CHAPTER 7

Non-Euclidean Geometry

Certainly one of the greatest mathematical discoveries of the nineteenth century was that of non-Euclidean geometry: seen but not revealed by Gauss, and developed in all its glory by Bolyai and Lobachevsky. The purpose of this chapter is to give an account of this theory, but we do not always follow the historical development. Rather, with hindsight we use those methods that seem to shed the most light on the subject. For example, continuity arguments have been replaced by a more axiomatic treatment.

There are actually three different approaches presented here. One begins with Saccheri's theory, dividing geometries into three classes, in Section 34, and the theorem of Saccheri–Legendre, using Archimedes' axiom, in Section 35. The second is the analytic model of a non-Euclidean geometry given in Section 39. Third is Hilbert's axiomatic approach based on the axiom of limiting parallels (L) in Section 40.

We start with a historical introduction to the problem of the parallels and the various futile attempts to prove Euclid's fifth postulate from the other axioms. Then we begin to explore this strange new world where the sum of the angles of a triangle can be less than two right angles. The defect of this angle sum provides a measure of area, which we exploit in Section 36.

To explain the Poincaré model of a non-Euclidean geometry, we need the Euclidean technique of circular inversion. This is developed in Section 37. It is a technique with many applications in Euclidean geometry. In particular, we make a digression in Section 38 to show how it can provide a solution to the classical problem of Apollonius, to construct a circle tangent to three given circles.

In Section 40 we present a development of non-Euclidean geometry based on the axiom of existence of limiting parallel rays, proposed by Hilbert. This allows us to avoid the appeal to continuity invoked by the founders of the subject and free ourselves from dependence on the real numbers. Then we give Hilbert's brilliant construction of an abstract field from the set of common ends of limiting parallel rays. This allows us to characterize hyperbolic planes by their associated fields without using the techniques of projective geometry.

We follow the principle, established earlier in this book, of systematically avoiding the use of real numbers. There is a slight cost, in that some familiar results will look different here, but I believe this approach is justified by keeping the intrinsic geometry in the foreground. For example, instead of taking logarithms to define a distance function, we use a multiplicative distance function μ. Then Bolyai's famous formula for the angle of parallelism α of a line segment PQ takes the form $\tan \alpha/2 = \mu(PQ)^{-1}$ (39.13) and (41.9). The arbitrary constant k that appears in some books, coming from the choice of a base for the logarithms in the distance function, is absent: In our approach, any two hyperbolic planes over the same field are isomorphic. Also, the hyperbolic trigonometric functions sinh, cosh, tanh do not appear in our formulae of hyperbolic trigonometry (42.2) and (42.3). As a result of this approach, the solution of any problem we consider can be found constructively, by ruler and compass, or, equivalently, by solving linear and quadratic equations in the coefficient field.

33 History of the Parallel Postulate

To set the background for the discovery of non-Euclidean geometry, a kind of geometry where there may be many lines through a point parallel to a given line, let us trace the history of attitudes toward the parallel postulate.

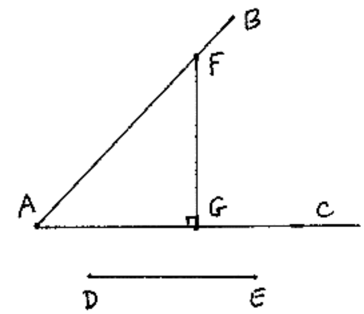
We have seen already that Euclid's fifth postulate, which we refer to as the parallel postulate, was of a much more sophisticated nature than the other postulates and axioms. Euclid seems to have recognized this himself, since he postponed using it as long as possible, and was careful to develop the standard congruence theorems for triangles without the parallel postulate.

Euclid was criticized for making this a postulate and not a theorem. Proclus (410–485), who represented the school of Plato in fifth-century Athens, has left an extensive commentary on the first book of Euclid's *Elements*. His opinion on the fifth postulate is unambiguous:

"This ought to be struck from the postulates altogether. For it is a theorem—one that invites many questions, which Ptolemy proposed to resolve in one of his books—and requires for its demonstration a number of definitions as well as theorems" (Proclus (1970), p. 150).

In his commentary on (I.29), Proclus gives Ptolemy's proof and points out its flaws, and then proceeds to give his own proof of the fifth postulate. First, he says, we must accept an axiom that was used earlier by Aristotle:

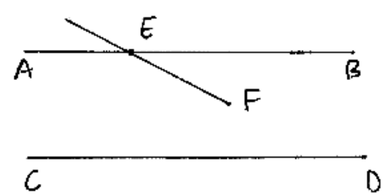
Aristotle's Axiom
If from a single point two straight lines making an angle are produced indefinitely, the interval between them will exceed any finite magnitude. In other words, given any angle BAC, and given a segment DE, there exists a point F on the ray AB such that the perpendicular FG from F to the line AC will be greater than DE.



