Hilbert's Axioms

CHAPTER



ur purpose in this chapter is to present (with minor modifications) a set of axioms for geometry proposed by Hilbert in 1899. These axioms are sufficient by modern standards of rigor to supply the foundation for Euclid's geometry. This will mean also axiomatizing those arguments where he used intuition, or said nothing. In particular, the axioms for betweenness, based on the work of Pasch in the 1880s, are the most striking innovation in this set of axioms.

Another choice has been to take the SAS theorem as an axiom, and thus bypass the method of superposition. It is possible to go the other route, and use motions of figures as a basic building block of geometry. This is what Hadamard does in his *Leçons de Géométrie Elémentaire* (1901–06), but the result is a step backward in logical clarity, because he never makes precise exactly what kind of motions he is allowing. See, however, Section 17 for a fuller discussion of rigid motions and SAS.

The first benefit of establishing the new system of axioms is, of course, to vindicate Euclid's *Elements*, and thus establish "Euclidean" geometry as a rigorous mathematical discipline. A second benefit is to pose carefully those problems that have bothered geometers for centuries, such as the question of the independence of the parallel postulate. Unless one has an exact understanding of precisely what is assumed and what is not, one risks going around in circles discussing these questions. In the development of our geometry with the new

axioms, we will keep the parallel postulate separate and note carefully what depends on it and what does not.

Besides presenting the axioms, this chapter will also contain the first consequences of the axioms, including different proofs of some of Euclid's early propositions, until we have established enough so that Euclid's later results can be deduced without difficulty from the new foundations we have established. In Sections 10, 11, 12, we show how to recover all the results of Euclid, Books I–IV, except for the theory of area, whose proof is postponed until Chapter 5.

6 Axioms of Incidence

The axioms of incidence deal with points and lines and their intersections. The points and lines are undefined objects. We simply postulate a set, whose elements are called *points*, together with certain subsets, which we call *lines*. We do not say what the points are, nor which subsets form lines, but we do require that these undefined notions obey certain axioms:

- **I1.** For any two distinct points *A*, *B*, there exists a unique line *l* containing *A*, *B*.
- I2. Every line contains at least two points.
- **I3.** There exist three noncollinear points (that is, three points not all contained in a single line).

Definition

A set whose elements are called points, together with a set of subsets called lines, satisfying the axioms (I1), (I2), (I3), will be called an *incidence geometry*. If a point P belongs to a line l, we will say that P lies on l, or that l passes through P.

From this modest beginning we cannot expect to get very interesting results, but just to illustrate the process, let us see how one can prove theorems based on these axioms.

Proposition 6.1

Two distinct lines can have at most one point in common.

Proof Let l, m be two lines, and suppose they both contain the points A, B, with $A \neq B$. According to axiom (I1), there is a unique line containing both A and B, so l must be equal to m.

Note that this fact, which was used by Euclid in the proof of (I.4) with the rather weak excuse that "two lines cannot enclose a space," follows here from the uniqueness part of axiom (I1). This should indicate the importance of stat-

ing explicitly the uniqueness of an object, which was rarely done in Euclid's *Elements*.

Now we have an axiom system, consisting of the undefined sets of points and lines, and the axioms (I1)-(I3). A *model* of that axiom system is a realization of the undefined terms in some particular context, such that the axioms are satisfied. You could also think of the model as an *example* of the *incidence geometry* defined above.

Example 6.1.1 (The real Cartesian

plane).

Here the set of points is the set \mathbb{R}^2 of ordered pairs of real numbers. The lines are those subsets of points P = (x, y) that satisfy a linear equation ax + by + c = 0 in the variables x, y. To verify that the axioms hold, for (I1) think of the "two-point formula" from analytic geometry: Given two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$. They lie on the line

$$y - a_2 = \frac{b_2 - a_2}{b_1 - a_1} (x - a_1)$$



if $a_1 \neq b_1$; if $a_1 = b_1$, they lie on the line $x = a_1$. To verify (I2), take any linear equation involving *y*. Substitute two different values of *x*, and solve for *y*. This gives two points on the line. If the equation did not involve *y*, say x = c, take the points (c, 0) and (c, 1). To verify (I3), consider the points (0, 0), (0, 1), (1, 0). One sees easily that there is no linear equation with all three points as solutions.

Example 6.1.2

One can also make models out of finite sets. For example, let the set of points be a set of three elements $\{A, B, C\}$, and take for lines the subsets $\{A, B\}, \{A, C\}$, and $\{B, C\}$. We represent this symbolically by the diagram, where the dots represent the elements of the set, and the lines drawn on the page show which subsets are to be taken as lines.



This diagram should be understood as purely symbolic, however, and has nothing to do with a triangle in the ordinary Cartesian plane. The verification of the axioms in this case is trivial.

Definition

Two distinct lines are *parallel* if they have no points in common. We also say that any line is parallel to itself.

The parallel postulate, in its equivalent form given by Playfair, can be stated as a further axiom about incidence of lines. However, we do not include this axiom in the definition of incidence geometry. Thus we may speak of an incidence geometry that does or does not satisfy Playfair's axiom.

P. (Playfair's axiom, also called the parallel axiom). For each point *A* and each line *l*, there is at most one line containing *A* that is parallel to *l*.

Note that the real Cartesian plane (6.1.1) satisfies (P), as you know, and the three-point geometry (6.1.2) satisfies (P) vacuously, because there are no distinct parallel lines at all. Next we give an example of an incidence geometry that does not satisfy (P).

Example 6.1.3

Let our set consist of five points A, B, C, D, E, and let the lines be all subsets of two points. It is easy to see that this geometry satisfies (I1)–(I3). However, it does not satisfy (P), because, for example, AB and AC are two distinct lines through the point A and parallel to the line DE.



Remember that the word *parallel* simply means that two lines have no points in common or are equal. It does not say anything about being in the same direction, or being equidistant from each other, or anything else.

We say that two models of an axiom system are *isomorphic* if there exists a 1-to-1 correspondence between their sets of points in such a way that a subset of the first set is a line if and only if the correspondence takes lines into lines." So for example, we see that (6.1.1), (6.1.2), and (6.1.3) are nonisomorphic models of incidence geometry, for the simple reason that their sets of points have different cardinality: There are no 1-to-1 correspondences between any of these sets.

On the other hand, we can show that any model of incidence geometry having just three points is isomorphic to the model given in (6.1.2). Indeed, let $\{1, 2, 3\}$ be a geometry of three points. By (I3), there must be three noncollinear points. Since there are only three points here, we conclude that there is no line containing all three. But by (I1), each subset of two points must be con-

tained in a line. Thus $\{1,2\}, \{2,3\}$, and $\{1,3\}$ are lines. Now by (I2), every line contains at least two points, so these are all the possible lines. In other words, the lines are just all subsets of two elements. Since (6.1.2) also has this property, any 1-to-1 correspondence between the sets $\{A, B, C\}$ and $\{1,2,3\}$ will give an isomorphism.

By the way, this proof shows that the isomorphism just found is not unique. There are six choices. This leads to the notion of automorphism.

Definition

An *automorphism* of an incidence geometry is an isomorphism of the geometry with itself, that is, it is a 1-to-1 mapping of the set of points onto itself, preserving lines.

Note that the composition of two automorphisms is an automorphism, and so is the inverse of an automorphism. Thus the set of automorphisms forms a group. In the example above, any 1-to-1 mapping of the set of three elements onto itself gives an automorphism of the geometry, so we see that the group of automorphisms of this geometry is the symmetric group on three letters, S_3 .

An important question about a set of axioms is whether the axioms are *independent* of each other. That is to say, that no one of them can be proved as a consequence of the others. For if one were a consequence of the others, then we would not need that one as an axiom. To try to prove directly that axiom A is not a consequence of axioms B, C, D, \ldots is usually futile. So instead, we search for a model in which axioms B, C, D, \ldots hold but axiom A does not hold. If such a model exists, then there can be no proof of A as a consequence of B, C, D, \ldots , so we conclude that A is independent of the others. This process must be repeated with each individual axiom, to show that each one is independent of all the others. With a long list of axioms this can become tedious and difficult, so we will forgo the process with our full list of axioms. But as an illustration of what is involved, let us show that the axioms (I1), (I2), (I3), and (P) are independent.

