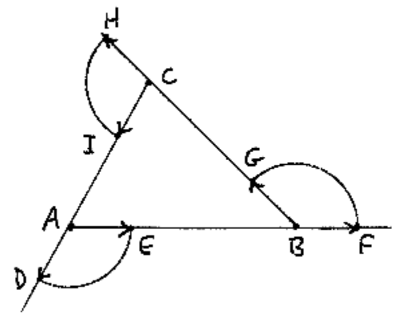
33.13 Discuss the following "proof" that the sum of the angles of a triangle is equal to two right angles, independent of the theory of parallels, due to Thibaut (1775–1832):

Let ABC be the given triangle. Take a segment AD on the line AC, pointing away from C. Rotate it to the position AE on the line AB. Then slide it along the line AB into the position BF. Rotate to BG, slide to CH, rotate to CI, and slide back to AD. In this process, the segment AD has made one complete rotation, which is 4 right angles. But the amount it has rotated is equal to the sum of the exterior angles DAE, FBG, and HCI. Replacing these by their supplementary angles, we find that the sum of the three interior angles of the triangle is equal to two right angles.



33.14 J.J. Callahan, then president of Duquesne University, in his book *Euclid or Einstein* (1931) claims to prove the parallel postulate of Euclid, and thus nullify the theories of Einstein based on non-Euclidean geometry. If you can locate a copy of his book, read his proof and find the flaw in his argument.

34 Neutral Geometry

Sir Henry Savile, in his public lectures on Euclid's *Elements* in Oxford in 1621, said, "In this most beautiful body of Geometry there are two moles, two blemishes, and no more, as far as I know, for whose removal and washing away, both older and more recent authors have shown much diligence." He was referring to the theory of parallels and the theory of proportion. Euclid's theory of proportion has been thoroughly vindicated, and receives its modern expression in the segment arithmetic that we have explained in Chapter 4.

The work on the theory of parallels, however, did not lead to the expected result. Instead of confirming Euclid's as the one true geometry, these researches showed that Euclid's was only one of many possible geometries. The others are what we now call non-Euclidean geometries. The story of this discovery is one of the most fascinating chapters in the history of mathematics, and has been amply told elsewhere. Here we will confine ourselves to the briefest outline.

We can distinguish four periods. The first, which we have elaborated in the previous section, might be called "dissatisfaction with Euclid." While fully accepting Euclid's *Elements* as the true geometry, critics said only that his treatment of this topic could have been better. So they tried to better Euclid, either by proving the parallel postulate, or by replacing it with some other more natural assumption.

The second period, exemplified by the work of Saccheri, Legendre, and Lambert, was based on the attitude, let us suppose the parallel postulate is false and see what conclusions we can draw. In this way they developed a collection of results that would be true if the parallel postulate were false, still expecting ultimately to find a contradiction and thus vindicate Euclid. So strong was the power of tradition that even after meticulously proving a whole series of propositions in this new geometry, each of these authors fell into error and deluded himself into thinking he had found a contradiction.

What a small step of the imagination, with what great consequences, was the transition to the third period! All it required was to think, yes it is possible to have a geometry in which the parallel postulate is false, and these are its first theorems. This step was taken independently by Carl Friedrich Gauss (1777–1855) in Germany, János Bolyai (1802–1860) in Hungary, and Nicolai Ivanovich Lobachevsky (1793–1856) in Russia. Although Gauss was the first to realize the existence of this new geometry, he published nothing of his researches, leaving Bolyai and Lobachevsky each to believe that he was the inventor of this new geometry. Bolyai exclaimed, in a letter to his father, "Out of nothing I have created a strange new universe."

The fourth period contains the confirmation of these new geometries by providing models for the axiom systems to show their consistency. This occurred only later, with the work of Beltrami, Klein, and Poincaré.

In this and the next section we will describe some work of the second period. Then in later sections we will give a model of the non-Euclidean geometry due to Poincaré, and a fuller axiomatic development of the theory, containing the results of Bolyai and Lobachevsky, in a logical framework provided by Hilbert.

A geometry satisfying Hilbert's axioms of incidence, betweenness, and congruence, in which we neither affirm nor deny the parallel axiom (P), will be called a *neutral geometry*. This is the same as a Hilbert plane, but the terminology emphasizes that we do not assume (P). Recall from Section 10 that the results of Euclid, Book I, up through (I.28), with the possible exception of (I.1) and (I.22), also hold in neutral geometry. A Hilbert plane in which (P) does not hold will be called a *non-Euclidean geometry*. We have already seen one example of a non-Euclidean geometry (18.4.3), but that one is semi-Euclidean, in the sense that the angle sum in a triangle is still equal to 2RA (two right angles) (Exercise 18.4). Now we will consider other geometries in which the angle sum of a triangle may be different from 2RA.

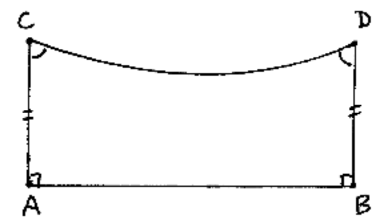
The results of this second period are mainly due to Girolamo Saccheri (1667–1733) and Adrien Marie Legendre (1752–1833). Saccheri's book *Euclides ab omni naevo vindicatus* was published in 1733. The title "Euclid freed of every blemish" recalls the quotation from Savile above. The first 32 propositions are a marvel of mathematical exposition. Unfortunately, after that his previously impeccable rigor lapses, and he says that he has proved the parallel postulate, because if it were false, there would be two lines having a common perpendicular at infinity, which is "repugnant to the nature of a straight line."

