So we proceed as follows: Bisect O_3E using G, H (3 steps), and get K. Draw the circle with center K, radius KE (1 step), and let it intersect γ_3' at L. Draw the circle Γ with center E through L (1 step). Then Γ will be orthogonal to γ_3', and so will be left fixed by circular inversion in Γ. Let Γ meet γ_2' in M, N, and draw the line $l = MN$ (1 step). Let Γ meet γ_1' in P, Q, and draw the line $m = PQ$ (1 step). Thus we have transformed $\gamma_1', \gamma_2', \gamma_3'$ into l, m, and γ_3', and we have the new problem of finding a circle tangent to these three. Furthermore, since γ_1' and γ_2' were tangent to O, their transforms l, m do not meet. In other words, l and m are parallel, so we have a case of (38.6) treated above. This portion of our construction was 7 steps.

Now perform (38.6) to find a circle σ tangent to l, m, γ_3', and let the points of tangency be R, S, T (9 steps). Actually, since we already have a line O, O_2 perpendicular to l and m, we can get the midline in 3 steps instead of 6, thus saving 3 steps. So this part of the construction counts 6 steps.

The last stage of the construction is to transport back σ by the circular inversion in Γ to get a circle tangent to $\gamma_1', \gamma_2', \gamma_3'$. Then for the same center we can draw the desired circle τ tangent to $\gamma_1, \gamma_2, \gamma_3$.

It is actually sufficient to pull back two of the points of tangency. Draw ER and let it intersect γ_1' at U (1 step). Then U is the inverse of R in Γ. Draw ET and let it meet γ_2' at V (1 step). Now U, V are two of the points of tangency of a circle (dotted) tangent to $\gamma_1', \gamma_2', \gamma_3'$. To find its center, draw O_1U and O_2V and let them meet at X (2 steps). Now the circle τ with center X and radius XY is the desired circle (1 step). This last part of the construction is 5 steps. (In the drawing I also found the inverse Z of S and drew O_3Z to check for accuracy, but this is not really part of the theoretical construction.) Total: 27 steps.

Exercises

Carry out the following ruler and compass constructions.

38.1 PLC. Treat as a special case of LCC.

38.2 LLC. Follow hint given earlier in text.

38.3 PCC. Treat as a special case of CCC.

38.4 LCC. Use a technique similar to the one we used for CCC to reduce to (38.6).

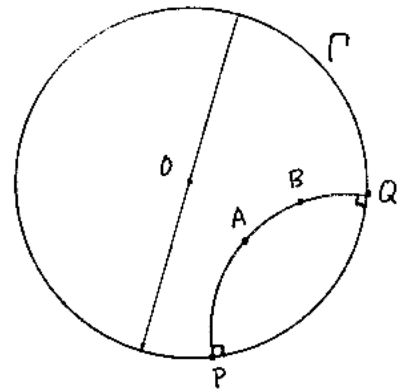
38.5 PPC. Do the general case, where the perpendicular bisector of AB does not meet γ.

38.6 Describe how you would construct all eight solutions to the problem of Apollonius.

39 The Poincaré Model

In this section we will show the *existence* of a non-Euclidean geometry, and hence the *consistency* of the axioms of non-Euclidean geometry, by exhibiting a model for a non-Euclidean geometry. Ironically, our model of a non-Euclidean geometry will be constructed within the logical framework of Euclidean geometry. So what we must do is to give an *interpretation* of the undefined notions of geometry in the model: point, line, betweenness, and congruence for line segments and angles, and then we must prove that the axioms all hold in this interpretation.

Our starting point will be the Cartesian plane Π over a Euclidean ordered field F. In this plane we fix a circle Γ with center O. (For a weakening of the Euclidean hypothesis on F, see Exercises 39.25 ff.)

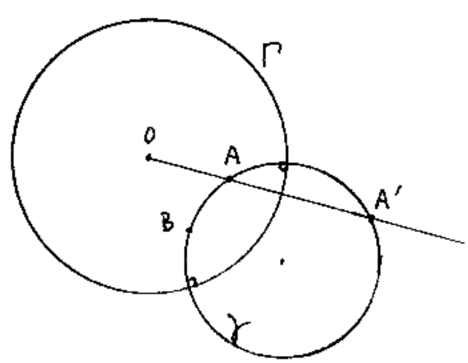


The *points* of our model (which we will call P-*points*) will be the set of points of Π *inside* Γ, not counting the points on Γ. A P-*line* will be the set of all P-points lying on a circle γ that is orthogonal to Γ, or that lie on a line through O. (To keep our language straight, the words point, line, circle will refer to the Euclidean notions in Π, and we will prefix a P to any word to mean the corresponding concept in the model we are building.)