Proposition 6.2

The axioms (I1), (I2), (I3), (P) are independent of each other.

Proof We have already seen that (6.1.3) is a model satisfying (I1), (I2), (I3), and not (P). Hence (P) is independent of the others.

For a model satisfying (I1), (I2), (P), and not (I3), take a set of two points and the one line containing both of them.

Note that (P) is satisfied trivially, because there are no points not on the line l.



For a model satisfying (I1), (I3), (P), and not (I2), take a set of three points A, B, C, and for lines take the subsets $\{A, B\}, \{A, C\}, \{B, C\}, \text{ and } \{A\}$. The existence of the one-point line $\{A\}$ contradicts (I2). Yet (P) is still fulfilled, because that one-point line is then the unique line through A parallel to $\{B, C\}$.

For a model satisfying (I2), (I3), (P) and not (I1), just take a set of three points and no lines at all.



While we are discussing axiom systems, there are a few more concepts we should mention. An axiom system is *consistent* if it will never lead to a contradiction. That is to say, if it is not possible to prove from the axioms a statement A and also to prove its negation not A. This is obviously a highly desirable property of a system of axioms. We do not want to waste our time proving theorems from a system of axioms that one day may lead to a contradiction. Unfortunately, however, the logician Kurt Gödel has proved that for any reasonably rich set of axioms, it will be impossible to prove the consistency of that system. So we will have to settle for something less, which is *relative consistency*. As soon as you can find a model for your axiom system within some other mathematical theory T, it follows that if T is consistent, then also your system of axioms is consistent. For any contradiction that might follow from your axioms would then also appear in the theory T, contradicting its consistency. So for example, if you believe in the consistency of the theory of real numbers, then you must accept the consistency of Hilbert's axiom system for geometry, because all of his axioms will hold in the real Cartesian plane. That is the best we can do about the question of consistency.

Another question about a system of axioms is whether it is *categorical*. This means, does it describe a unique mathematical object? Or in other words, is there a *unique* model (up to isomorphism) for the system of axioms? In fact, it will turn out that if we take the entire list of Hilbert's axioms, including the parallel axiom (P) and Dedekind's axiom (D), the system will be categorical, and the unique model will be the real Cartesian plane. (We will prove this result later (21.3).) Also, if we take all of Hilbert's axioms, together with (D) and the hyperbolic axiom (L) (see Section 40), we will have another categorical system, whose unique model is the non-Euclidean Poincaré model over the real numbers (Exercise 43.2).

However, from the point of view of this book, it is more interesting to have an axiom system that is not categorical, and then to investigate the different possible geometries that can arise. Therefore, we will almost never assume Dedekind's axiom (D), and we will only sometimes assume Archimedes' axiom (A), or the parallel axiom (P). Finally, one can ask whether the axiom system is *complete*, which means, can every statement that is true in every model of the axiom system be proved as a consequence of the axioms? Again, Gödel has shown that any axiomatic system of reasonable richness cannot be complete. For a fuller discussion of these questions, see Chapter 51 of Kline (1972) on the foundations of mathematics.

Exercises

- 6.1 Describe all possible incidence geometries on a set of four points, up to isomorphism. Which ones satisfy (P)?
- 6.2 The Cartesian plane over a field F. Let F be any field. Take the set F² of ordered pairs of elements of the field F to be the set of points. Define lines to be those subsets defined by linear equations, as in Example 6.1.1. Verify that the axioms (I1), (I2), (I3), and (P) hold in this model. (See Section 14 for more about Cartesian planes over fields.)
- 6.3 A *projective plane* is a set of points and subsets called lines that satisfy the following four axioms:
 - P1. Any two distinct points lie on a unique line.
 - P2. Any two lines meet in at least one point.
 - P3. Every line contains at least three points.
 - P4. There exist three noncollinear points.

Note that these axioms imply (I1)-(I3), so that any projective plane is also an incidence geometry. Show the following:

(a) Every projective plane has at least seven points, and there exists a model of a projective plane having exactly seven points.

- (b) The projective plane of seven points is unique up to isomorphism.
- (c) The axioms (P1), (P2), (P3), (P4) are independent.
- 6.4 Let *F* be a field, and let $V = F^3$ be a three-dimensional vector space over *F*. Let Π be the set of 1-dimensional subspaces of *V*. We will call the elements of Π "points." So a "point" is a 1-dimensional subspace $P \subseteq V$. If $W \subseteq V$ is a 2-dimensional subspace of *V*, then the set of all "points" contained in *W* will be called a "line." Show that the set Π of "points" and the subsets of "lines" forms a projective plane (Exercise 6.3).
- 6.5 An *affine plane* is a set of points and subsets called lines satisfying (11), (12), (13), and the following stronger form of Playfair's axiom.

P'. For every line l, and every point A, there exists a unique line m containing A and parallel to l.

(a) Show that any two lines in an affine plane have the same number of points (i.e., there exists a 1-to-1 correspondence between the points of the two lines).

(b) If an affine plane has a line with exactly n points, then the total number of points in the plane is n^2 .

(c) If *F* is any field, show that the *Cartesian plane* over F (Exercise 6.2) is a model of an affine plane.

(d) Show that there exist affine planes with 4, 9, 16, or 25 points. (The nonexistence of an affine plane with 36 points is a difficult result of Euler.)

6.6 In an incidence geometry, consider the relationship of parallelism, "l is parallel to m," on the set of lines.

(a) Give an example to show that this need not be an equivalence relation.

(b) If we assume the parallel axiom (P), then parallelism is an equivalence relation.

(c) Conversely, if parallelism is an equivalence relation in a given incidence geometry, then (P) must hold in that geometry.

6.7 Let Π be an affine plane (Exercise 6.5). A *pencil* of parallel lines is the set of all the lines parallel to a given line (including that line itself). We call each pencil of parallel lines an "ideal point," or a "point at infinity," and we say that an ideal point "lies on" each of the lines in the pencil. Now let Π' be the enlarged set consisting of Π together with all these new ideal points. A *line* of Π' will be the subset consisting of a line of Π plus its unique ideal point, or a new line, called the "line at infinity," consisting of all the ideal points.

(a) Show that this new set Π' with subsets of lines as just defined forms a projective plane (Exercise 6.3).

(b) If Π is the Cartesian plane over a field *F* (Exercise 6.2), show that the associated projective plane Π' is isomorphic to the projective plane constructed in Exercise 6.4.

- 6.8 If there are n + 1 points on one line in a projective plane Π, then the total number of points in Π is n² + n + 1.
- 6.9 Kirkman's schoolgirl problem (1850) is as follows: In a certain school there are 15 girls. It is desired to make a seven-day schedule such that each day the girls can walk in the garden in five groups of three, in such a way that each girl will be in the same group with each other girl just once in the week. How should the groups be formed each day?

To make this into a geometry problem, think of the girls as points, think of the groups of three as lines, and think of each day as describing a set of five lines, which we call a pencil. Now consider a *Kirkman geometry*: a set, whose elements we call *points*, together with certain subsets we call *lines*, and certain sets of lines we call *pencils*, satisfying the following axioms:

K1. Two distinct points lie on a unique line.

K2. All lines contain the same number of points.

K3. There exist three noncollinear points.

K4. Each line is contained in a unique pencil.

K5. Each pencil consists of a set of parallel lines whose union is the whole set of points.

(a) Show that any affine plane gives a Kirkman geometry where we take the pencils to be the set of all lines parallel to a given line. (Hence by Exercise 6.5 there exist Kirkman geometries with 4, 9, 16, 25 points.)

(b) Show that any Kirkman geometry with 15 points gives a solution of the original schoolgirl problem.

(c) Find a solution for the original problem. (There are many inequivalent solutions to this problem.)