Saccheri's work was perhaps before its time, because it did not receive the recognition it deserved, and lay hidden in obscurity until the end of the nineteenth century. Essentially equivalent results were discovered independently half a century later by Legendre, whose book *Eléments de Géométrie* was first published in 1794. It was followed by many new editions, reprints, and translations, which had a wide influence on the teaching of geometry and revitalized interest in the question of parallels.

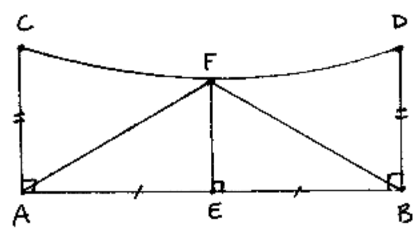
We start with a figure extensively studied by Saccheri, which goes back to Clavius, in his commentary on Euclid's (I.29), where he proposes the axiom that we discussed earlier (Section 33). Since it was Clavius's edition of Euclid that was recommended to Saccheri by the Jesuit mathematician Tommaso Ceva, we may assume that Saccheri was inspired by Clavius to study this figure further.

Proposition 34.1

In a Hilbert plane, suppose that two equal perpendiculars AC, BD stand at the ends of an interval AB, and we join CD. (This is called a Saccheri quadrilateral.) *Then the angles at C and D are equal, and furthermore, the line joining the midpoints of AB and CD, the* midline, *is perpendicular to both.*



Proof Given $ABCD$ as above, let E be the midpoint of AB and let l be the perpendicular to AB at E. Since l is the perpendicular bisector of AB, the points A, C lie on one side of l, while B, D lie on the other side. Hence l meets the segment CD in a point F. By (SAS) the triangles AEF and BEF are congruent. Hence the angles $\angle FAE$ and $\angle FBE$ are equal, and $AF = FB$.

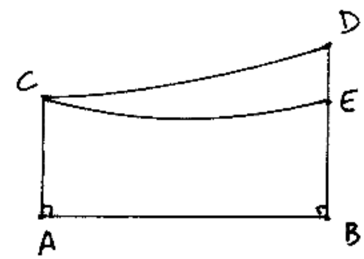


By subtraction from the right angles at A and B we find that the angles $\angle CAF$ and $\angle DBF$ are equal. So by (SAS) again, the triangles CAF and DBF are congruent. This shows that the angles at C and D are equal, and that F is the midpoint of CD.

The two pairs of congruent triangles also imply that the angles $\angle CFE$ and $\angle DFE$ are equal. So by definition, both of these angles are right angles.

Remark 34.1.1
From the equality of the angles at C and D, Saccheri distinguished three cases, which he called the hypothesis of the acute angle, the hypothesis of the right angle, and the hypothesis of the obtuse angle, according to whether C and D were acute, right, or obtuse. He showed that if any one of these holds for one such quadrilateral, it holds for all. His proofs used continuity (in the form of the intermediate value theorem), but we will show in the following propositions that his result is also true in an arbitrary Hilbert plane.

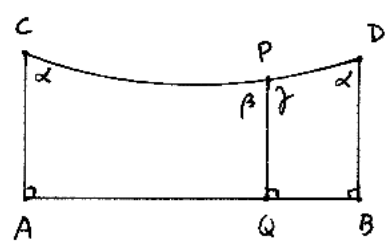
Proposition 34.2
Let $ABCD$ be a quadrilateral with right angles at A and B, and unequal sides AC, BD. Then the angle at C is greater than the angle at D if and only if $AC < BD$.



Proof Suppose $AC < BD$, and choose E on BD such that $AC = BE$. Then $ABCE$ is a Saccheri quadrilateral and $\angle ACE = \angle BEC$, by the previous proposition. Now, the angle $\angle ACD$ is bigger than $\angle ACE$, and $\angle BEC$ is bigger than the angle at D by the exterior angle theorem (I.16), so we find that the angle at C is bigger than the angle at D, as required.

On the other hand, if $AC > BD$, the same argument with roles reversed shows that the angle at C is less than the angle at D. Hence we obtain the "if and only if" conclusion of the proposition.

Proposition 34.3
Let $ABCD$ be a Saccheri quadrilateral, let P be a point on the segment CD, and let PQ be the perpendicular to AB. Let α be the angle at C (equal to the angle at D).



(a) *If $PQ < BD$, then α is acute.*
(b) *If $PQ = BD$, then α is right.*
(c) *If $PQ > BD$, then α is obtuse.*

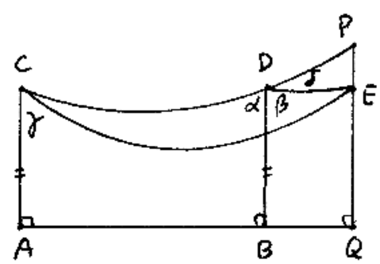
Proof Let β, γ be the two angles at P. In case (a), if $PQ < BD$, then $PQ < AC$ also, and from the previous proposition we obtain $\alpha < \beta$ and $\alpha < \gamma$. Hence $2\alpha < \beta + \gamma = 2\text{RA}$. Thus α is acute. The proofs of cases (b), (c) are analogous.

