stage, but they raised no general questions about this procedure. The situation changed in the late 1830s and early 1840s. In the first place, Peacocke, Gregory, and de Morgan began to conceive of algebra in such abstract mathematical terms that it was no longer necessary to construe the *relata* of its operations as numbers at all. Secondly, Hamilton began work on an algebra of hypercomplex numbers, which, while they are defined by algebraic operations, do not satisfy all the rules that hold up to complex numbers. These two developments suggested that algebra may be more general than had been thought. It was in this context that George Boole, regarded by many as the founder of modern formal logic, was able to devise an abstract calculus for logic. Showing how the laws of algebra can be formally stated without interpretation, and how the laws governing numbers up to complex numbers need not all hold together in every algebraic system, he was able to go on to develop a limited algebra which represented the operations of traditional syllogistic.

Freed of its exclusively numerical interpretation, algebra becomes a much more powerful apparatus and its application to logic takes it directly to the most fundamental issues. Such a development is a continuation of Descartes' work on algebra, but it is a continuation completely alien to Descartes' own approach. To understand why this is the case, we must consider what Descartes thinks is methodologically distinctive about algebra.

Algebra, deduction, and Cartesian "analysis"[14]

As we have seen, Descartes maintains that whereas earlier mathematicians were exclusively concerned with computing particular numerical solutions to equations, he abstracts from numbers because he is concerned with structural features of the equations themselves. Now it is possible to draw a direct analogy with logic here. If we are to think of logic in algebraic terms, in the same way that Descartes thinks of arithmetic algebraically, what we must do is to abstract from particular *truths* (just as Descartes abstracts from particular *numbers*) and explore the relations between truths, independently of their content, in abstract structural terms. But this move to a higher level of abstraction, which Leibniz glimpsed and which is constitutive of modern logic and the philosophy of mathematics, was utterly alien to Descartes. Descartes was blind to the possibility of logic being construed in terms of an extension of his algebraic techniques because he conceived of logic (which for him was Aristotelian syllogistic) as being a redundant method of presentation of already achieved results, whereas algebra, he thought, was something completely different, namely a method of discovery of new results. The question of method has been dealt with elsewhere in this volume, but a few words on how it specifically relates to algebra would not be out of place here.

When, in Rule IV of the *Rules for the Direction of Our Native Intelligence,* Descartes discusses the need for 'a method of finding out the truth', he turns his attention to mathematics. When he first studied mathematics, he tells us, he found it unsatisfactory. Although the results that mathematicians obtained were true, they did not make it clear how they had come by their results, and in many cases it seemed that it was a matter of luck rather than skill. Consequently, many had quite understandably rejected mathematics as empty and childish. But the founders of philosophy in antiquity had made mathematics a prerequisite for the study of wisdom. This indicates to Descartes that they must have had a "species of mathematics different from ours" (AT X 376: CSM I 18), and he claims to find traces of this "true mathematics" in the writings of Pappus and Diophantus. But these authors feared "that their method [of discovery], being so easy and simple, would become cheapened if it were divulged, and so, in order to gain our admiration, they put in its place sterile truths which, with some ingenuity, they demonstrated deductively" (AT X 376–7: CSM I 19).

The art of discovery that Descartes believes he had rediscovered is what he calls "analysis." In antiquity, analysis and synthesis were complementary procedures, and Pappus distinguished two kinds of analysis: "theoretical analysis," in which we attempt to establish the truth of a theorem, and "problematical analysis," in which we attempt to discover something unknown. If these procedures are successfully completed, we must then prove our result by synthetic means, whereby we start from definitions, axioms, and rules and deduce our result solely from these. The mathematical texts of antiquity, concerned as they were with rigorous demonstration, presented only synthetic proofs. Descartes does two things: he effectively restricts "analysis" to problematical analysis, and he completely rejects the need for synthesis. The latter is evident as soon as one glances at the *Geometry*. The traditional lists of definitions, postulates, etc. are completely absent, and we are immediately introduced to problem-solving techniques. For Descartes, the aim of the exercise, an aim he believes only algebra can enable one to achieve in a systematic way, is to solve problems. Once one has solved the problem, the presentation of the result in synthetic terms is, for Descartes, completely redundant. In more general terms, what this amounts to is a rejection of the value of deductive inference in mathematics.

This is one of the most problematic parts of Descartes' conception of algebra, and he parts company on this issue not only with modern mathematicians but also with his contemporaries. The source of the problem lies in his view that deductive inference can never have any epistemic value, and can never play any role in furthering knowledge. Leibniz was the first philosopher to respond fully to this view, and he pointed out that whereas analysis may be valuable as a way of solving particular problems, in the synthetic or deductive presentation of results in mathematics we set in train a systematic structuring of and extension of knowledge which enables gaps, difficulties, flaws, etc. to be recognized, precisely identified and solved.