Then Proclus proposes to prove the following lemma of Proclus.

Lemma of Proclus
If a straight line cuts one of two parallel lines, it cuts the other also.

His proof goes like this. If AB and CD are two parallel lines, and if EF cuts AB, with F on the side toward CD, then we apply Aristotle's axiom to the angle BEF. As we extend the ray EF indefinitely, its interval from the line AB will exceed the distance between the parallel lines, and so it must cut the line CD.



From this lemma (which is essentially the same as what we now call Playfair's axiom), Proclus easily proves the parallel postulate.

Proclus's reasoning was apparently accepted for some time, since it is reproduced without critical comment by F. Commandino in his edition of Euclid (1575).

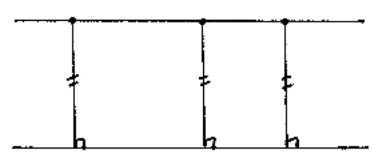
We can observe two things about the argument of Proclus. First of all, he assumes another axiom (the axiom of Aristotle) in the course of his proof. This is not uncommon in various attempted proofs of the parallel postulate. Often, one ends up assuming (consciously or unconsciously) something else that turns out to be equivalent to the parallel postulate. In this particular case, it is not so bad: We will see that Aristotle's axiom is a consequence of Archimedes' axiom, and does not imply the parallel postulate by itself (35.6).

The more serious flaw in Proclus's argument is that he speaks of "the distance between the parallel lines" as if all the points of one line were at the same distance from the other line. Since the definition of parallel lines is lines in the same plane that do not meet, however far extended, it does not follow from the definition that they are at a constant distance from each other. In fact, this assumption of constant distance is enough to prove the parallel postulate (in the presence of Aristotle's axiom), as Proclus shows. Thus, in view of (I.34) it is equivalent to the parallel postulate.

This confusion of the definition of parallel lines as lines that do not meet with the common-sense notion of parallel lines as equidistant from each other (like railroad tracks) has persisted. For example, in the edition of Euclid's first six books by J. Peletier (1557), definition 35 says, "Parallels, or equidistant straight lines, are those which being in the same plane, and extended arbitrarily in either direction, do not meet." However, Peletier follows Euclid's proofs in Book I, and does not make use of the equidistant property.

A more striking example is the very popular edition of the *Elements of Geometry* by the Jesuit Andrea Tacquet (1612–1660), first published in 1654 and reprinted many times over the next hundred and fifty years (Tacquet (1738)). Tacquet's book is not a strict translation of Euclid, but an arrangement, to make the study of geometry easier for beginners. Though he preserves the numbering of Euclid's propositions, he takes great liberties with their proofs. For example, he says that there is no point in proving (I.16), because it is a special case of (I.32)! He apparently does not care about the fact that Euclid's proof of (I.16) is independent of the parallel postulate, while (I.32) depends on it.

Tacquet says that since there are various species of lines (such as the hyperbola and a straight line) that approach each other indefinitely but never meet, so Euclid's definition of parallel lines does not satisfactorily reflect the nature of parallels.



He takes as his definition that two lines are parallel if the points of one are all equidistant from the other, as measured by perpendiculars from points on the first line to the second line.

There is no harm, of course, in using any definition you like of parallel lines, though this one places a great burden on the proof of existence of parallels. Tacquet misses the subtlety, however, because in the next sentence he says that you can generate parallel lines as the locus of points at a fixed distance from a given line as the perpendicular moves along. Here he is implicitly using another axiom, which was in fact stated explicitly and used earlier by Christoph Clavius (1537–1612) as a substitute for Euclid's parallel postulate:

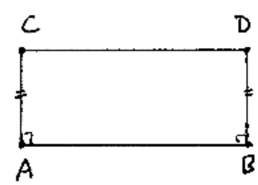
Clavius's Axiom
The set of points equidistant from a given line on one side of it form a straight line.

This axiom, as one can easily show, is almost equivalent to the parallel postulate that Tacquet was trying to avoid (Exercise 33.7).