Having thus defined the P-points and P-lines of our model, we can verify the incidence axioms (I1), (I2), (I3).

Proposition 39.1
The P-*model satisfies* (I1), (I2), *and* (I3).

Proof For (I1), suppose we are given two P-points, A, B. If the line AB passes through O, then it is a P-line containing them and is the only such. If A, B, and O are not collinear, let A' be the inverse of A under inversion in the circle Γ (cf. Section 37). Then there is a unique circle γ passing through A, A', and B. By (37.3), γ is orthogonal to Γ, so that portion of γ that is inside Γ becomes a P-line containing A and B. It is unique, because again by (37.3), any circle γ orthogonal to Γ that contains A also contains A'.

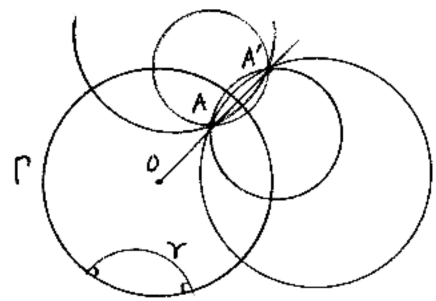


The other two axioms (I2), existence of at least two points on a line and (I3) existence of three noncollinear points, are obvious.

We see immediately that this geometry will be non-Euclidean because the parallel axiom (P) does not hold.

Proposition 39.2

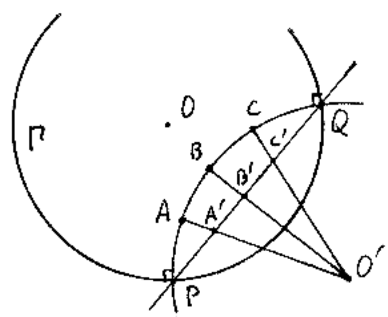
The parallel axiom (P) *does not hold in the* P-*model: There is a* P-*line* γ *and a* P-*point* A *such that there is more than one* P-*line through* A *that is* P-*parallel to* γ. (*Of course,* P-*parallel means that two* P-*lines do not intersect.*)



Proof Take a P-line γ in one part of our P-plane and take a point A far away. Let A' be the inverse of A. Then (by 37.3)), any circle through A and A' will be orthogonal to Γ, so it gives a P-line passing through A. There are many of these that do not meet γ, and these are all P-lines through A that are P-parallel to γ.

Definition

If A, B, C are P-points on a P-line γ, we define the P-*betweenness* relation $A * B * C$ as follows. Let O' be the center of γ (which is always outside Γ), draw the line PQ, and project the points A, B, C to points $A', B', C' \in PQ$ from the point O'. Then we will say $A * B * C$ (P-betweenness) if and only if $A' * B' * C'$ on the line PQ (usual betweenness). If A, B, C lie on a P-line that is an ordinary line through O, we take the usual notion of betweenness.



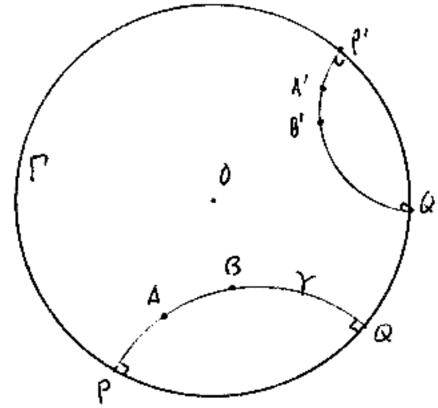
Proposition 39.3

The notion of P-*betweenness for* P-*points satisfies axioms* (B1)–(B4).

Proof Axioms (B1), (B2), and (B3) follow immediately from the corresponding statements for ordinary betweenness on the line PQ. For (B4), taking into account the circle–circle intersection property (E) in Π and noting that two circles orthogonal to Γ can meet at most once inside Γ, we see that to say that P-points A, B are on the same P-side of a P-line γ is equivalent to saying that A, B as ordinary points are either both inside γ or both outside γ. Thus we can define the inside of a P-triangle, and (B4) is clear.

Definition

We define *congruence* in our P-model as follows. Two P-angles are P-*congruent* if the Euclidean angles they define are congruent in the usual sense. For line segments, we proceed as follows. Given two P-points, let the P-line joining them be the circle γ orthogonal to Γ. Let γ meet Γ in two points P, Q, and label them so that P is the one closer to A. For another pair of points A', B' lying on a P-line γ', label P', Q' similarly. Then we say that the P-segment AB is P-*congruent* to the P-segment $A'B'$ if the cross-ratio (AB, PQ) is equal to the cross-ratio $(A'B', P'Q')$ (cf. Section 37 for cross-ratios).