The problem is a deep one, however, and many philosophers have questioned the standing of deduction. Sextus Empiricus, one of the most important of the ancient skeptics, offered the following ingenious argument against deductive inference.[15] Compare the following arguments:

	A	*B*
(1)	If it is day, it is light	It is day
(2)	It is day	It is light
(3)	It is light	

A is a deductive argument, *B* a nondeductive one. Sextus' argument is that deductive arguments are always, by their own criteria, flawed. In the present case, for example, either (3) follows from (2) or it does not. If it does, then *B* is a perfectly acceptable argument for in *B* we simply infer (3) from (2). But if this is the case then (1) is clearly redundant. On the other hand, if (3) does not follow from (2) then (1) is false, since (1) clearly asserts that it does. So deductive proof is impossible: what *A* tells us over and above *B* is either redundant or false. Not many philosophers have been prepared to go quite so far as Sextus, but many have raised general worries about the point of deduction. Some, such as J. S. Mill, have held that the premises contain the same assertion as the conclusion in deductive arguments, and that this is in effect what makes them valid.[16] Here, a question must be raised about the point of deductive arguments. Others, such as the logical positivists, have held that logical truths are analytically true and hence we can never learn anything new from them.

This surely cannot be right, for we do sometimes learn something new from deductive proofs. Consider, for example, Hobbes's first encounter with Euclid's *Elements*, as reported by Aubrey in his *Brief Lives:*

> Being in a Gentelman's Library, Euclid's Elements lay open, and 'twas the 47 El. libri I. He read the proposition. By G –, sayd he (he would now and then sweare an emphatical oath by way of emphasis), this is impossible! So he reads the Demonstration of it, which referred him back to another, which he also read. [And so on] that at last he was demonstratively convinced of that trueth. This made him in love with Geometry.[17]

Here Hobbes begins not only by not believing something, but by not even believing it to be possible, and a chain of deductive reasonings convinces him otherwise. This is a clear case of epistemic advance, i.e. Hobbes ends up with a belief he would not otherwise have had, and it is a purely deductive argument that is responsible for his having this new belief. Now it is true that not all deductive arguments bring with them epistemic advance: the argument "if p, then p" clearly involves no epistemic advance, although it is a formally valid deductive argument. Where Descartes goes wrong is to deny that *any* deductive argument involves epistemic advance. This is simply not plausible, as the Hobbes case shows.

Moreover, even if deductive arguments *could* never bring about epistemic advance, we would still have good reason to be interested in the systematic relations between truths, e.g. the truths of geometry or the truths of arithmetic, since it is of some importance that we know in what way some follow from others and what precisely this "following from" consists in. But Descartes assumes that epistemic advance is the only criterion of worth, and this leads him to dismiss anything he does not believe to be a method of discovery. Algebra he sees as a method of discovery *par excellence*, and it is precisely because he sees it in this way that the possibility of thinking about deduction in algebraic terms is closed off to him.

3.3 THE APPLICATION OF MATHEMATICS TO REALITY

The abstract nature of algebra, as Descartes realizes, is the source of its power. But it is also a potential source of difficulties, for if mathematics is as abstract as Descartes maintains, its relation to the material world may become a problem. This is an especially important question for Descartes since he is concerned to develop a mathematical physics, an account of the material world that is completely mathematical. Descartes deals with the question of a mathematical physics in the *Rules for the Direction of Our Native Intelligence*, in a way that ties together mathematics, epistemology and natural science, and his account here is useful not just in helping us understand in what way he thinks something as abstract as algebra can relate to the natural world, but also in throwing some light on what he thinks this abstraction consists in.

Simple natures

Throughout the *Rules*, Descartes insists that knowledge must begin with what he calls "simple natures," which are those things that are

not further analyzable and which we can grasp in a direct and intuitive way. Such simple natures can only be grasped by the intellect, although "while it is the intellect alone that is capable of knowledge, it can be helped or hindered by three other faculties; namely, the imagination, sense-perception and memory." (AT X 398: CSM I 32) In Rule XIV the connection between the intellect and the imagination is elaborated upon in a rather interesting way:

> By 'extension' we mean whatever has length, breadth and depth, leaving to one side whether it is a real body or merely a space. This notion does not, I think, need further elucidation, for there is nothing more easily perceived by our imagination. . . . For even though someone may convince himself, if we suppose every object in the universe annihilated, that this would not prevent extension *per se* existing, his conception would not use any corporeal image, but would be merely a false judgement of the intellect working alone. He will admit this himself if he reflects attentively on this image of extension which he tries to form in his imagination. For he will notice that he does not perceive it in isolation from every subject, and that his imagination of it and his judgement of it are quite different. Consequently, whatever our intellect may believe as to the truth of the matter, these abstract entities are never formed in the imagination in isolation from subjects.
> (AT X 442–3: CSM I 59)

Descartes goes on to argue that, whereas "extension" and "body" are represented by one and the same idea in the imagination, this is not true of the intellect. When we say that "number is not the thing counted" or "extension or shape is not body," the meanings of "number" and "extension" here are such that there are no special ideas corresponding to them in the imagination. These two statements are "the work of the pure intellect, which alone has the ability to separate out abstract entities of this type" (AT X 444: CSM I 60). Descartes insists that we must distinguish statements of this kind, in which the meanings of the terms are separated from the content of the ideas in the imagination, from statements in which the terms, albeit "employed in abstraction from their subjects, do not exclude or deny anything which is not really distinct from what they denote" (AT X 445: CSM I 61).

The intellect and the imagination

This distinction between the two kinds of proposition is perhaps most clearly expressed in the distinction between their proper objects, i.e. the objects of the intellect and the objects of the imagination, respectively. The proper objects of the intellect are completely abstract entities and are free of images or "bodily representations." Indeed, while engaging in its proper activity, the intellect "turns itself toward itself" (AT VII 73: CSM II 51) and beholds those things that are purely intellectual such as thought and doubt, as well as those "simple natures" which are common to both mind and body, such as existence, unity, and duration. But the intellect can also apply itself to "ideas" in the imagination. In doing so it also carries out an operation which is proper to it, but which the imagination cannot carry out, namely, that of separating out components of these ideas by abstraction.

It is here that the necessity for the imagination arises, because the intellect by itself has no relation at all to the world. Entities conceived in the intellect are indeterminate. The imagination is required to render them determinate. When we speak of numbers, for example, the imagination must be employed to represent to ourselves something that can be measured by a multitude of objects. The intellect understands "fiveness" as something separate from five objects (or line segments, or points, or whatever), and hence the imagination is required if this "fiveness" is to correspond to something in the world. What we are effectively dealing with here, as far as the intellect is concerned, is algebra. It is insofar as the objects of algebra, the indeterminate content of which has been separated out by the intellect, can be represented and conceived symbolically as lines and planes that they can be identified with the real world. Algebra deals with completely abstract entities, conceived in the intellect, but these abstract entities must be represented symbolically, and thus rendered determinate, which requires the aid of the imagination. The imagination thereby represents *general* magnitudes (abstract entities) as *specific* magnitudes (which are not distinct from what they are the magnitudes of).

However, not any kind of specific magnitude will do here. The privileged specific magnitude that Descartes wishes to single out is spatial extension. There are two reasons for this. First, algebraic entities can be represented geometrically, i.e. purely in terms of spatial extension. Secondly, Descartes argues (e.g. in Rule 12) that when we consider the physiological, physical and optical aspects of perception it is clear that what we see in no way resembles bodies in

the world. The world itself contains no colors, odors etc. (no secondary qualities) but only spatially extended body. The secondary qualities that we perceive are simply a feature of the interaction of our sense organs, cognitive apparatuses, etc., with the external world. They are psychic additions of a perceiving mind. So the world is simply spatially extended body, and what is registered in the imagination is no less simply spatially extended magnitudes.

In sum, then, the corporeal world and the abstract entities of algebra are represented in the imagination as extended magnitudes and measures of extended magnitudes respectively, the former then being mapped onto the latter:

Intellect	abstract entities (algebra)
	↓
Imagination	lines, line lengths, etc. (geometry and arithmetic)
	extended magnitudes
	↑
Corporeal world	material objects (material extension)

In this schema, the pure thought characteristic of algebra which the intellect engages in does not map directly onto the corporeal world. Rather, a representation of it in the form of arithmetic and geometry maps onto a representation of the corporeal world, a representation consisting exclusively of two-dimensional shapes. This conception is subject to many problems, as might be expected from an account which deals with such fundamental questions, but it does provide us with the first explicit epistemological and metaphysical basis for a mathematical physics in the history of philosophy, and in many ways its role in Descartes' thought is more central than even the "Cogito."