6.10 In a finite incidence geometry, the number of lines is greater than or equal to the number of points.

7 Axioms of Betweenness

In this section we present axioms to make precise the notions of betweenness (when one point is in between two others), on which is based the notion of sidedness (when a point is on one side of a line or the other), the concepts of inside and outside, and also the concepts of *order*, when one segment or angle is bigger than another. We have seen the importance of these concepts in reading Euclid's geometry, and we have also seen the dangers of using these concepts intuitively, without making their meaning precise. So these axioms form an important part of our new foundations for geometry. At the same time, these axioms and their consequences may seem difficult to understand for many readers, not because the mathematical concepts are technically difficult, but because the notions of order and separation are so deeply ingrained in our daily experience of life that it is difficult to let go of our intuitions and replace them with axioms. It is an exercise in forgetting what we already know from our inner nature, and then reconstituting it with an open mind as an external logical structure.

Throughout this section we presuppose axioms (I1)-(I3) of an incidence geometry. The geometrical notions of betweenness, separation, sidedness, and order will all be based on a single undefined relation, subject to four axioms. We postulate a *relation* between sets of three points *A*, *B*, *C*, called "*B* is between *A* and *C*." This relation is subject to the following axioms.

- **B1.** If *B* is between *A* and *C*, (written *A* * *B* * *C*), then *A*, *B*, *C* are three distinct points on a line, and also *C* * *B* * *A*.
- **B2.** For any two distinct points A, B, there exists a point C such that A * B * C.
- **B3.** Given three distinct points on a line, one and only one of them is between the other two.

B4. (Pasch). Let A, B, C be three noncollinear points, and let l be a line not containing any of A, B, C. If lcontains a point D lying between A and B, then it must also contain either a point lying between A and C or a point lying between B and C, but not both.



Definition

If A and B are distinct points, we define the *line segment* \overline{AB} to be the set consisting of the points A, B and all points lying between A and B. We define a *triangle* to be the union of the three line segments $\overline{AB}, \overline{BC}$, and \overline{AC} whenever A, B, C are three noncollinear points. The points A, B, C are the *vertices* of the triangle, and the segments $\overline{AB}, \overline{BC}, \overline{AC}$ are the *sides* of the triangle.

Note: The segments \overline{AB} and \overline{BA} are the same sets, because of axiom (B1). The *endpoints* A, B of the segment \overline{AB} are uniquely determined by the segment \overline{AB} (Exercise 7.2). The vertices A, B, C, and the sides $\overline{AB}, \overline{AC}, \overline{BC}$ of a triangle ABC are uniquely determined by the triangle (Exercise 7.3).

With this terminology, we can rephrase (B4) as follows: If a line *l* that does not contain any of the vertices A, B, C of a triangle meets one side \overline{AB} , then it must meet one of the other sides \overline{AC} or \overline{BC} , but not both.

From these axioms together with the axioms of incidence (I1)-(I3) we will deduce results about the separation of the plane by a line, and the separation of a line by a point.

Proposition 7.1 (Plane separation)

Let *l* be any line. Then the set of points not lying on *l* can be divided into two nonempty subsets S_1, S_2 with the following properties:

- (a) Two points A, B not on l belong to the same set $(S_1 \text{ or } S_2)$ if and only if the segment \overline{AB} does not intersect l.
- (b) Two points A, C not on l belong to the opposite sets (one in S₁, the other in S₂) if and only if the segment AC intersects l in a point.

We will refer to the sets S_1, S_2 as the two sides of l, and we will say "A and B are on the same side of l," or "A and C are on opposite sides of l."



Proof We start by defining a relation ~ among points not on *l*. We will say $A \sim B$ if either A = B or if the segment \overline{AB} does not meet *l*. Our first step is to show that ~ is an equivalence relation. Clearly, $A \sim A$ by definition, and $A \sim B$ implies $B \sim A$ because the set \overline{AB} does not depend on the order in which we write *A* and *B*. The nontrivial step is to show the relation is transitive: If $A \sim B$ and $B \sim C$, we must show $A \sim C$.

Case 1 Suppose A, B, C are not collinear. Then we consider the triangle *ABC*. Since $A \sim B$, *l* does not meet \overline{AB} . Since $B \sim C$, *l* does not meet \overline{BC} . Now by Pasch's axiom (B4), it follows that *l* does not meet \overline{AC} . Hence $A \sim C$.

Case 2 Suppose A, B, C lie on a line m. Since A, B, C do not lie on l, the line m is different from l. Therefore l and m can meet in at most one point (6.1). But by (I2) every line has at least two points. Therefore, there exists a point D on l, D not lying on m.



Now apply axiom (B2) to find a point *E* such that D * A * E. Then D, A, E are collinear (B1); hence *E* is not on *l*, since *A* is not on *l*, and the line *DAE* already meets *l* at the point *D*. Furthermore, the segment \overline{AE} cannot meet *l*. For if it did, the intersection point would be the unique point in which the line *AE* meets *l*, namely *D*. In that case *D* would be between *A* and *E*. But we constructed *E* so that D * A * E, so by (B3), *D* cannot lie between *A* and *E*. Thus $\overline{AE} \cap l = \emptyset$, so $A \sim E$. Note also that *E* does not lie on the line *m*, because if *E* were on *m*, then the line *AE* would be equal to *m*, so *D* would lie on *m*, contrary to our choice of *D*. Therefore, *A*, *B*, *E* are three noncollinear points. Then by Case 1 proved above, from $A \sim E$ and $A \sim B$ we conclude $B \sim E$. By Case 1 again, from $B \sim E$ and $B \sim C$ we conclude $C \sim E$. Applying Case 1 a third time to the three noncollinear points *A*, *C*, *E*, from $A \sim E$ and $C \sim E$ we conclude $A \sim C$ as required.

Thus we have proved that \sim is an equivalence relation. An equivalence relation on a set divides that set into a disjoint union of equivalence classes, and these equivalence classes will satisfy property (a) by definition. To complete the proof it will be sufficient to show that there are exactly two equivalence classes S_1 , S_2 for the relation \sim . Then to say that \overline{AC} meets l, which is equivalent to $A \neq C$, will be the same as saying that A, C belong to the opposite sets.

By (I3) there exists a point not on l, so there is at least one equivalence class S_1 . Given $A \in S_1$, let D be any point on l, and choose by (B2) a point C such that

A * D * C. Then A and C do not satisfy \sim , so there must be at least two equivalence classes S_1 and S_2 .

The last step is to show that there are at most two equivalence classes. To do this, we will show that if $A \neq C$ and $B \neq C$, then $A \sim B$.

Case 1 If A, B, C are not collinear, we consider the triangle *ABC*. From $A \neq C$ we conclude that \overline{AC} meets *l*. From $B \neq C$ we conclude that \overline{BC} meets *l*. Now by Pasch's axiom (B4) it follows that \overline{AB} does not meet *l*. So $A \sim B$ as required.

Case 2 Suppose *A*, *B*, *C* lie on a line *m*. As in Case 2 of the first part of the proof above, choose a point *D* on *l*, not on *m*, and use (B2) to get a point *E* with D * A * E. Then $A \sim E$ as we showed above.



Now, $A \neq C$ by hypothesis, and $A \sim E$, so we conclude that $C \neq E$, since \sim is an equivalence relation (if $C \sim E$, then $A \sim C$ by transitivity: contradiction). Looking at the three noncollinear points B, C, E, from $E \neq C$ and $B \neq C$ we conclude using Case 1 that $B \sim E$. But also $A \sim E$, so by transitivity, $A \sim B$ as required.

Proposition 7.2 (Line separation)

Let A be a point on a line l. Then the set of points of l not equal to A can be divided into two nonempty subsets S_1, S_2 , the two sides of A on l, such that

- (a) B, C are on the same side of A if and only if A is not in the segment BC;
- (b) *B*, *D* are on opposite sides of *A* if and only if *A* belongs to the segment \overline{BD} .