Now the real work begins, to verify the congruence axioms. We start with the easy ones.

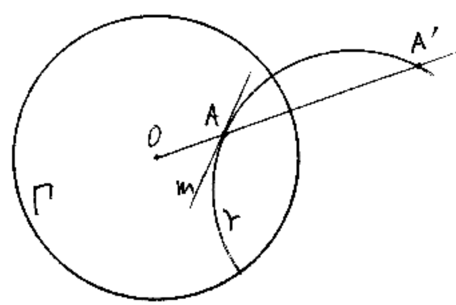
Proposition 39.4

P-*congruence satisfies axioms* (C2)–(C5).

Proof (C2) is obvious from the definition, since congruence of segments is defined by equivalence of associated quantities in the field.

(C3) requires a calculation. From the definition of cross-ratio it follows that $(AB, PQ) \cdot (BC, PQ) = (AC, PQ)$ (verify!). So when two segments are added together, the associated cross-ratios multiply. From this (C3) follows immediately.

To prove (C4), laying off angles, first suppose that we are given a point A inside Γ and a line m through A. Let A' be the inverse of A. Then there exists a unique circle γ passing through A and A' and tangent to the line m. By (37.3) γ is orthogonal to Γ. This shows that there exists a P-line at A with any given tangent direction. Now, if an angle α is given and a P-line δ given at A, by (C4) in Π there is a unique line m forming the angle α with δ at A (and on a given side of δ). Then the P-line with tangent m gives the required P-angle at A, and is unique.



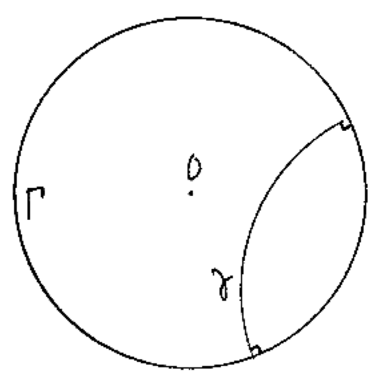
(C5) follows from the same statemnt for Euclidean angles, because congruence of angles is the same.

Before proceeding to a discussion of the remaining axioms (C1), (C6), (E), (A), and (D), we will establish the existence of rigid motion (ERM) in this model. Recall from Section 17 that a *rigid motion* is a transformation of the geometry that preserves the undefined notions of point, line, betweenness, and congruence. In our case, a P-*rigid motion* will be a transformation of the set of points inside Γ that is 1-to-1 and onto, sends P-lines to P-lines, and preserves P-betweenness and P-congruence of angles and segments.

Proposition 39.5 (Existence of rigid motions (ERM) for the Poincaré model) *There are enough* P-*rigid motions of the Poincaré model so that:*

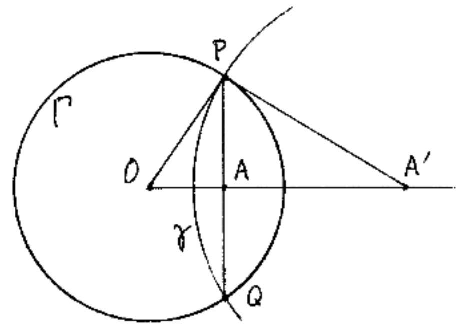
(1) *For any two* P-*points* A, A', *there is a* P-*rigid motion sending* A *to* A'.
(2) *Given* P-*points* A, B, B', *there is a* P-*rigid motion leaving* A *fixed and sending the ray* $\overrightarrow{AB}$ *to the ray* $\overrightarrow{AB}'$.
(3) *For any* P-*line* γ *there is a P-rigid motion leaving all the points of* γ *fixed and interchanging the two sides of* γ.

Proof We start with the last property. Given a P-line γ, let ρ_γ be the circular inversion in γ. Since Γ is orthogonal to γ, ρ_γ sends Γ to itself (37.3). Also, the inside of Γ is sent to the inside of Γ, so that the P-plane is mapped to itself, in a way that is clearly 1-to-1 and onto. Since circular inversion sends circles into circles (37.4) and is conformal (37.5), a circle orthogonal to Γ will be sent to another circle orthogonal to Γ, in other words, ρ_γ sends P-lines into P-lines. (Note that this works also for the limiting case of a line through O, which is also orthogonal to Γ.)



Circular inversion clearly preserves betweenness (Exercise 39.1). It preserves P-congruence of angles because this is the same as usual congruence of angles, and inversion is conformal (37.5). Also, ρ_γ preserves P-congruence of P-segments, because this is defined by the cross-ratio, which is invariant under circular inversion (37.6). Finally, note that ρ_γ interchanges that part of the P-plane that is inside γ with that part that is outside γ, so ρ_γ is a P-rigid motion as required for the third statement of (ERM). Since it leaves the points of γ fixed and interchanges the sides of γ, it is the P-*reflection* in γ (Section 17).