Remark 34.3.1
Once we have proved all three cases (a), (b), and (c), it follows that each one is an equivalence, not only an implication.

Proposition 34.4
Again let $ABCD$ be a Saccheri quadrilateral, but this time let P be a point on the line CD outside the interval CD. Let PQ be the perpendicular to the line AB, and let α be the angle at C (equal to the angle at D).

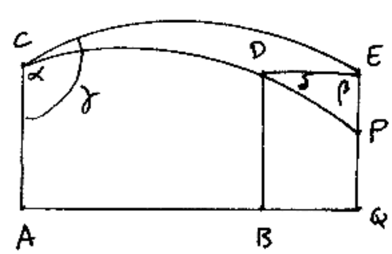
(a) *If $PQ > BD$, then α is acute.*
(b) *If $PQ = BD$, then α is right.*
(c) *If $PQ < BD$, then α is obtuse.*

Proof In case (a), assuming $PQ > BD$, choose E in PQ such that $BD = QE$. Draw CE and DE. Then we have three Saccheri quadrilaterals. We will compare their angles. Let α, β, γ be the top angles of the quadrilaterals $ABCD$, $BQDE$, $AQCE$, respectively. Let $\delta = \angle EDP$. Then δ is an exterior angle of the triangle CDE, so by (I.16), $\delta > \angle DCE = \alpha - \gamma$. On the other hand, looking at the angles at E, we see that $\beta > \gamma$. Now, $2\text{RA} = \alpha + \beta + \delta > \alpha + \gamma + \alpha - \gamma = 2\alpha$, so α is acute.



For case (b), when $PQ = BD$, then $AQCP$ is a Saccheri quadrilateral, so by (34.3b) its angle, which is equal to the angle of $ABCD$, is right.

In case (c), when $PQ < BD$, the proof is similar. Extend PQ to E with $BD = QE$ and join CE, DE. This gives three Saccheri quadrilaterals, with upper angles α, β, γ as marked. Let $\delta = \angle PDE$. Then by the exterior angle theorem (I.16), $\delta > \angle DCE = \gamma - \alpha$. Looking at E we see that $\gamma > \beta$. On the other hand, looking at D we see that $\alpha + \beta - \delta = 2\text{RA}$. So, combining these results, we obtain



$$2\text{RA} = \alpha + \beta - \delta < \alpha + \gamma - \delta < 2\alpha.$$

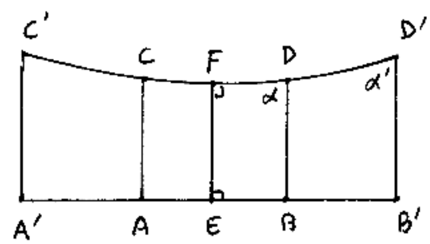
Hence α is obtuse, as required.

Remark 34.4.1
As in the previous proposition, once we have proved all three cases, they each become equivalences, not just implications.

Theorem 34.5 (Saccheri)
In any Hilbert plane, if one Saccheri quadrilateral has acute angles, so do all Saccheri quadrilaterals. If one has right angles, so do they all. If one has obtuse angles, so do they all.

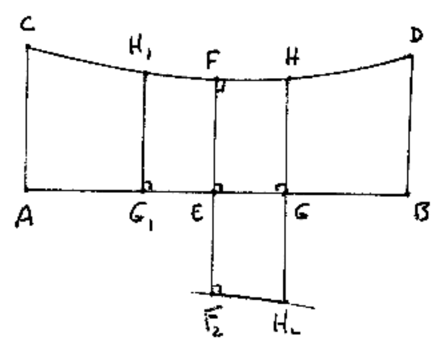
Proof We will give the proof only in the acute case, since the proofs in the two other cases are identical.

Suppose $ABCD$ is a Saccheri quadrilateral with acute angles, and let EF be its midline (34.1). If $A'B'C'D'$ is another Saccheri quadrilateral with midline equal to EF, then it can be moved by a rigid motion to make the midlines coincide. Suppose $AB < A'B'$. We obtain a figure as shown, with α acute. Hence, by (34.4), $BD < B'D'$. Then by (34.3), α' is acute. If $AB > A'B'$, we run the same argument in the reverse order. It follows that all Saccheri quadrilaterals with midline equal to EF have acute angles.



Next we show that for any other segment, there exists a Saccheri quadrilateral with acute angles and midline equal to that segment.

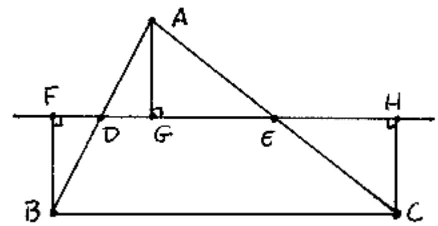
Lay off the given segment as EG on the ray EB. Let the perpendicular to AB at G meet CD in H. Reflect G and H in EF to get G_1, H_1. Reflect F and H in AB to get F_2, H_2. Now, G_1GH_1H is a Saccheri quadrilateral with midline EF, so by the previous argument, its angle β is acute. But then FF_2HH_2 is another Saccheri quadrilateral with the same acute angle β and midline EG. Now by the earlier argument, every other Saccheri quadrilateral with midline equal to EG has acute angles. But EG was arbitrary, so the theorem is proved.