Proof Given the line l and a point A on l, we know from (I3) that there exists a point E not on l. Let m be the line containing A and E. Apply (7.1) to the line m. If m has two sides S'_1, S'_2 , we define S_1 and S_2 to be the intersections of S'_1 and S'_2 with l. Then properties (a) and (b) follow immediately from the previous proposition.



The only mildly nontrivial part is to show that S_1 and S_2 are nonempty. By (I2), there is a point *B* on *l* different from *A*. And by (B2) there exists a point *D* such that B * A * D. Then *D* will be on the opposite side of *A* from *B*, and will lie on *l*, so both sides are nonempty.

Now that we have some basic results on betweenness, we can define rays and angles.

Definition

Given two distinct points A, B, the ray \overrightarrow{AB} is the set consisting of A, plus all points on the line AB that are on the same side of A as B. The point A is the origin, or vertex, of the ray. An angle is the union of two rays \overrightarrow{AB} and \overrightarrow{AC} originating at the same point, its vertex, and not lying on the same line. (Thus there is no "zero angle," and there is no "straight angle" (180°).) Note that the vertex of a ray or angle is uniquely determined by the ray or angle (proof similar to Exercises 7.2, 7.3).

The inside (or interior) of an angle $\angle BAC$ consists of all points *D* such that *D* and *C* are on the same side of the line *AB*, and *D* and *B* are on the same side of the line *AC*. If *ABC* is a triangle, the *inside* (or *interior*) of the triangle *ABC* is the set of points that are simultaneously in the insides of the three angles $\angle BAC, \angle ABC, \angle ACB$.

Proposition 7.3 (Crossbar theorem)

Let $\angle BAC$ be an angle, and let D be a point in the interior of the angle. Then the ray \overrightarrow{AD} must meet the segment \overrightarrow{BC} .



Proof This is similar to Pasch's axiom (B4), except that we must consider a line AD that passes through one vertex of the triangle ABC. We will prove it with Pasch's axiom and several applications of the plane separation theorem (7.1).

Let us label the lines AB = l, AC = m, AD = n. Let *E* be a point on *m* such that E * A * C (B2). We will apply Pasch's axiom (B4) to the triangle *BCE* and the line *n*. By construction *n* meets the side *CE* at *A*. Also, *n* cannot contain *B*, because it meets the line *l* at *A*. We will show that *n* does not meet the segment \overline{BE} , so as to conclude by (B4) that it must meet the segment \overline{BC} .



So we consider the segment \overline{BE} . This segment meets the line l only at B, so all points of the segment, except B, are on the same side of l. By construction, C is on the opposite side of l from E, so by (7.1) all points of \overline{BE} , except B, are on the opposite side of l from C. On the other hand, since D is in the interior of the angle $\angle BAC$, all the points of the ray \overline{AD} , except A, are on the same side of l as C. Thus the segment \overline{BE} does not meet the ray \overline{AD} .

A similar reasoning using the line *m* shows that all points of the segment \overline{BE} , except *E*, lie on the same side of *m* as *B*, while the points of the ray of *n*, opposite the ray \overrightarrow{AD} , lie on the other side of *m*. Hence the segment \overline{BE} cannot meet the opposite ray to \overrightarrow{AD} . Together with the previous step, this shows that the segment \overline{BE} does not meet the line *n*. We conclude by (B4) that *n* meets the segment \overline{BC} in a point *F*.

It remains only to show that *F* is on the ray \overrightarrow{AD} of the line *n*. Indeed, *B* and *F* are on the same side of *m*, and also *B* and *D* are on the same side of *m*, so (7.1) *D* and *F* are on the same side of *m*, and so *D* and *F* are on the same side of *A* on the line *n*. In other words, *F* lies on the ray \overrightarrow{AD} .

Example 7.3.1

We will show that the real Cartesian plane (6.1.1), with the "usual" notion of betweenness, provides a model for the axioms (B1)-(B4).

First, we must make precise what we mean by the usual notion of betweenness. For three distinct real numbers $a, b, c \in \mathbb{R}$, let us define a * b * c if either a < b < c or c < b < a. Then it is easy to see that this defines a notion of betweenness on the real line \mathbb{R} that satisfies (B1), (B2), and (B3).

If $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ are three points in \mathbb{R}^2 , let us define A * B * C to mean that A, B, C are three distinct points on a line, and that either $a_1 * b_1 * c_1$ or $a_2 * b_2 * c_2$, or both. In fact, if either the *x*- or the *y*coordinates satisfy this betweenness condition, and if the line is neither horizontal nor vertical, then the other coordinates will also satisfy it, because the points lie on a line, and linear operations (addition, multiplication) of real numbers either preserve or reverse inequalities. Thus linear operations preserve betweenness. So we can verify easily that this notion of betweenness in \mathbb{R}^2 satisfies (B1), (B2), and (B3). For (B4), let *l* be a line, and let *A*, *B*, *C* be three noncollinear points not on *l*. The line *l* is defined by some linear equation ax + by + c = 0. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be the linear function defined by $\varphi(x, y) = ax + by + c$. Since φ is a linear function, φ will preserve betweenness. For example, if *l* meets the segment \overline{AB} , then 0 will lie between $\varphi(A)$ and $\varphi(B)$. In other words, one of $\varphi(A), \varphi(B)$ will be positive and the other negative. Suppose $\varphi(A) > 0$ and $\varphi(B) < 0$. Consider $\varphi(C)$. If $\varphi(C) > 0$, then *l* will meet \overline{BC} but not \overline{AC} . If $\varphi(C) < 0$, then *l* will meet \overline{AC} but not \overline{BC} . This proves (B4).

Exercises

- 7.1 Using the axioms of incidence and betweenness and the line separation property, show that sets of four points *A*, *B*, *C*, *D* on a line behave as we expect them to with respect to betweenness. Namely, show that
 - (a) A * B * C and B * C * D imply A * B * D and A * C * D.
 - (b) A * B * D and B * C * D imply A * B * C and A * C * D.
- 7.2 Given a segment \overline{AB} , show that there do not exist points $C, D \in \overline{AB}$ such that C * A * D. Hence show that the endpoints A, B of the segment are uniquely determined by the segment.
- 7.3 Given a triangle *ABC*, show that the sides \overline{AB} , \overline{AC} , and \overline{BC} and the vertices *A*, *B*, *C* are uniquely determined by the triangle. *Hint*: Consider the different ways in which a line can intersect the triangle.
- 7.4 Using (I1)–(I3) and (B1)–(B4) and their consequences, show that every line has infinitely many distinct points.
- 7.5 Show that the line separation property (Proposition 7.2) is not a consequence of (B1), (B2), (B3), by constructing a model of betweenness for the set of points on a line, which satisfies (B1), (B2), (B3) but has only finitely many points. (Then by Exercise 7.4, line separation must fail in this model.) For example, in the ring $\{0, 1, 2, 3, 4\}$ of integers (mod 5), define a * b * c if $b = \frac{1}{2}(a + c)$.
- 7.6 Prove directly from the axioms (11)–(13) and (B1)–(B4) that for any two distinct points *A*, *B*, there exists a point *C* with *A* * *C* * *B*. (*Hint*: Use (B2) and (B4) to construct a line that will be forced to meet the segment AB but does not contain *A* or *B*.)
- 7.7 Be careful not to assume without proof statements that may appear obvious. For example, prove the following:

(a) Let A, B, C be three points on a line with C in between A and B. Then show that $\overline{AC} \cup \overline{CB} = \overline{AB}$ and $\overline{AC} \cap \overline{CB} = \{C\}$.







- 7.9 Show that the interior of a triangle is nonempty.
- 7.10 Suppose that a line *l* contains a point *D* that is in the inside of a triangle *ABC*. Then show that the line *l* must meet (at least) one of the sides of the triangle.



- 7.11 A set *U* of points in the plane is a *convex set* if whenever *A*, *B* are distinct points in *U*, then the segment \overline{AB} is entirely contained in *U*. Show that the inside of a triangle is a convex set.
- 7.12 A subset *W* of the plane is *segment-connected* if given any two points $A, B \in W$, there is a finite sequence of points $A = A_1, A_2, \ldots, A_n = B$ such that for each $i = 1, 2, \ldots, n-1$, the segment $\overline{A_i A_{i+1}}$ is entirely contained within *W*.