CONCLUSION

What is remarkable about Descartes' work in algebra is its level of abstraction. This achievement has often been obscured, either by Descartes' own statement that all he was doing was rediscovering a secret method of discovery known to the mathematicians of antiquity, or by the widely held modern view that these mathematicians had a 'geometrical algebra', i.e. an algebraic interpretation of arithmetic that employed geometrical notation. I have given some reasons why I believe these, and especially the latter, to be wrong. In fact what the mathematicians of antiquity had was not an especially abstract algebraic interpretation of arithmetic but an especially concrete geometrical interpretation of it. The abstract interpretation comes only when the resources of arithmetic and geometry are combined to produce something far more powerful and abstract than either of them, and this is Descartes' achievement. He inaugurates (with Vieta and others) what I have identified as the first stage in the development of algebra, namely the freeing of number from spatial intuitions. This opened the way to the second stage, the freeing of algebra itself from an exclusively numerical interpretation. The move to this second stage was, however, one that went completely against the whole tenor of Descartes' approach. This was not so much because it takes one to a level of abstraction that even he was not prepared to countenance, for his early idea of a "universal mathematics" involves an extremely abstract (but unworkable) conception of mathematics that transcends any specific content, dealing only with whatever has order and magnitude (AT X 378: CSM I 19). It is rather because of his requirement that it be a method of discovery, which in turn means it must be epistemically informative. Deductive inference, he thinks (wrongly), can never be epistemically informative, so he rejects any connection between algebra and logic. Yet the second stage in the development of algebra comes about largely as a result of its application to systems of deductive reasoning.

Descartes was, then, not at all worried by the very abstract nature of his algebra in a mathematical context. But in many ways it is even more remarkable that he was not worried by it in a physical context either. His chief aim was to develop a mathematical physics and mathematics is, ultimately, algebra for Descartes. Well aware, at least after his early "universal mathematics" phase, that it could not just be a matter of applying a system as abstract as algebra to something as concrete and specific as the real world, he tried to establish that they do have one crucial thing in common: geometry. The only real properties of matter are those that can be understood wholly in geometrical terms, and algebra is represented in the imagination in purely geometrical terms. It is therefore geometry that ties the two

together. This may not be the most fruitful way of establishing a basis for a mathematical physics,[18] but the sheer daring and ingenuity of the conception is breathtaking, and indeed it is the first explicit philosophical attempt to come to terms in any detail with how a mathematical physics might be possible.

In sum, Descartes' work in algebra is something whose interest extends far beyond mathematics. This work made him one of the greatest mathematicians of the seventeenth century. But in following through its consequences for the development of a quantitative mechanical understanding of the corporeal world, he became one of the greatest natural scientists of the seventeenth century; and in following through its consequences for the question of method, he became its greatest philosopher.

NOTES

1 For a full account of the *Geometry* see Scott, *The Scientific Work of René Descartes*, chs. 6–9.

2 Readers who find the mathematics in what follows difficult may wish to omit this section.

3 See Grosholz, "Descartes' Unification of Algebra and Geometry," in Gaukroger (ed.), *Descartes, Philosophy, Mathematics, and Physics*, pp. 156–68.

4 See in particular Vuillemin, *Mathématiques et métaphysique chez Descartes.*

5 Klein, *Greek Mathematical Thought and the Origin of Algebra.*

6 Szabó, *The Beginnings of Greek Mathematics*, especially the Appendix.

7 See especially the discussion of Proposition 5 of Book II of the *Elements*, ibid., pp. 332–53.

8 Unguru, "On the need to rewrite the history of Greek mathematics."

9 What follows is derived from Gaukroger, "Aristotle on intelligible matter," where a much fuller account can be found.

10 On the early development of metrical geometry, see Knorr, *The Evolution of the Euclidean Elements*, pp. 170ff.

11 Klein, *Greek Mathematical Thought*, pp. 133ff.

12 The only exception to this constraint on multiplication occurs in a relatively late Alexandrian work, Heron's *Metrica* I 8, where two squares, i.e., areas, are multiplied together.

13 Kneale and Kneale, *The Development of Logic*, pp. 390ff.

14 For a full discussion of the issues raised in this section, see Gaukroger, *Cartesian Logic.*

15 Sextus Empiricus, *Outlines of Pyrrhonism*, II, 159 (Loeb edition: Vol 1, pp.253–5).
16 See Book II of Mill, *A System of Logic* [1843] (London: Longmans, 1967).
17 *Aubrey's Brief Lives*, ed. Oliver Lawson Dick (London: Secker and Warburg, 1960), p. 150.
18 See Gaukroger, "Descartes' project for a mathematical physics," in Gaukroger (ed.), *Descartes*, pp. 97–140.