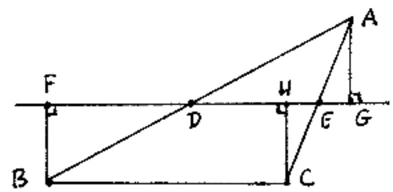
Next we will show how to interpret this result on Saccheri quadrilaterals in terms of the sum of the angles in a triangle.

Proposition 34.6
Given a triangle ABC, there is a Saccheri quadrilateral for which the sum of its two top angles is equal to the sum of the three angles of the triangle.

Proof Let ABC be the given triangle. Let D and E be the midpoints of AB and AC, and draw the line DE, which we call the *midline* of the triangle. Drop perpendiculars BF, AG, CH to DE.



Now, $AD = DB$, and the vertical angles at D are equal, so by (AAS) the triangles ADG and BDF are congruent. Similarly, $AE = EC$ and the vertical angles at E are congruent, so the triangles AEG and CEH are congruent. From congruent triangles we obtain $BF = AG = CH$. The quadrilateral $FHBC$ has right angles at F and H, so it is a Saccheri quadrilateral (upside down). The angles of the quadrilateral at B and C are composed of the angles of the triangle at B and C, plus angles that are congruent to the two parts of the angle at A, divided by the line AG. Hence the angles at B and C of the quadrilateral equal the angle sum of the triangle. It follows that the triangle and the quadrilateral have equal defect.



If G happens to fall outside the interval FH, the same argument works, except that we use differences instead of sums of angles.

Theorem 34.7
In any Hilbert plane:

(a) *If there exists a triangle whose angle sum is less than* 2RA, *then every triangle has angle sum less than* 2RA.
(b) *The following conditions are equivalent:*
   (i) *There exists a triangle with angle sum* = 2RA.
   (ii) *There exists a rectangle.*
   (iii) *Every triangle has angle sum* = 2RA.
(c) *If there exists a triangle whose angle sum is greater than* 2RA, *then every triangle has angle sum greater than* 2RA.

Proof (a) If there exists a triangle with angle sum less than 2RA, then the associated Saccheri quadrilateral of (34.6) must have acute angles. By (34.5) it follows that every Saccheri quadrilateral has acute angles, and then by (34.6) again, every triangle must have angle sum less than 2RA.

The proof of (b) is the same, where we note that a rectangle is just the same thing as a Saccheri quadrilateral with right angles. The proof of (c) is the same as the proof of (a).

Definition

In case (a) of the theorem, we say that the geometry is *semihyperbolic*. In case (b) we say that it is *semi-Euclidean*, and in case (c) we say that it is *semielliptic*.

Remark 34.7.1

Note that these three cases are equivalent to what Saccheri called the hypothesis of the acute angle, the hypothesis of the right angle, and the hypothesis of the obtuse angle. Thus all Hilbert planes can be divided into these three classes. Of course, a Euclidean plane, or more generally any Hilbert plane satisfying (P), is semi-Euclidean, by (I.32). On the other hand, we have seen an example of a semi-Euclidean plane that does not satisfy (P) in Exercise 18.4.

We reserve the term *hyperbolic* for geometries satisfying Hilbert's hyperbolic axiom (cf. Section 40). Those geometries will be semihyperbolic, but there are also semihyperbolic geometries that are not hyperbolic (Exercise 39.24).

As for the semielliptic case, these were first discovered in 1900 by Dehn, who called them non-Legendrean. The term elliptic is usually applied to geometries like a projective plane in which there are no parallel lines at all. These do not satisfy Hilbert's axioms, so fall outside our realm of inquiry. However, a suitably small patch of a spherical geometry over a non-Archimedean field gives an example of a semielliptic Hilbert plane (Exercise 34.14).

Definition

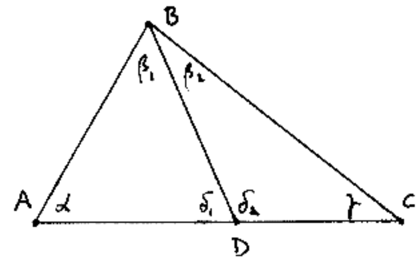
We say that a triangle is *Euclidean* if the sum of its angles is equal to 2RA. Otherwise, we call it *non-Euclidean*. To measure the divergence of a triangle from the Euclidean case, we define the *defect* of any triangle to be 2RA$-$(sum of angles in the triangle). Thus $\delta = 0$ for a Euclidean triangle, δ is a positive angle for a triangle in a semihyperbolic plane, and δ is the negative of an angle for a triangle in a semielliptic plane.