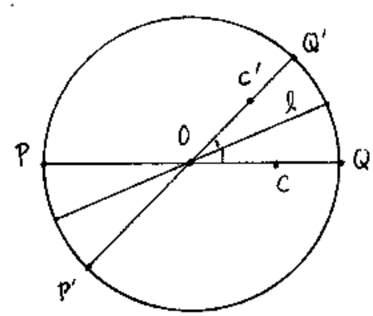
Next we will show that for any $A \neq O$, there is a circle γ orthogonal to Γ (a P-line) such that the P-reflection in γ interchanges O and A. Let A' be the inverse of A; let γ be the circle with center A' that is orthogonal to Γ. Then the construction (37.1) for the circle γ, using the same diagram (!), shows that inversion in γ sends A to O. Thus the P-reflection in γ interchanges A and O.



Now, since a composition of P-rigid motions is again a P-rigid motion, given two points A, A', we can first send A to O as above, then send O to A'. The composition of these two reflections will be a P-rigid motion sending A to A', which proves (1).

Now suppose that we are given three points A, B, B'. Let ρ be a P-rigid motion taking A to O, and let $\rho(B) = C$, $\rho(B') = C'$. If we can solve problem (2) for O, C, C', in other words, if there is a P-rigid motion θ leaving O fixed and sending the ray $\overrightarrow{OC}$ to the ray $\overrightarrow{OC}'$, then $\rho^{-1}\theta\rho$ will solve the problem (2) for A, B, B'. So we reduce to solving the problem for O, C, C'.

Let l be the angle bisector of angle COC'. Then l is a line through O, which is also a P-line. The ordinary reflection in l is clearly a P-rigid motion that leaves O fixed and sends the ray $\overrightarrow{OC}$ to the ray $\overrightarrow{OC}'$.



This completes the proof of (ERM) for the Poincaré model.

Proposition 39.6
Axioms (C1) *and* (C6) *hold in the Poincaré model.*

Proof Suppose it is required to find a point B' on a P-ray emanating from a point A' such that $A'B'$ is P-congruent to a given P-segment AB. By (ERM) = (39.5), there is a *P*-rigid motion φ taking A to A'. There is also a *P*-rigid motion ψ taking the ray $\varphi(\overrightarrow{AB})$ to the given ray from A'. Then $B' = \psi\varphi(A')$ is a point on the given ray, and $AB \cong A'B'$ because rigid motions preserve congruence. Thus (C1) holds in the Poincaré model.

To show that (C6) holds, since we have already established (C1)–(C5), we simply apply (17.1), which shows that under those circumstances, (ERM) implies (C6).

In order to discuss (E) in the Poincaré model, we first need to identify what is a P-circle. By definition, of course, it is the set of all P-points B' such that the P-segment $A'B'$, for a certain fixed point A', is P-congruent to a given P-segment AB. Since the definition of P-congruence of segments is not very intuitive, it is not easy to see immediately what kind of curves these are. First we need a lemma.

Lemma 39.7
If C, C' are two points inside Γ, not equal to the center of Γ, O, then the P-*segment OC (which is equal to the Euclidean segment OC, since the* P-*line joining O and C is just the usual line OC) is* P-*congruent to the* P-*segment OC' if and only if OC is congruent to OC' in the ambient Euclidean plane* Π.

Proof Let P and Q be the endpoints of the diameter of Γ passing through O and C. Then the P-congruence of OC is determined by the cross-ratio

$$(OC, PQ) = \frac{OP}{OQ} \div \frac{CP}{CQ}.$$

Let $r =$ radius of Γ and let $x =$ Euclidean distance from O to C. Then the cross-ratio is

$$\frac{r}{r} \div \frac{r+x}{r-x} = \frac{r-x}{r+x}.$$

If C' is another point, and if the distance from O to C' is y, then we obtain similarly

$$(OC', P'Q') = \frac{r-y}{r+y}.$$

Thus, to say that OC is P-congruent to OC' is to say that

$$\frac{r-x}{r+x} = \frac{r-y}{r+y}.$$

Cross multiplying, we obtain

$$r^2 - rx + ry - xy = r^2 + rx - ry - xy,$$

so

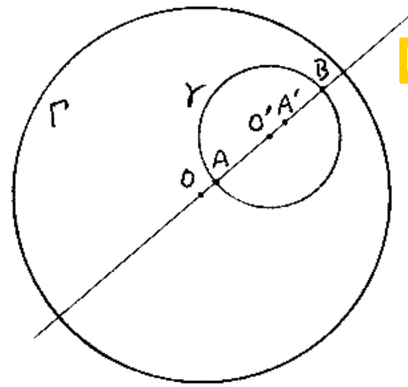
$$2rx = 2ry.$$

Since our field has characteristic 0, this is equivalent to $x = y$, i.e., OC is congruent to OC' in the usual sense.