If *ABC* is a triangle, show that the *exterior* of the triangle, that is, the set of all points of the plane lying neither on the triangle nor in its interior, is a segment-connected set.

7.13 Let A, B, C, D be four points, no three collinear, and assume that the segments \overline{AB} , \overline{BC} , \overline{CD} , \overline{DA} have no intersections except at their endpoints. Then the union of these four segments is a *simple closed quadrilateral*. The segments \overline{AC} and \overline{BD} are the *diagonals* of the quadrilateral. There are two cases to consider.

Case 1 \overline{AC} and \overline{BD} meet at a point *M*. In this case, show that for each pair of consecutive vertices (e.g., *A*, *B*), the remaining two vertices (*C*, *D*) are on the same side of the line *AB*. Define the *interior* of the quadrilateral to be the set of points *X* such that for each side (e.g., \overline{AB}), *X* is on the same side of the line *AB* as the remaining vertices (*C*, *D*). Show that the interior is a convex set.



Case 2 \overline{AC} and \overline{BD} do not meet. In this case, show that one of the diagonals (\overline{AC} in the picture) has the property that the other two vertices B, D are on the same side of the line AC, while the other diagonal \overline{BD} has the property that A and C are on the opposite sides of the line BD. Define the *interior* of the quadrilateral to be the union of the interiors of the triangles ABD and CDB plus the interior of the segment \overline{BD} . Show in this case that the interior is a segment-connected set, but is not convex.



(For a generalization to *n*-sided figures, see Exercise 22.11.)

- 7.14 (*Linear ordering*) Given a finite set of distinct points on a line, it is possible to label them A_1, A_2, \ldots, A_n in such a way that $A_i * A_j * A_k$ if and only if either i < j < k or k < j < i.
- 7.15 Suppose that lines a, b, c through the vertices A, B, C of a triangle meet at three points inside the triangle. Label them

$$X = a \cdot c,$$

$$Y = a \cdot b,$$

$$Z = b \cdot c.$$

Show that one of the two following arrangements must occur:

(i) A * X * Y and B * Y * Z and C * Z * X (shown in diagram), or

(ii) A * Y * X and B * Z * Y and C * X * Z.



8 Axioms of Congruence for Line Segments

To the earlier undefined notions of point, line, and betweenness, and to the earlier axioms (I1)–(I3), (B1)–(B4), we now add an undefined notion of congruence for line segments, and further axioms (C1)–(C3) regarding this notion. This congruence is what Euclid called equality of segments. We postulate an undefined notion of *congruence*, which is a relation between two line segments \overline{AB}

and \overline{CD} , written $AB \cong CD$. For simplicity we will drop the bars over AB in the notation for a line segment, so long as no confusion can result. This undefined notion is subject to the following three axioms

C1. Given a line segment *AB*, and given a ray *r* originating at a point *C*, there exists a unique point *D* on the ray *r* such that $AB \cong CD$.



C2. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

C3. (Addition). Given three points *A*, *B*, *C* on a line satisfying A * B * C, and three further points D, E, F on a line satisfying D * E * F, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.



Let us observe how these axioms are similar to Euclid's postulates and how they are different. First of all, while Euclid phrases some of his postulates in terms of constructions ("to draw a line through any two given points," and "to draw a circle with any given center and radius"), Hilbert's axioms are existential. (I1) says for any two distinct points there exists a unique line containing them. And here, in axiom (C1), it is the existence of the point D (corresponding to Euclid's construction (I.3)) that is taken as an axiom. Hilbert does not make use of ruler and compass constructions. In their place he puts the axiom (C1) of the existence of line segments and later (C4) the existence of angles. If you like, you can think of (C1) and (C4) as being tools, a "transporter of segments" and a "transporter of angles," and consider some of Hilbert's theorems as constructions with these tools.

The second congruence axiom (C2) corresponds to Euclid's common notion that "things equal to the same thing are equal to each other." This is one part of the modern notion of an equivalence relation, so to be comfortable in using congruence, let us show that it is indeed an equivalence relation.

Proposition 8.1

Congruence is an equivalence relation on the set of line segments.

Proof To be an equivalence relation, congruence must satisfy three properties.

(1) *Reflexivity*: Every segment is congruent to itself. This is explicitly stated in (C2). And by the way, this corresponds to Euclid's fourth common notion that "things which coincide with each other are equal to each other."

$EYKAEI \Delta OY \sum_{XEION \Pi POTON} \Sigma TOI-$

EVCLIDIS ELEMENTORVM GEOmetricorum liber primus.



St hic liber primus totus ferè elementarius, nontantum ad reliquos frquentes huius Operis libros, fed etiam ad aliorum Geometrarum fcripta intelligenda necellarius, Nam in hoc libro communium uocabulorum, que fubinde in geometria uerfanti occurrunt, definitiones continentur. Preceptiones deinde ducendi perpendicula-

rem, quomodo item Trilatere figure, fecundum latera uel angulos di uerfæ, & Quadrilateræ, formari debeant. Figura item aliqua propofita, quomodo illa in alterius formæ figuram permutanda fit, præceptiones, ut diximus, traduntur. Cum igitur talia doceantur, & plura etiam alia, quàm hoc loco commemorare uoluimus, facile erit cuiuís, non folum quàm fit necellarius, fed etiam ad reliqua perdifcenda liber iste quàm utilís, perspicere.

DEFINITIONES.

Punctum est, cuius pars nulla. 2. Linea uerd, longitudo latitudinis expers. Lineg autem termini puncta. 3. Recta linea est, que equabiliter inter sua puncta iacet.



Επιφάνατα ές μύχο και πλαίτο μόνορ έχη. Επιφανείας σε πε γαπα, γραμμαί. Επίπεσος επιφαίνως ες μ. Επις εξίου Ταϊς εφ' εαυς? ευθείαις κειται.



4. Superficies est, que longitudinem & latitudinem tantum habet. K 3 Super-

Plate IV. The beginning of the *Elements* in the edition of Scheubel (1550), showing the Greek text of the first few definitions.

(2) Symmetry: If $AB \cong CD$, then $CD \cong AB$. This is a consequence of (C2): Given $AB \cong CD$, and writing $AB \cong AB$ by reflexivity, we conclude from (C2) that $CD \cong AB$.

(3) *Transitivity*: If $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$. This follows by first using symmetry to show $CD \cong AB$, and then applying (C2). Notice that Hilbert's formulation of (C2) was a clever way of including symmetry and transitivity in a single statement.

The third axiom (C3) is the counterpart of Euclid's second common notion, that "equals added to equals are equal." Let us amplify this by making a precise definition of the sum of two segments, and then showing that sums of congruent segments are congruent.

Definition

Let *AB* and *CD* be two given segments. Choose an ordering *A*, *B* of the endpoints of *AB*. Let *r* be the ray on the line l = AB consisting of *B* and all the points of *l* on the other side of *B* from *A*. Let *E* be the unique point on the ray *r* (whose existence is given by (C1)) such that $CD \cong BE$.



We then define the segment *AE* to be the *sum* of the segments *AB* and *CD*, depending on the order *A*, *B*, and we will write AE = AB + CD.

Proposition 8.2 (Congruence of sums)

Suppose we are given segments $AB \cong A'B'$ and $CD \cong C'D'$. Then $AB + CD \cong A'B' + C'D'$.

Proof Let E' be the point on the line A'B' defining the sum A'E' = A'B' + C'D'. Then A * B * E by construction of the sum AB + CD, because E is on the ray from B opposite A. Similarly, A' * B' * E'. We have $AB \cong A'B'$ by hypothesis. Furthermore, we have $CD \cong C'D'$ by hypothesis, and $CD \cong BE$ and $C'D' \cong$ B'E' by construction of E and E'. From (8.1) we know that congruence is an equivalence relation, so $BE \cong B'E'$. Now by (C3) it follows that $AE \cong A'E'$ as required.