Lemma 34.8

If a triangle ABC is cut into two triangles by a single transversal BD, the defect of the big triangle is equal to the sum of the defects of the two small triangles:

$$\delta(ABC) = \delta(ABD) + \delta(BCD).$$

Proof Label the angles as shown in the diagram. Then



$$\delta(ABD) = 2\text{RA} - \alpha - \beta_1 - \delta_1,$$

$$\delta(BCD) = 2\text{RA} - \beta_2 - \delta_2 - \gamma.$$

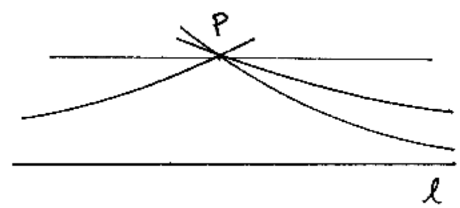
Since $\delta_1 + \delta_2 = 2\text{RA}$, by adding we obtain

$$\delta(ABD) + \delta(BCD)$$
$$= 2\text{RA} - \alpha - \beta_1 - \beta_2 - \gamma = \delta(ABC),$$

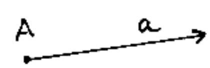
as required.

The Theory of Parallels in Neutral Geometry

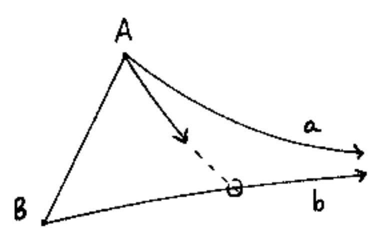
Given a line l and a point P not on l, we know from (I.31) that there exists a line through P parallel to l. If the Hilbert plane satisfies Playfair's axiom (P), that parallel is unique. But in the non-Euclidean case, there may be more than one parallel to l through P. Among all these parallels, there may be one that is closer to l than all the others on one side. To make a formal definition, it matters which end of the line we look at, so we will phrase it in terms of rays.



We denote a ray by the symbol Aa, where A is its endpoint, and a denotes the line carrying the ray, together with a choice of one of the two directions on the line. Two rays are *coterminal* if they lie on the same line and "go in the same direction." This can be made precise by saying that one ray is a subset of the other. Thus if Aa is a ray and A' is another point on the line carrying a, we denote by $A'a$ the corresponding coterminal ray.



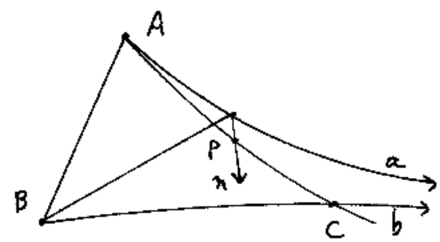
Definition
A ray Aa is *limiting parallel* to a ray Bb if either they are coterminal, or if they lie on distinct lines not equal to the line AB, they do not meet, and every ray in the interior of the angle BAa meets the ray Bb. In symbols we write $Aa \,|||\, Bb$.



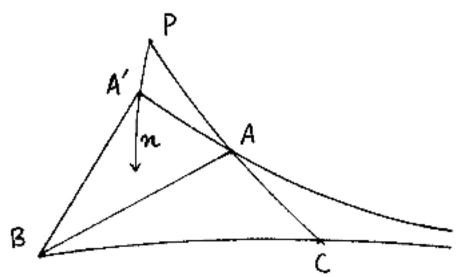
It requires some work, in the following propositions, to show that this notion is an equivalence relation. Note that we say nothing about the existence of such limiting parallels. All the following results should be understood in the sense that they hold whenever the limiting parallels exist. Later, in Section 40, we will introduce the hyperbolic axiom, which postulates the existence of limiting parallels from any point to any given ray.

Proposition 34.9
If $Aa \mathbin{|||} Bb$, and if $A'a$, $B'b$ are rays coterminal to Aa, Bb respectively, then $A'a \mathbin{|||} B'b$.

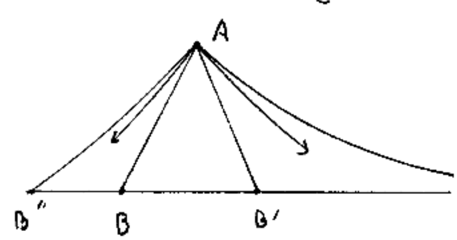
Proof It is sufficient to replace one ray at a time by a coterminal ray. So first, suppose that A' is on the ray Aa. We must show that every ray n in the interior of the angle $BA'a$ meets the ray Bb. Take a point P on the ray n, different from A'. Then the ray $\overrightarrow{AP}$ lies in the interior of the angle BAa, so by hypothesis it meets the ray Bb in a point C. Now, the ray n cuts one side of the triangle ABC, so by Pasch's axiom (B4) it must cut another. The side AB is impossible, so n meets BC, which is contained in the ray Bb, as required.



Next, suppose A' is on the line a, but not in the ray Aa. Let $A'n$ be a ray in the angle $BA'a$, and take a point P on the line n, but not in the ray $A'n$. Then the ray $\overrightarrow{PA}$, after it passes through A, is in the interior of the angle BAa, so meets Bb in a point C. By the crossbar theorem (7.3) $A'n$ will meet AB, and then by Pasch's axiom it will meet BC.



If we replace B by a point B' in the ray Bb, or by a point B'' on the line b outside the ray Bb, the proof is easier. Any ray from A in the interior of the appropriate angle must meet the ray $B'b$ or $B''b$ either by the crossbar theorem or by the property $Aa \mathbin{|||} Bb$.