Proposition 39.8
Every P-*circle is an ordinary circle that is entirely contained in the inside of* Γ, *and conversely, every circle entirely inside* Γ *is a* P-*circle.* (Warning: *The* P-*center of a* P-*circle is usually not equal to its ordinary center.*)

Proof Given a P-circle ζ with P-center A', consider a rigid motion θ that takes A' to O. This will transform ζ into a P-circle with P-center O. Since P-congruence and ordinary congruence are the same for segments beginning at O by the lemma, this image $\theta(\zeta)$ is an ordinary circle with center O. Then θ^{-1} will carry this ordinary circle back to the given P-circle ζ. Now observe that in the proof of (ERM), all the rigid motions we needed were made out of compositions of P-reflections (which are circular inversions in suitable circles) or reflections in a line through O. Since all of these transformations send circles into circles (37.4), it follows that ζ is a circle. Since the transformed circle was a circle around O entirely contained inside Γ, the image is also entirely contained inside Γ.

Conversely, given an ordinary circle ζ completely contained inside Γ, with (ordinary) center O', draw OO'. Let it meet ζ at A, B. P-bisect the segment AB at A', and choose a P-reflection ρ_γ that sends A' to O. Then $\rho_\gamma(\zeta)$ will be a circle, the images of A and B will be equidistant from O, and this circle will be symmetric about the line $l = OO'$, which is sent into itself by ρ_γ. Hence $\rho_\gamma(\zeta)$ is a circle with center O, which is also a P-circle. Applying ρ_γ^{-1}, it follows that the original circle ζ is a P-circle with P-center C.



Proposition 39.9
The circle–circle intersection property (E) *holds in the Poincaré model over a Euclidean ordered field F.*

Proof Since P-lines and P-circles are all either usual circles or lines through O, and since betweenness is the same in the P-model as in the ambient Euclidean space, (E) in the P-model follows directly from (E) in the Cartesian plane Π, and this in turn follows from the Euclidean hypothesis on F (16.2). Since P-circles are usual circles entirely contained inside Γ, there is no problem about any of the intersections falling outside Γ.

For the next proposition it will be convenient to introduce the notion of a distance function. In ordinary Euclidean geometry the distance function assigns to each interval a positive number, and adding segments corresponds to adding numbers. More generally, we make the following definition.

Definition
A *distance function* on a Hilbert plane is a function d that to each segment assigns an element of an ordered abelian group G such that

(1) $d(AB) > 0$ for any segment AB.
(2) $d(AB) = d(A'B')$ if and only if $AB \cong A'B'$.
(3) if $A * B * C$, then $d(AC) = d(AB) + d(BC)$.

If the group happens to be written multiplicatively, we will call it a *multiplicative* distance function. The usual distance function on the Cartesian plane over a field F (Section 16) is an additive distance function with values in the additive group of the field $(F, +)$

Lemma 39.10
In the Poincaré model over a field F, the function $\mu(AB) = (AB, PQ)^{-1}$ is a multiplicative distance function with values in the multiplicative group of the field $(F_{>0}, \cdot)$.

Proof Because of our convention that P is the endpoint closer to A, the cross-ratio (AB, PQ) is in the interval $(0, 1)$ in F. Therefore, $\mu(AB) > 1$. We have already used it to define congruence, and we have seen that it is multiplicative (proof of 39.4). Hence μ is a multiplicative distance function.

Proposition 39.11
Archimedes' axiom (A) *will hold in the* P-*model if we assume Archimedes' axiom* (A′) *for the field F. Similarly, Dedekind's axiom* (D) *will hold if we assume* (D′) *in the field.* (*Cf.* (15.4) *for* (A′) *and* (D′).)

Proof Using the multiplicative distance function μ of (39.10), Archimedes' axiom in the P-plane is equivalent to the following statement in F: Given $c, d \in F$, $c, d > 1$, $\exists n > 0$ such that $c^n > d$.

We will show that this property is a consequence of Archimedes' axiom (A′) for F. Write $c = 1 + x$, so $x \in F, x > 0$. Then

$$c^n = (1 + x)^n = 1 + nx + \text{positive terms} \geq 1 + nx.$$

Now (A′) says that for some $n, nx > d$. Hence also $c^n > d$, as required.