The French mathematician Alexis Claude Clairaut (1713–1765) wrote an *Elémens de Géométrie* (first published in 1741) in which he tried to make geometry more accessible for students. He complained about the usual method of teaching the elements, in which "one always starts with a great number of definitions, postulates, axioms, and first principles, which appear to offer nothing but dryness to the reader." He thought that Euclid's careful reasoning was merely to satisfy a fussy audience: "That Euclid went to the trouble to prove that two circles which cut each other do not have the same center; that a triangle contained inside another triangle has the sum of its sides less than that of the triangle in which it is enclosed—one should not be surprised. For this geometer had to convince the obstinate sophists who glorified in finding fault with the most evident truths: so it was necessary that geometry, like logic, make use of proper reasoning, to close the mouths of its critics."

Clairaut's purpose is to introduce the concepts of geometry simply and naturally in the context of practical questions such as measurement of terrain. So he talks of straight lines to measure the distance between points, and how to construct perpendicular lines. Then he says, what is more easy than to use this method to construct a rectangle? One has only to take a segment AB, and at its endpoints raise perpendiculars AC and BD of equal length, and then join CD. From here he develops the theory of parallels. The hidden assumption is that his construction makes a rectangle. So we will call this assumption Clairaut's axiom.

Clairaut's Axiom
Given a segment AB, let AC and BD be equal segments perpendicular to AB. Then the angles at C and D are right angles, i.e., $ABCD$ is a rectangle.



Robert Simson, M.D. (1687–1768), professor of mathematics in the University of Glasgow, made an important edition of Euclid's elements, in Latin and in English, first published in 1756, which went through some thirty successive editions. Simson railed against the errors introduced by earlier editors, and wished to "restore the principal Books of the Elements to their original Accuracy.... This I have endeavored to do by taking away the inaccurate and false Reasonings which unskilful Editors have put into the place of some of the genuine Demonstrations of Euclid, who has ever been justly celebrated as the most accurate of Geometers, and by restoring to him those Things which Theon or others have suppressed, and which have these many ages been buried in Oblivion" (Simson (1803), Preface). Simson's restorations were not so much based on textual studies as on his faith that anything mathematically true and accurate must have been Euclid's, while anything incorrect or not rigorous must have been inserted by "some unskilful editor." About the parallel postulate, he says, "It seems not to be properly placed among the Axioms, as, indeed, it is not self-evident; but it may be demonstrated thus." Simson then introduces an axiom,

Simson's Axiom
A straight line cannot first come nearer to another straight line, and then go further from it, before it cuts it; and, in like manner, a straight line cannot go further from another straight line, and then come nearer to it; nor can a straight line keep the same distance from another straight line, and then come nearer to it, or go further from it (Simson (1803), p. 295).

From this axiom, and implicitly making use of Archimedes' axiom, Simson proves (correctly) five propositions, of which the last is Euclid's parallel postulate.

So here we have a clear case of an author substituting another axiom that seems more natural to him, and then using it to prove the parallel postulate.

John Playfair (1748–1819), professor of natural philosophy, formerly of mathematics, in the University of Edinburgh, published a new edition of the first six books of Euclid's *Elements* that first appeared in 1795. He says that Dr. Simson has done a fine job of restoring Euclid's *Elements*, and that his purpose in presenting a new edition is to give them the form that may "render them most useful." He says, "A new axiom is also introduced in the room of the 12th [which we call the fifth postulate], for the purpose of demonstrating more easily some of the properties of parallel lines" (Playfair (1795), Preface). This is Playfair's axiom.

Playfair's Axiom
Two straight lines that intersect one another cannot be both parallel to the same straight line.

In his notes to (I.29), Playfair has an interesting discussion of the problem of parallels. He agrees with Proclus that Euclid's postulate should be proved, and not taken as an axiom. He then reviews the three methods by which geometers "have attempted to remove this blemish from the *Elements*. . .

(1) by a new definition of parallel lines;
(2) by introducing a new Axiom concerning parallel lines, more obvious than Euclid's;
(3) by reasoning merely from the definition of parallels, and the properties of lines already demonstrated, without the assumption of any new Axiom."

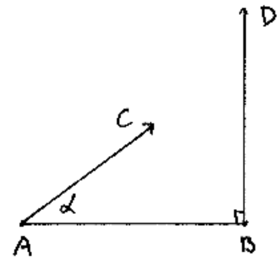
Exercises

Throughout these exercises, we assume the axioms of a Hilbert plane.

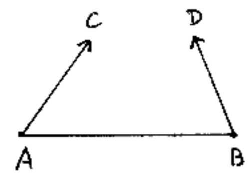
33.1 Show that the lemma of Proclus is equivalent to Playfair's axiom (P).

33.2 Consider the following special case of Euclid's parallel postulate, which we will call the **right triangle axiom**: Given a right angle ABD and an acute angle $\alpha = CAB$ on the same side of the line AB, the ray AC when extended will meet the ray BD extended.