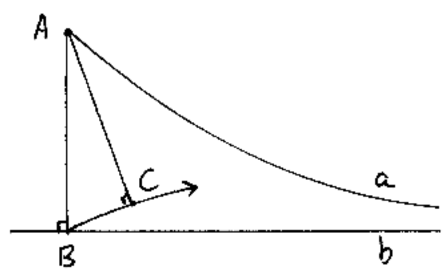


In this proof we passed over in silence a small point, namely to show that after replacing Aa, Bb by coterminal rays $A'a, B'b$, we still have satisfied the condition that the rays $A'a$ and $B'b$ do not meet, and they lie on lines not equal to $A'B'$. For this it is sufficient to show that if $Aa \parallel\!| Bb$, then the lines supporting those rays do not meet. We leave this as Exercise 34.6.

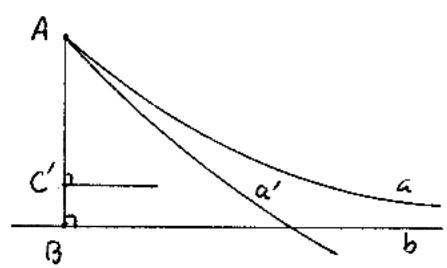
Proposition 34.10
If a ray Aa is limiting parallel to another ray Bb, then also Bb is limiting parallel to Aa.

Proof If the rays are coterminal, this is trivial, so we may assume that a and b are distinct lines. Drop a perpendicular AB' to the line b. Then by the previous proposition, Aa is limiting parallel to $B'b$, and it will be sufficient to prove $B'b$ limiting parallel to Aa. In other words, changing notation, we may assume that the angle ABb is a right angle.

We must show that any ray Bn in the interior of the angle at B meets the ray Aa. Suppose it does not. Drop the perpendicular AC from A to n. Since the angle ABn is acute, by the exterior angle theorem, C must lie on the ray Bn, not on the other side of B. In the triangle ABC, the angle at C is right, while the other two angles are acute. Hence by (I.19), $AC < AB$. (Why is the angle at A acute? Because it is less than the angle BAa, and this angle must be less than or equal to RA. Otherwise, the perpendicular to BA at A would lie inside the angle BAa and be parallel to Bb, contradicting our hypothesis.)



Rotate C, n, and a around the point A until C lands on a point C' of AB, and n', a' are the images of n, a. Then Aa' will meet Bb, and n' will be parallel to b, so by Pasch's axiom, it will meet a'. Rotating back, we find that n meets Aa, a contradiction.



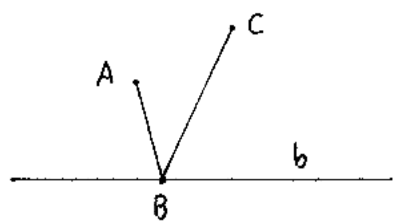
Proposition 34.11
Given three rays Aa, Bb, Cc, if $Aa \parallel\!| Bb$ and $Bb \parallel\!| Cc$, then $Aa \parallel\!| Cc$.

Proof If any two are coterminal, the result follows from the previous propositions, so we may assume that they lie on distinct lines.

Lemma 34.12
Given three rays Aa, Bb, Cc lying on distinct lines, with $Aa \parallel\!| Bb$ and $Bb \parallel\!| Cc$, after replacing one by a coterminal ray if necessary, we may assume that A, B, and C are collinear.

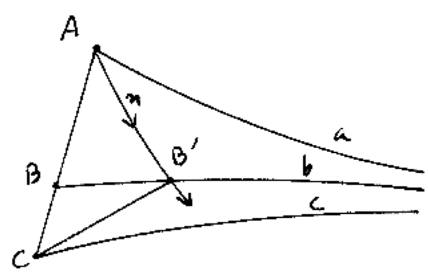
Proof If A, C lie on opposite sides of the line b, then the segment AC meets the line b in a point B'. Replacing Bb by the coterminal ray $B'b$, we have A, B', C collinear.

If A, C lie on the same side of the line b, we consider the angles ABb and CBb. If these angles are equal, then A, B, C are collinear. If they are not equal, one must be smaller, say CBb is smaller. Then the ray $\overrightarrow{BC}$ is in the interior of the angle ABb, and $Bb \parallel\!| Aa$ by (34.10), so the ray $\overrightarrow{BC}$ meets Aa in a point A'. Replacing Aa by $A'a$ we have A', B, C collinear. If ABb is smaller, the same argument works replacing C by a point C'.

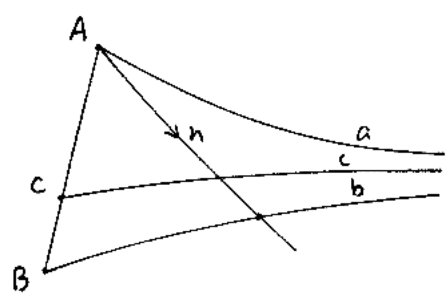


Proof of (34.11), continued By the lemma, we may assume A, B, C collinear. It follows immediately from the hypotheses that the rays Aa, Bb, Cc are all on the same side of the line ABC.

Case 1 If B is between A and C, take any ray An in the interior of the angle CAa. Since $Aa \parallel\!| Bb$, this ray meets Bb in a point B'. Then $B'b \parallel\!| Cc$ by (34.9), so the continuation of that ray will meet Cc. Hence $Aa \parallel\!| Cc$.