Note: Since the segment AB is equal to the segment BA, it follows in particular that the sum of two segments is independent of the order A, B chosen, up to congruence. Thus addition is well-defined on congruence equivalence classes of line segments. So we can speak of addition of line segments or congruent segments without any danger (cf. also Exercise 8.1, which shows that addition of line segments is associative and commutative, up to congruence). Later (Section

19) we will also define multiplication of segments and so create a field of segment arithmetic.

Euclid's third common notion is that "equals subtracted from equals are equal." Bearing in mind that subtraction does not always make sense, we can interpret this common notion as follows.

Proposition 8.3

Given three points A, B, C on a line such that A * B * C, and given points E, F on a ray originating from a point D, suppose that $AB \cong DE$ and $AC \cong DF$. Then E will be between D and F, and $BC \cong EF$. (We regard BC as the difference of AC and AB.)



Proof Let F' be the unique point on the ray originating at E, opposite to D, such that $BC \cong EF'$. Then from $AB \cong DE$ and $BC \cong EF'$ we conclude by (C3) that $AC \cong DF'$. But F and F' are on the same ray from D (check!) and also $AC \cong DF$, so by (C2) and the uniqueness part of (C1), we conclude that F = F'. It follows that D * E * F and $BC \cong EF$, as required.

Note the role played by the uniqueness part of (C1) in the above proof. We can regard this uniqueness as corresponding to Euclid's fifth common notion, "the whole is greater than the part." Indeed, this statement could be interpreted as meaning, if A * B * C, then *AB* cannot be congruent to *AC*. And indeed, this follows from (C1), because *B* and *C* are on the same ray from *A*, and if $AB \cong AC$, then *B* and *C* would have to be equal by (C1).

So we see that Euclid's common notions, at least in the case of congruence of line segments, can be deduced as consequences of the new axioms (C1)-(C3). Another notion used by Euclid without definition is the notion of inequality of line segments. Let us see how we can define the notions of greater and lesser also using our axioms.

Definition

Let *AB* and *CD* be given line segments. We will say that *AB* is *less than CD*, written AB < CD, if there exists a point *E* in between *C* and *D* such that $AB \cong$ *CE*. In this case we say also that *CD* is *greater than AB*, written *CD* > *AB*.



In the next proposition, we will see that this notion of less than is compatible with congruence, and gives an order relation on congruence equivalence classes of line segments.

Proposition 8.4

(a) Given line segments $AB \cong A'B'$ and $CD \cong C'D'$, then AB < CD if and only if A'B' < C'D'.

(b) The relation < gives an order relation on line segments up to congruence, in the following sense:

- (i) If AB < CD, and CD < EF, then AB < EF.
- (ii) Given two line segments AB, CD, one and only one of the three following conditions holds: AB < CD, AB ≅ CD, AB > CD.

Proof (a) Given $AB \cong A'B'$ and $CD \cong C'D'$, suppose that AB < CD. Then there is a point E such that $AB \cong CE$ and C * E * D. Let E' be the unique point on the ray $\overline{C'D'}$ such that $CE \cong$ C'E'. It follows from (8.3) that C' * E' *D'. Furthermore, by transitivity of congruence, $A'B' \cong C'E'$, so A'B' < C'D' as required. The "if and only if" statement follows by applying the same argument starting with A'B' < C'D'.

(b) (i) Suppose we are given AB < CD and CD < EF. Then by definition, there is a point $X \in CD$ such that $AB \cong CX$, and there is a point $Y \in EF$ such that $CD \cong EY$. Let $Z \in \overrightarrow{EF}$ be such that $CX \cong EZ$. Then by (8.3) we have E * Z * Y. It follows that E * Z * F (Exercise 7.1) and that $AB \cong EZ$. Hence AB < EF as required.

(ii) Given line segments AB and CD, let E be the unique point on the ray \overrightarrow{CD} for which $AB \cong CE$. Then either D = Eor C * E * D or C * D * E. We cannot have D * C * E because D and E are on the same side of C. These conditions are equivalent to $AB \cong CD$, or AB < CD, or AB > CD, respectively, and one and only one of them must hold.



Example 8.4.1

Let us define congruence for line segments in the real Cartesian plane \mathbb{R}^2 , so that it becomes a model for the axioms (I1)–(I3), (B1)–(B4), and (C1)–(C3) that we have introduced so far. We have already seen how to define lines and betweenness (7.3.1). Given two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$, we define the *distance* d(A, B) by



 $d(A,B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$

This is sometimes called the *Euclidean distance* or the *Euclidean metric* on \mathbb{R}^2 . Note that $d(A, B) \ge 0$, and d(A, B) = 0 only if A = B.

Now we can give an interpretation of the undefined notion of congruence in this model by *defining* $AB \cong CD$ if d(A, B) = d(C, D). Let us verify that the axioms (C1), (C2), (C3) are satisfied.

For (C1), we suppose that we are given a segment *AB*, and let d = d(A, B). We also suppose that we are given a point $C = (c_1, c_2)$ and a ray emanating from *C*. For simplicity we will assume that the ray has slope m > 0and that it is going in the direction of increasing *x*-coordinate (we leave the other cases to the reader). Then any point *D* on this ray has coordinates D = $(c_1 + h, c_2 + mh)$ for some $h \ge 0$. The corresponding distance is

$$d(C,D) = h\sqrt{1+m^2}.$$



To find a point *D* with $AB \cong CD$ is then equivalent to solving the equation (in a variable h > 0)

$$h\sqrt{1+m^2} = d,$$

where *m* and d > 0 are given. Clearly, there is a unique solution $h \in \mathbb{R}$, h > 0, for given *d*, *m*. This proves (C1).

The second axiom (C2) is trivial from the definition of congruence using a distance function.

To prove (C3), it will be sufficient to prove that the distance function is additive for points in a line: If A * B * C, then

$$d(A,B) + d(B,C) = d(A,C).$$

Suppose the line is y = mx + b, and $A = (a_1, a_2)$ is the point with smallest *x*-coordinate.

Then there are h, k > 0 such that

$$B = (a_1 + h, a_2 + mh),$$

$$C = (a_1 + h + k, a_2 + m(h + k))$$



In this case

$$d(A, B) = h\sqrt{1 + m^2},$$

$$d(B, C) = k\sqrt{1 + m^2},$$

$$d(A, C) = (h + k)\sqrt{1 + m^2}$$

so the additivity of the distance function follows.

We will sometimes call this model, the real Cartesian plane with congruence of segments defined by the Euclidean distance function, the *standard model* of our axiom system.

Exercises

The following exercises (unless otherwise specified) take place in a geometry with axioms (I1)-(I3), (B1)-(B4), (C1)-(C3).

8.1 (a) Show that addition of line segments is associative: Given segments AB, CD, EF, and taking A, B in order, then (AB + CD) + EF = AB + (CD + EF). (This means that we obtain the same segment as the sum, not just congruent segments.)

(b) Show that addition of line segments is commutative up to congruence: Given segments AB, CD, then $AB + CD \cong CD + AB$.

- 8.2 Show that "halves of equals are equal" in the following sense: if $AB \cong CD$, and if *E* is a *midpoint* of *AB* in the sense that A * E * B and $AE \cong EB$, and if *F* is a midpoint of *CD*, then $AE \cong CF$. (Note that we have not yet said anything about the existence of a midpoint: That will come later (Section 10).) Conclude that a midpoint of *AB*, if it exists, is unique.
- 8.3 Show that addition preserves inequalities: If AB < CD and if EF is any other segment, then AB + EF < CD + EF.

8.4 Let *r* be a ray originating at a point *A*, and let *s* be a ray originating at a point *B*. Show that there is a 1-to-1 mapping $\varphi: r \to s$ of the set *r* onto the set *s* that preserves congruence and betweenness. In other words, if for any $X \in r$ we let $X' = \varphi(X) \in s$, then for any X, Y, $Z \in r$, $XY \cong X'Y'$, and $X * Y * Z \Leftrightarrow$ X' * Y' * Z'.