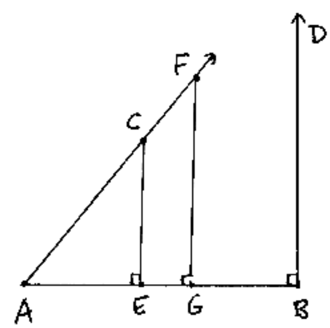
Show that the right triangle axiom is equivalent to (P).



33.3 Show directly that the right triangle axiom implies the special case of Euclid's parallel postulate that says, given acute angles $\alpha = CAB$ and $\beta = ABD$ on the same side of the line AB, the rays AC and BD will meet.

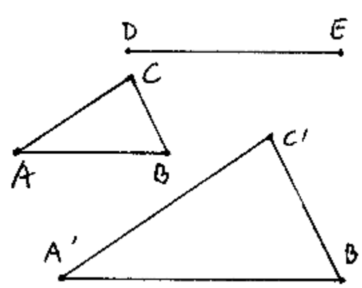


33.4 Discuss the following "proof" of the right triangle axiom due to Franceschini (1756–1840): Given A, B, C, D as in Exercise 33.2, drop a perpendicular CE from C to the line AB. Since α is an acute angle, E will lie between A and B. Now take a point F further out on the ray AC. Drop a perpendicular FG from F to AB. Then G is between E and B. As the point F moves out the ray AC without bound, so the point G must move along the ray AE without bound, and thus it must eventually reach B. Then F will be the intersection of AC and BD.



33.5 John Wallis (1616–1703) gave a proof of the parallel postulate based on the principle that to every figure there is always a similar figure of arbitrary size. To be precise, we state Wallis's axiom as follows:

Wallis's Axiom
Given a triangle ABC and given a line segment DE, there exists a *similar* tri-

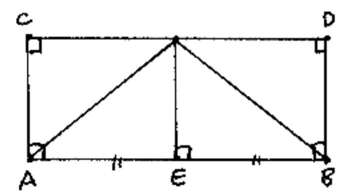


angle $A'B'C'$ (that is, a triangle with the same angles as the triangle ABC) having side $A'B' \geq DE$.

(a) Show that Wallis's axiom implies (P).

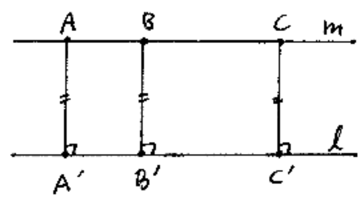
(b) In the non-Archimedean geometry of (18.4.3) show that there are similar triangles of different sizes, but that Wallis's axiom fails. (We will see later that in a semihyperbolic or semielliptic non-Euclidean geometry, the only similar triangles are congruent triangles (Exercise 34.4).)

33.6 In a Hilbert plane, show that opposite sides of a *rectangle* (i.e., a figure with four right angles) are equal. *Hint*: Bisect one side AB at E, erect a perpendicular to AD at E, and use the accompanying diagram. Your goal: to show $AB \cong CD$.

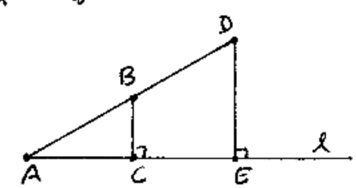


33.7 In this exercise we explore the consequences of Clavius's axiom.

(a) Let l be a line, and let m be a set of equidistant points, which by Clavius's axiom is a line. Thus for points A, B, C in m, the perpendiculars AA', BB', CC' to l are all equal. Show that the angles at A, B, C are also right angles.



(b) Let ABC be a right triangle. Extend AB to D so that $AB \cong BD$, and drop the perpendicular DE to AC. Assuming Clavius's axiom, show that $DE \cong 2BC$.



(c) Show that Clavius's axiom, together with Archimedes' axiom (A), implies (P).

(d) Show that Clavius's axiom holds in the non-Archimedean plane of (18.4.3) even though (P) does not.

33.8 (a) Show that Aristotle's axiom holds in the Cartesian plane over a field F, even if F is not Archimedean.

(b) Show that Aristotle's axiom fails in the plane of (18.4.3).

33.9 Show that Clairaut's axiom is equivalent to Clavius's axiom.

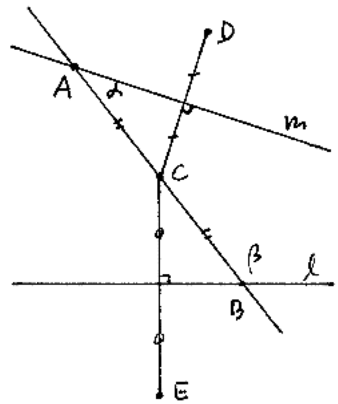
33.10 Show that Simson's axiom is equivalent to Clavius's axiom.