Case 2 C is between A and B. In this case a ray An in the interior of the angle CAa meets b in a point B'. Then Cc must meet n by Pasch's axiom.



Case 3 A is between C and B. The proof is the same, taking into account $Cc \parallel\!| Bb$ by (34.10).

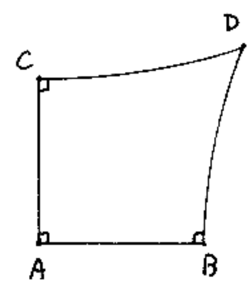
Remark 34.12.1
The proof of Case 2 of (34.11) actually shows a stronger result: If $Aa \parallel\!| Bb$, if C is between A and B, and if Cc is any ray entirely in the interior of the angles BAa and ABb, then Cc is also limiting parallel to Aa and Bb.

Corollary 34.13
The relation "limiting parallel" for rays is an equivalence relation, which includes the equivalence relation of being coterminal. We define an end *to be an equivalence class of limiting parallel rays.*

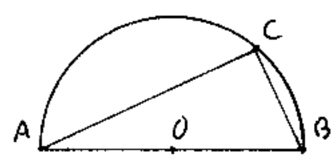
Exercises

34.1 If $ABCD$ is a Saccheri quadrilateral, show that $CD > AB$ if and only if the angles at C, D are acute.

34.2 Define a *Lambert quadrilateral* to be a quadrilateral $ABCD$ with right angles at A, B, C. Show that the fourth angle D is acute, right, or obtuse according as the geometry is semihyperbolic, semi-Euclidean, or semielliptic.



34.3 Let AB be the diameter of a circle, and let ABC be a triangle inscribed in the semicircle. Show that the angle at C is acute, right, or obtuse, according as the geometry is semihyperbolic, semi-Euclidean, or semielliptic.



34.4 In a semihyperbolic or a semielliptic plane, prove the (AAA) congruence theorem for triangles: If two triangles ABC and $A'B'C'$have $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, then the two triangles are congruent.

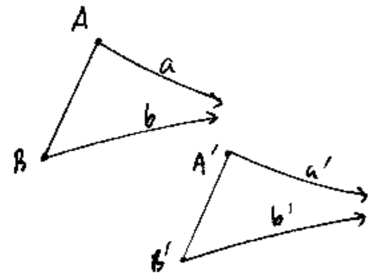
34.5 In a semihyperbolic or a semielliptic plane, show that for any line l and any point A not on l, there are infinitely many lines through A parallel to l. (*Hint*: Use Saccheri quadrilaterals.)

34.6 In Aa and Bb are limiting parallel rays lying on distinct lines, show directly from the definition that the lines carrying these rays do not meet.

34.7 In a Hilbert plane satisfying Dedekind's axiom (D), show that for any point A and any ray Bb, there exists a ray Aa from A, limiting parallel to Bb.

34.8 In the Hilbert plane of (18.4.3) show that there do not exist any pairs of limiting parallel rays lying on distinct lines.

34.9 (ASAL) Given four rays Aa, Bb, $A'a'$, $B'b'$, assume that $\angle BAa = \angle B'A'a'$, $AB = A'B'$, and $\angle ABb = \angle A'B'b'$. Show that $Aa \parallel\!| Bb$ if and only if $A'a' \parallel\!| B'b'$.

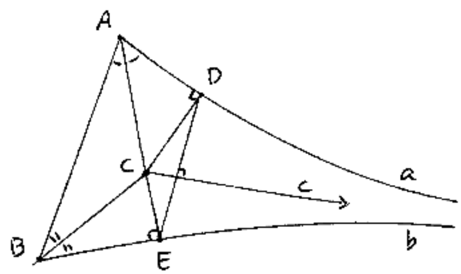


34.10 (ASL) Given $Aa \parallel\!| Bb$ and $A'a' \parallel\!| B'b'$, assume $\angle BAa = \angle B'A'a'$ and $AB = A'B'$. Then $\angle ABb = \angle A'B'b'$. We call the figure consisting of the segment AB and the two limiting parallel rays Aa and Bb a *limit triangle*.

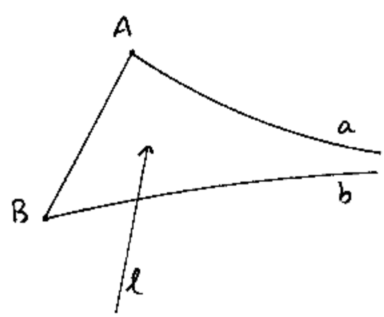
34.11 Given a limit triangle $aABb$, construct its *midline* as follows. Let the angle bisectors at A, B meet at a point C. Drop perpendiculars CD, CE from C to a, b. Join DE, and let c be the perpendicular from C to DE.

(a) Show that Cc is limiting parallel to Aa and Bb.

(b) Show that reflection in the line c interchanges a and b. Thus c plays a role for the rays Aa and Bb similar to the role of the angle bisector of an angle, which interchanges the two sides of an angle by reflection. So we can think of C as the intersection of the three (generalized) angle bisectors of the limit triangle.