For Dedekind's axiom, (D′) in F implies (D) in Π (15.4), and this clearly implies (D) in the P-plane because of the way we defined betweenness by projecting onto a line segment. (For a converse to (39.11), see Exercise 39.7.)

Proposition 39.12
For any point A and any ray Bb in the Poincaré model, there exists a limiting parallel ray (*cf.* Section 34) *Aa to Bb.*

Proof Let the P-ray Bb meet the defining circle Γ of the Poincaré model in a point Q. Let A' be the circular inverse of A in Γ, and let γ be the circle through

A, Q, A'. Then γ defines a P-line, and we take Aa to be the P-ray of that P-line having Q at its end. Then it is clear that Aa and Bb are limiting parallel rays in the Poincaré model.

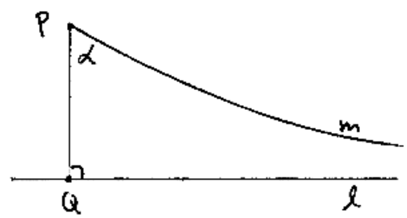
Using a little Euclidean geometry in the ambient Cartesian plane, we can derive a marvelous relationship between the length of a segment and the angle it makes with a limiting parallel.

Proposition 39.13 (Bolyai's formula) *Suppose we are given in the Poincaré model a point P, a line l, the perpendicular PQ to l, and a limiting parallel line m, making an angle α with PQ.*

Then

$$\tan\frac{\alpha}{2} = \mu(PQ)^{-1},$$

where the tangent is understood to be of the corresponding Euclidean angle, and μ is the multiplicative distance function. The equality takes place in the field F.



Proof We may assume that the Poincaré model is made with a circle Γ of radius 1 (cf. Exercise 39.23). We can move P, Q, l, m so that Q becomes the center of Γ, the line l becomes a radius QA, and P lies on an orthogonal radius QB. The limiting parallel through P to l will be part of a circle Δ, orthogonal to Γ at A. Its center therefore is at a point $C = (1, c)$ on the line $x = 1$. Let P be the point $(0, y)$. Then $CP = CA$, so

$$c^2 = (c - y)^2 + 1.$$

Therefore,

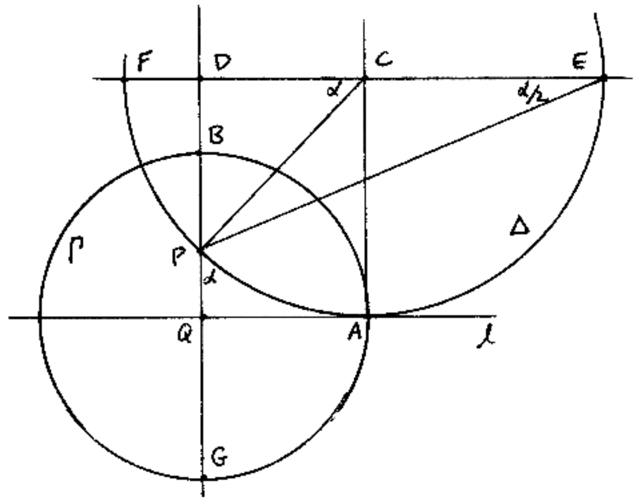
$$c = \frac{1 + y^2}{2y}. \qquad (1)$$

Draw a diameter EF of Δ parallel to the x-axis. Then the angle α between our limiting parallel and PQ, called the *angle of parallelism* of the segment PQ, is equal to the angle PCF. If we draw EP, then the angle $PEF = \alpha/2$ (III.20). Now

$$\tan\frac{\alpha}{2} = DP/DE = \frac{c - y}{c + 1}.$$

Substituting from (1) we obtain

$$\tan\frac{\alpha}{2} = \frac{1-y}{1+y}. \tag{2}$$



On the other hand, the multiplicative distance function is

$$\begin{aligned}\mu(PQ) &= (PQ, BG)^{-1}\\ &= \left(\frac{PB}{PG} \div \frac{QB}{QG}\right)^{-1}\\ &= \left(\frac{1-y}{1+y} \div \frac{1}{1}\right)^{-1}\\ &= \frac{1+y}{1-y}. \end{aligned}\tag{3}$$

From (2) and (3) we conclude that

$$\tan\frac{\alpha}{2} = \mu(PQ)^{-1},$$

as required.

Remark 39.13.1
From this it follows that given any angle α less than a right angle, there exists a segment PQ with angle of parallelism equal to α. Indeed, $\tan(\alpha/2)$ will be an element of the field F, and then we can find a $y \in F$ satisfying (2) above. In particular if we take $\alpha = \frac{1}{2}$ RA (one-half right angle), there will be a corresponding

segment PQ uniquely determined up to congruence. In this sense there is an absolute standard of length in the Poincaré model, whereas in Euclidean geometry the choice of unit length is arbitrary.