33.11 Farkas Bolyai, the father of János, proposed the following axiom.

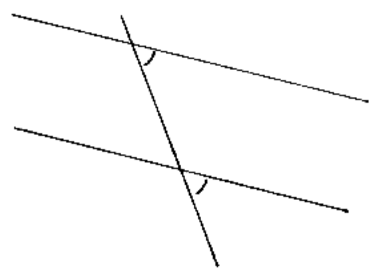
Bolyai's Axiom
For any three noncollinear points A, B, C there exists a circle containing them.

(a) Use the following construction to show that Bolyai's axiom implies Euclid's parallel postulate. Given two lines l, m and a transversal AB, assume that the angles α, β on one side add up to less than two right angles. Let C be the midpoint of AB. From C drop perpendiculars to l and m, and extend each an equal distance on the far side to obtain D and E. Show that C, D, E are not collinear, and then use Bolyai's axiom to prove that l and m must meet.

(b) Show that Bolyai's axiom holds in any Hilbert plane with (P).



33.12 Dr. Anton Bischof in his thesis (1840) proposed to free the theory of parallels from its dependence on Euclid's parallel postulate by giving a different definition of parallel lines. Discuss his theory, which goes like this: Lines are parallel if they have the same direction.

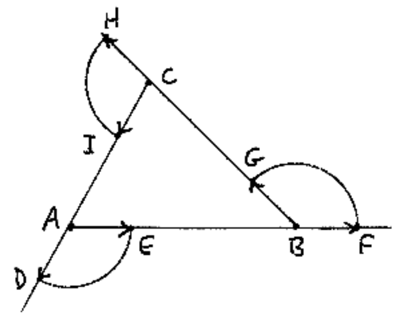


The direction of a line can be measured by the angle it makes with another line. So we define "parallelism is the equality of direction of similar lines against every other straight line." In other words, two lines are parallel if they make equal angles with every other line that meets them both.

Then it is clear that parallel lines cannot meet, because a transversal line through the point of intersection would make the same angle with both of them, so they would be equal. By the same reasoning it is clear that there can be only one parallel to a given line through a given point. If two lines make the same angle with a line that cuts them, they will be parallel. "Similarly one obtains all the other corollaries which one finds in all the textbooks."

33.13 Discuss the following "proof" that the sum of the angles of a triangle is equal to two right angles, independent of the theory of parallels, due to Thibaut (1775–1832):

Let ABC be the given triangle. Take a segment AD on the line AC, pointing away from C. Rotate it to the position AE on the line AB. Then slide it along the line AB into the position BF. Rotate to BG, slide to CH, rotate to CI, and slide back to AD. In this process, the segment AD has made one complete rotation, which is 4 right angles. But the amount it has rotated is equal to the sum of the exterior angles DAE, FBG, and HCI. Replacing these by their supplementary angles, we find that the sum of the three interior angles of the triangle is equal to two right angles.



33.14 J.J. Callahan, then president of Duquesne University, in his book *Euclid or Einstein* (1931) claims to prove the parallel postulate of Euclid, and thus nullify the theories of Einstein based on non-Euclidean geometry. If you can locate a copy of his book, read his proof and find the flaw in his argument.

34 Neutral Geometry

Sir Henry Savile, in his public lectures on Euclid's *Elements* in Oxford in 1621, said, "In this most beautiful body of Geometry there are two moles, two blemishes, and no more, as far as I know, for whose removal and washing away, both older and more recent authors have shown much diligence." He was referring to the theory of parallels and the theory of proportion. Euclid's theory of proportion has been thoroughly vindicated, and receives its modern expression in the segment arithmetic that we have explained in Chapter 4.

The work on the theory of parallels, however, did not lead to the expected result. Instead of confirming Euclid's as the one true geometry, these researches showed that Euclid's was only one of many possible geometries. The others are what we now call non-Euclidean geometries. The story of this discovery is one of the most fascinating chapters in the history of mathematics, and has been amply told elsewhere. Here we will confine ourselves to the briefest outline.

We can distinguish four periods. The first, which we have elaborated in the previous section, might be called "dissatisfaction with Euclid." While fully accepting Euclid's *Elements* as the true geometry, critics said only that his treatment of this topic could have been better. So they tried to better Euclid, either