- 8.5 Given two distinct points O, A, we define the *circle* with *center* O and *radius* OA to be the set Γ of all points B such that $OA \cong OB$.
 - (a) Show that any line through O meets the circle in exactly two points.
 - (b) Show that a circle contains infinitely many points.

(Warning: It is not obvious from this definition whether the center *O* is uniquely determined by the set of points Γ that form the circle. We will prove that later (Proposition 11.1).)

- 8.6 Consider the *rational Cartesian plane* \mathbf{Q}^2 whose points are ordered pairs of rational numbers, where lines are defined by linear equations with rational coefficients and betweenness and congruence are defined as in the standard model (Examples 7.3.1 and 8.4.1). Verify that (II)–(I3) and (B1)–(B4) are satisfied in this model. Then show that (C2) and (C3) hold in this model, but (C1) fails.
- 8.7 Consider the real Cartesian plane \mathbb{R}^2 , with lines and betweenness as before (Example 7.3.1), but define a different notion of congruence of line segments using the distance function given by the sum of the absolute values:

$$d(A,B) = |a_1 - b_1| + |a_2 - b_2|,$$

where $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Some people call this "taxicab geometry" because it is similar to the distance by taxi from one point to anther in a city where all streets run east–west or north–south. Show that the axioms (C1), (C2), (C3) hold, so that this is another model of the axioms introduced so far. What does the circle with center (0, 0) and radius 1 look like in this model?

8.8 Again consider the real Cartesian plane \mathbb{R}^2 , and define a third notion of congruence for line segments using the sup of absolute values for the distance function:

$$d(A,B) = \sup\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Show that (C1), (C2), (C3) are also satisfied in this model. What does the circle with center (0,0) and radius 1 look like in this case?

8.9 Following our general principles, we say that two models M, M' of our geometry are *isomorphic* if there exists a 1-to-1 mapping $\varphi : M \to M'$ of the set of points of M onto the set of points of M', written $\varphi(A) = A'$, that sends lines into lines, preserves betweenness, i.e., A * B * C in $M \Leftrightarrow A' * B' * C'$ in M', and preserves congruence of line segments, i.e., $AB \cong CD$ in $M \Leftrightarrow A'B' \cong C'D'$ in M'.

Show that the models of Exercise 8.7 and Exercise 8.8 above are isomorphic to each

90 ____2. Hilbert's Axioms

other, but they are not isomorphic to the standard model (Example 8.4.1). Note: To show that the two models of Exercise 8.7 and Exercise 8.8 are isomorphic, you do not need to make the distance functions correspond. It is only the notion of congruence of line segments that must be preserved. To show that two models are not isomorphic, one method is to find some statement that is true in one model but not true in the other model.

8.10 Nothing in our axioms relates the size of a segment on one line to the size of a congruent segment on another line. So we can make a weird model as follows. Take the real Cartesian plane \mathbb{R}^2 with the usual notions of lines and betweenness. Using the Euclidean distance function d(A, B), define a new distance function

 $d'(A,B) = \begin{cases} d(A,B) & \text{if the segment } AB \text{ is either horizontal or vertical,} \\ 2d(A,B) & \text{otherwise.} \end{cases}$

Define congruence of segments $AB \cong CD$ if d'(A, B) = d'(C, D). Show that (C1), (C2), (C3) are all satisfied in this model. What does a circle with center (0, 0) and radius 1 look like?

- 8.11 The triangle inequality is the statement that if A, B, C are three distinct points, then $AC \leq AB + BC$.
 - (a) The triangle inequality always holds for collinear points.

(b) The triangle inequality holds for any three points in the standard model (Example 8.4.1) and also in taxicab geometry (Exercise 8.7).

(c) The triangle inequality does not hold in the model of Exercise 8.10. Thus the triangle inequality is not a consequence of the axioms of incidence, betweenness, and congruence of line segments (C1)-(C3). (However, we will see in Section 10 that the triangle inequality, in the form of Euclid (I.20), is a consequence of the full set of axioms of a Hilbert plane.)

9 Axioms of congruence for Angles

Recall that we have defined an *angle* to be the union of two rays originating at the same point, and not lying on the same line. We postulate an undefined notion of *congruence* for angles, written \cong , that is subject to the following three axioms:



C5. For any three angles α, β, γ , if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

C6. (SAS) Given triangles *ABC* and *DEF*, suppose that $AB \cong DE$ and $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely, $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.



Note that Hilbert takes the existence of an angle congruent to a given one (C4) as an axiom, while Euclid proves this by a ruler and compass construction (I.23). Since Hilbert does not make use of the compass, we may regard this axiom as a tool, the "transporter of angles," that acts as a substitute for the compass.

As with (C2), we can use (C5) to show that congruence is an equivalence relation.

Proposition 9.1

Congruence of angles is an equivalence relation.

Proof The proof is identical to the proof of (8.1), using (C5) in place of (C2).

As in the case of congruence of line segments, we would like to make sense of Euclid's common notions in the context of congruence of angles. This proposition (9.1) is the analogue of the first common notion, that "things equal to the same thing are equal to each other." The second common notion, that "equals added to equals are equal," becomes problematic in the case of angles, because in general we cannot define the sum of two angles.

If $\angle BAC$ is an angle, and if a ray \overrightarrow{AD} lies in the interior of the angle $\angle BAC$, then we will say that the angle $\angle BAC$ is the *sum* of the angles $\angle DAC$ and $\angle BAD$.



However, if we start with the two given angles, there may not be an angle that is their sum in this sense. For one thing, they may add up to a straight line, or "two right angles" as Euclid says, but this is not an angle. Or their sum may be greater than 180°, in which case we get an angle, but the two original angles will not be in the interior of the new angle. So we must be careful how we state results having to do with sums of angles.

Note that we do not have an axiom about congruence of sums of angles analogous to the axiom (C3) about addition of line segments. That is because we

can prove the corresponding result for angles. But in order to do so, we will need (C6).

Hilbert's use of (C6) = (SAS) as an axiom is a recognition of the insufficiency of Euclid's proof of that result (I.4) using the method of superposition. To justify the method of superposition by introducing axioms allowing motion of figures in the plane would be foreign to Euclid's approach to geometry, so it seems prudent to take (C6) as an axiom. However, we will show later (17.5) that the (SAS) axiom is essentially equivalent to the existence of a sufficiently large group of rigid motions of the plane. The axiom (C6) is necessary, since it is independent of the other axioms (Exercise 9.3). This axiom is essentially what tells us that our plane is homogeneous: Geometry is the same at different places in the plane.

Now let us show how to deal with sums of angles and inequalities among angles based on these axioms.

Definition

If $\angle BAC$ is an angle, and if *D* is a point on the line *AC* on the other side of *A* from *C*, then the angles $\angle BAC$ and $\angle BAD$ are supplementary.



Proposition 9.2

If $\angle BAC$ and $\angle BAD$ are supplementary angles, and if $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles, and if $\angle BAC \cong \angle B'A'C'$, then also $\angle BAD \cong \angle B'A'D'$.

Proof Replacing B', C', D' by other points on the same rays, we may assume that $AB \cong A'B'$, $AC \cong A'C'$, and $AD \cong A'D'$. Draw the lines BC, BD, B'C', and B'D'.

First we consider the triangles *ABC* and *A'B'C'*. By hypothesis we have $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$. So by (C6) we conclude that the triangles are congruent. In particular, $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$.

Next we consider the triangles BCDand B'C'D'. Since $AC \cong A'C'$ and $AD \cong$ A'D', and C * A * D and C' * A' * D', we conclude from (C3) that $CD \cong C'D'$. Using $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$ proved above, we can apply (C6) again to see that the triangles BCD and B'C'D'are congruent. In particular, $BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$.



Now we consider the triangles *BDA* and *B'D'A'*. From the previous step we have $BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$. But by hypothesis we have $DA \cong D'A'$. So a third application of (C6) shows that the triangles *BDA* and *B'D'A'* are congruent. In particular, $\angle BAD \cong \angle B'A'D'$, which was to be proved.