Exercises

All exercises take place in the Poincaré model over a Euclidean ordered field F, unless otherwise noted. Proofs should be based on the Euclidean geometry of the Cartesian plane over F. In particular, do not use any of the results of Section 34 or Section 35 that depend on Archimedes' axiom.

39.1 Verify that circular inversion preserves betweenness in the Poincaré model (cf. proof of Proposition 39.5).

39.2 Show that the angle sum of any triangle in the Poincaré model is less than 2RA so this geometry is semihyperbolic (Section 34).

39.3 For any angle α, show the existence of a line entirely contained inside the angle α (cf. Exercise 35.4).

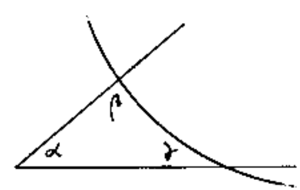
39.4 Show that for any angle $\alpha < 60°$ there exists an equilateral triangle with all of its angles equal to α.

39.5 If an equilateral triangle has sides equal to AB and angles equal to α, show that

$$\frac{2a}{1+a^2} = \frac{2t^2}{1-t^2},$$

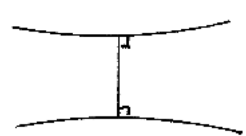
where $a = \mu(AB)$ is its multiplicative length, and where $t = \tan(\alpha/2)$ (cf. Example 42.3.2).

39.6 Given any three angles α, β, γ with $\alpha + \beta + \gamma < 2RA$, show that there exists a triangle with angles α, β, γ in the Poincaré model. *Hint*: First show in the Cartesian plane that you can find an angle α meeting a circle at angles β and γ. Then shrink or expand this figure so that it becomes a triangle in the Poincaré model.

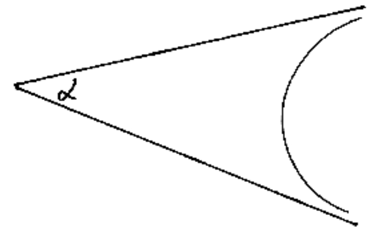


39.7 Prove the converse of Proposition 39.11, namely, if (A) or (D) holds in the Poincaré model, then (A′) resp. (D′) holds in F.

39.8 If two lines are parallel, but not limiting parallel, then they have a unique common orthogonal line.



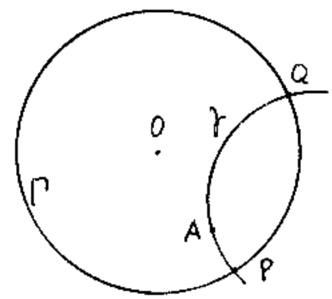
39.9 For any angle α, there is an *enclosing line*, which is a line limiting parallel to both arms of α.



39.10 Give an alternative proof of (C1) in the Poincaré model, without using rigid motions, as follows. Given a point A, a P-line γ, and given a quantity $b \in F$, $0 < b < 1$, we need to find a point $B \in \gamma$ such that

$$(AB, PQ) = b.$$

Do this by showing that in Euclidean geometry, the locus of points B such that BP/BQ is a given ratio $k \in F$ is a *circle*. Then use (E), in the Cartesian plane, to show that this circle intersects γ and thus find the required B.

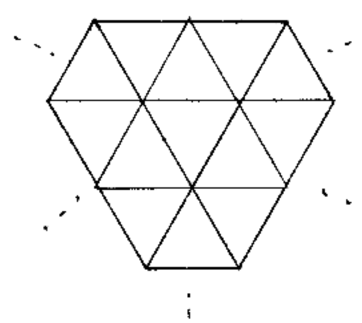


39.11 Given the circle Γ, its center O, and another circle ζ entirely contained inside Γ, give a ruler and compass construction (in the ambient Euclidean plane) of the P-center ζ regarded as a P-circle (cf. Proposition 39.8).

39.12 (Euclidean geometry). Find all possible ways of filling the entire Euclidean plane with triangles satisfying the following conditions:

(a) The triangles are all congruent to each other. There is no overlap, and they fill the entire plane.

(b) At each vertex of the triangulation, all the angles are the same (though they may be different from the angles at a different vertex).