34.12 Show that the analogue of Pasch's axiom (B4) holds for a limit triangle $aABb$: If l is a line that does not contain A or B, and does not contain a ray limiting parallel to Aa or Bb, and if l meets one side AB, Aa, or Bb, then it must meet a second side, but not all three.



34.13 *Spherical geometry*. Let F be a Euclidean ordered field. In the Cartesian 3-space over F consider the sphere of radius r given by the equation $x^2 + y^2 + z^2 = r^2$. We can make a geometry, called *spherical geometry*, as follows. Our *s-points* are the points of F^3 on the surface of the sphere. Our *s-lines* are *great circles* on the sphere, that is to say, the intersections of the sphere with planes of F^3 passing through the origin $O = (0, 0, 0)$. On any piece of an s-line that is less than half of a great circle, we can define betweenness by projecting the points from O into any plane. We say that two segments of s-lines are congruent if the chords joining their endpoints, as line segments of F^3 inside the sphere, are congruent. We say that angles are congruent if the projected angles on the tangent planes to the sphere at their vertices are congruent.

Which of Hilbert's axioms hold in this geometry? You will see right away that (I1) fails and betweenness does not make very good sense, so it is not a Hilbert plane. Show, however, that the congruence axioms (C1)–(C6) and (ERM) do hold.

34.14 (a) Now suppose that we take F to be a non-Archimedean field, such as the one in (18.4). Let t be an infinite element in F, take the sphere of radius t, and take our geometry Π_0 to consist of only those points on the surface of the sphere that are at finite distance from a fixed point A on the sphere. Show that this geometry satisfies all of Hilbert's axioms, so it is a Hilbert plane. Show also that the sum of the angles of any triangle in this geometry is *greater* than two right angles. This is an example of a semi-elliptic Hilbert plane.

(b) Again take F to be a non-Archimedean field, and let Π_1 be the set of points on a sphere of radius 1 whose distance from a fixed point A is infinitesimal. Show that Π_1 is another semielliptic Hilbert plane, and show that Π_1 is not isomorphic to the plane Π_0 of part (a). *Hint*: cf. Exercise 18.6.

34.15 In any Hilbert plane, show that the three angle bisectors of a triangle meet in a point.

34.16 In any Hilbert plane, if two of the perpendicular bisectors of the sides of a triangle meet, then all three perpendicular bisectors meet in the same point.

34.17 We say that two lines in a Hilbert plane are *strictly parallel* if every transversal line makes equal alternate interior angles. Show that the following conditions are equivalent:

(i) The plane is semi-Euclidean.

(ii) For every point P and every line l, there exists a unique line m through P strictly parallel to l.

(iii) There exists at least one pair of distinct strictly parallel lines.

34.18 Show that strictly parallel lines (Exercise 34.17) behave in many of the same ways as parallel lines in Euclidean geometry:

(a) If l is strictly parallel to m, and m strictly parallel to n, then l is strictly parallel to n (analogue of (I.30)).

(b) If both pairs of opposite sides of a quadrilateral are strictly parallel, then opposite sides and opposite angles are equal (analogue of (I.34)).

34.19 In a semi-Euclidean plane, show that if two of the altitudes of a triangle meet, then all three altitudes meet in the same point.

34.20 In a semi-Euclidean plane, show that the medians of a triangle all meet in a point.

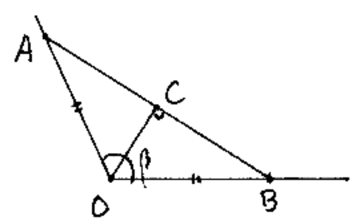
34.21 In any Hilbert plane, show that the line joining the midpoints of two sides of a triangle is orthogonal to the perpendicular bisector of the third side.

35 Archimedean Neutral Geometry

If we add Archimedes' axiom to the axioms of neutral geometry, we have the remarkable fact that the angle sum of a triangle is always less than or equal to two right angles. In other words, the semielliptic case is impossible. Saccheri's proof of this result uses a continuity argument, so we prefer the method of Legendre, using a repeated application of the construction Euclid used in (I.16) for the proof of the exterior angle theorem. In either case, the proof makes essential use of Archimedes' axiom. To begin with, we show that the analogue of Archimedes' axiom holds for angles.

Lemma 35.1
In a Hilbert plane with (A), *let* α, β *be given angles. Then there exists an integer* $n > 0$ *such that* $n\alpha > \beta$, *or else* $n\alpha$ *becomes undefined by exceeding* 2RA.

Proof First we make a reduction. Given the angle β at O, measure off equal segments OA and OB on the two arms, and draw AB. The line OC joining O to the midpoint of AB will bisect the angle β and will make a right angle at C.



Since it is just as good to prove the lemma for $\frac{1}{2}\beta$, we reduce to studying the case of an angle contained in a right triangle.

So now let OAB be a right triangle with the angle β at O and a right angle at A. Suppose, by way of contradiction, that $n\alpha \leq \beta$ for all n. Lay off the angle α inside the triangle, and let that angle cut off a segment AA_1 on the line AB. Again lay off the angle α at O to cut a segment A_1A_2 on the line AB. Continuing in this manner, we obtain a sequence of points $A_1, A_2, A_3, \ldots$ on AB, with each successive segment A_iA_{i+1} subtending an angle α to O.