Note: We may think of this result as a replacement for (I.13), which says that the angles made by a ray standing on a line are either right angles or are equal to two right angles. We cannot use Euclid's statement directly, because in our terminology, the sum of two right angles is not an angle. However, in applications, Euclid's (I.13) can be replaced by (9.2). So for example, we have the following corollary.

Corollary 9.3

Vertical angles are congruent.

Proof Recall that *vertical* angles are defined by the opposite rays on the same two lines. The vertical angles α and α' are each supplementary to β , and β is congruent to itself, so by the proposition, α and α' are congruent.



Proposition 9.4 (Addition of angles)

Suppose $\angle BAC$ is an angle, and the ray \overline{AD} is in the interior of the angle $\angle BAC$. Suppose $\angle D'A'C' \cong \angle DAC$, and $\angle B'A'D' \cong \angle BAD$, and the rays $\overline{A'B'}$ and $\overline{A'C'}$ are on opposite sides of the line A'D'. Then the rays $\overline{A'B'}$ and $\overline{A'C'}$ form an angle, and $\angle B'A'C' \cong \angle BAC$, and the ray $\overline{A'D'}$ is in the interior of the angle $\angle B'A'C'$. For short, we say "sums of congruent angles are congruent."

Proof Draw the line *BC*. Then the ray \overrightarrow{AD} must meet the segment \overrightarrow{BC} , by the crossbar theorem (7.3). Replacing the original *D* by this intersection point, we may assume that *B*, *D*, *C* lie on a line and B * D * C. On the other hand, replacing B', C', D' by other points on the same rays, we may assume that $AB \cong A'B'$, and $AC \cong A'C'$, and $AD \cong A'D'$. We also have $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$ by hypothesis.

By (C6) we conclude that the triangles ΔBAD and $\Delta B'A'D'$ are congruent. In particular, $BD \cong B'D'$ and $\angle BDA \cong B'D'A'$.



Again by (C6) we conclude that the triangles ΔDAC and $\Delta D'A'C'$ are congruent. In particular, $DC \cong D'C'$ and $\angle ADC \cong \angle A'D'C'$.

Let E' be a point on the line B'D' with B' * D' * E'. Then $\angle A'D'E'$ is supplementary to $\angle A'D'B'$, which is congruent to $\angle ADB$. So by (9.2) and transitivity of congruence, we find that $\angle A'D'E' \cong \angle A'D'C'$. Since these angles are on the same side of the line A'D', we conclude from the uniqueness part of (C4) that they are the same angle. In other words, the three points B', D', and C' lie on a line.

Then from (C3) we conclude that $BC \cong B'C'$. Since $\angle ABD \cong \angle A'B'D'$ by the first congruence of triangles used in the earlier part of the proof, we can apply (C6) once more to the triangles ABC and A'B'C'. The congruence of these triangles implies $\angle BAC \cong \angle B'A'C'$ as required. Since B', D', and C' are collinear and D'A'C' is an angle, it follows that A', B', C' are not collinear, so B'A'C' is an angle. Since B' and C' are on opposite sides of the line A'D', it follows that B' * D' * C', and so the ray $\overrightarrow{A'D'}$ is in the interior of the angle $\angle B'A'C'$, as required.

Next, we will define a notion of inequality for angles analogous to the inequality for line segments in Section 8.

Definition

Suppose we are given angles $\angle BAC$ and $\angle EDF$. We say that $\angle BAC$ is *less than* $\angle EDF$, written $\angle BAC < \angle EDF$, if there exists a ray \overrightarrow{DG} in the interior of the angle $\angle EDF$ such that $\angle BAC \cong \angle GDF$. In this case we will also say that $\angle EDF$ is greater than $\angle BAC$.



Proposition 9.5

- (a) If $\alpha \cong \alpha'$ and $\beta \cong \beta'$, then $\alpha < \beta \Leftrightarrow \alpha' < \beta'$.
- (b) Inequality gives an order relation on angles, up to congruence. In other words:
- (i) If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.
- (ii) For any two angles α and β, one and only one of the following holds: α < β;
 α ≃ β; α > β.

Proof The proofs of these statements are essentially the same as the corresponding statements for line segments (8.4), so we will leave them to the reader.

Definition

A right angle is an angle α that is congruent to one of its supplementary angles β .



Note: In this definition, it does not matter which supplementary angle to α we consider, because the two supplementary angles to α are vertical angles, hence congruent by (9.3). Two lines are *orthogonal* if they meet at a point and one, hence all four, of the angles they make is a right angle.

Proposition 9.6

Any two right angles are congruent to each other.

Proof Suppose that $\alpha = \angle CAB$ and $\alpha' = \angle C'A'B'$ are right angles. Then they will be congruent to their supplementary angles β, β' , by definition. Suppose α and α' are not congruent. Then by (9.5) either $\alpha < \alpha'$ or $\alpha' < \alpha$. Suppose, for example, $\alpha < \alpha'$. Then by definition of inequality there is a ray $\overrightarrow{A'E'}$ in the interior of angle α' such that $\alpha \cong \angle E'A'B'$.

It follows (check!) that the ray $\overline{A'C'}$ is in the interior of $\angle E'A'D'$, so that $\beta' < \angle E'A'D'$. But $\angle E'A'D'$ is supplementary to $\angle E'A'B'$, which is congruent to α , so by (9.2), $\angle E'A'D' \cong \beta$. Therefore, $\beta' < \beta$. But $\alpha \cong \beta$ and $\alpha' \cong \beta'$, so we conclude that $\alpha' < \alpha$, which is a contradiction.



Note: Thus the congruence of all right angles can be proved and does not need to be taken as an axiom as Euclid did (Postulate 4). The idea of this proof already appears in Proclus.

Example 9.6.1

We will show later that the real Cartesian plane \mathbb{R}^2 provides a model of all the axioms listed so far. You are probably willing to believe this, but the precise definition of what we mean by congruence of angles in this model, and the proof that axioms (C4)–(C6) hold, requires some work. We will postpone this work until we make a systematic study of Cartesian planes over arbitrary fields, and then we will show more generally that the Cartesian plane over any ordered field satisfying a certain algebraic condition gives a model of Hilbert's axioms (17.3).

The other most important model of Hilbert's axioms is the non-Euclidean Poincaré model, which we will discuss in Section 39.

Exercises

- 9.1 (Difference of angles). Suppose we are given congruent angles $\angle BAC \cong \angle B'A'C'$. Suppose also that we are given a ray \overrightarrow{AD} in the interior of $\angle BAC$. Then there exists a ray A'D' in the interior of $\angle B'A'C'$ such that $\angle DAC \cong \angle D'A'C'$ and $\angle BAD \cong \angle B'A'D'$. This statement corresponds to Euclid's Common Notion 3: "Equals subtracted from equals are equal," where "equal" in this case means congruence of angles.
- 9.2 Suppose the ray \overrightarrow{AD} is in the interior of the angle $\angle BAC$, and the ray \overrightarrow{AE} is in the interior of the angle $\angle DAC$. Show that \overrightarrow{AE} is also in the interior of $\angle BAC$.



- 9.3 Consider the real Cartesian plane where congruence of line segments is given by the absolute value distance function (Exercise 8.7). Using the usual congruence of angles that you know from analytic geometry (Section 16), show that (C4) and (C5) hold in this model, but that (C6) fails. (Give a counterexample.)
- 9.4 Provide the missing betweenness arguments to complete Euclid's proof of (I.7) in the case he considers. Namely, assuming that the ray \overrightarrow{AD} is in the interior of the angle $\angle CAB$, and assuming that D is outside the triangle ABC, prove that \overrightarrow{CB} is in the interior of the angle $\angle ACD$ and \overrightarrow{DA} is in the interior of the angle $\angle CDB$.



10 Hilbert Planes

We have now introduced the minimum basic notions and axioms on which to found our study of geometry.