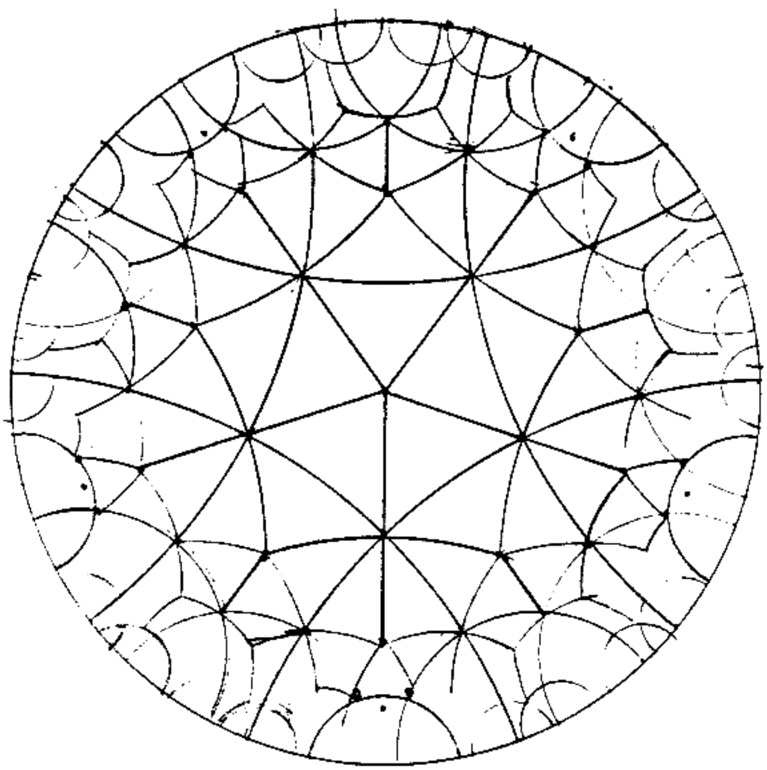
We consider two "ways" of filling the plane "the same" if one can be moved to the other by a dilation followed by a rigid motion.

One such triangulation is shown, where the angles at each vertex are all 60°. This is the only possibility if all angles are equal. Expect to find three more ways, allowing angles at different vertices to be different, and *prove* that you have found all possibilities.

39.13 In the Poincaré model of non-Euclidean geometry, show, in contrast to the Euclidean situation described in Exercise 39.12 above, that there are infinitely many different ways to cover the P-plane by congruent P-triangles satisfying properties (a) and (b).

In particular, prove that the plane can be covered by equilateral triangles with all angles equal to 45° and with eight meeting at each vertex. If AB is a side of one of these triangles, find $\mu(AB)$.

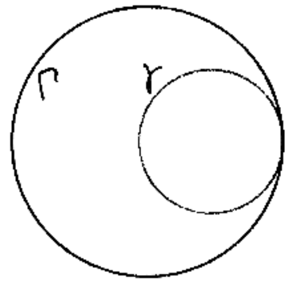
Draw a big circle Γ on a piece of paper, and then accurately draw enough of these P-triangles inside Γ to show how they cover the whole P-plane. (This drawing can be accomplished entirely by ruler and compass, but don't bother listing the steps, except to show how you got the first triangle.)



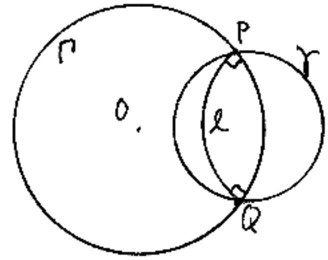
Congruent, isosceles, 72°–45°–45° triangles, filling up the Poincaré model of the non-Euclidean plane (cf. Exercise 39.13).

39.14 In the Poincaré model made inside a circle Γ in the Cartesian plane over F, we have seen that any Euclidean circle γ entirely contained inside Γ is a P-circle (Proposition 39.8).

(a) If γ is a Euclidean circle inside Γ and tangent to Γ, show that there is a pencil of limiting parallel lines (a *pencil* means the set of all lines that are mutually limiting parallels at one end) such that the curve γ is orthogonal to all the lines of the pencil. Such a curve is called a *horocycle* in the Poincaré model.



(b) If γ is a Euclidean circle that cuts Γ at points P, Q, let l be the P-line having the endpoints P, Q. Show that the points of γ inside Γ form a curve of points equidistant from the P-line l. Such a curve is called an *equidistant curve* or *hypercycle*.

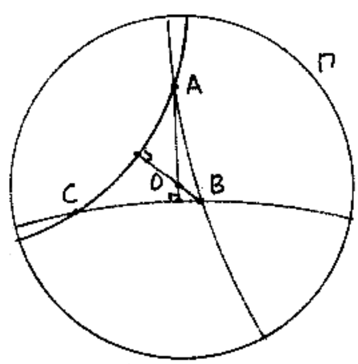


(c) Given any three distinct points A, B, C in the Poincaré model, show that they are contained in a unique P-line or P-circle or horocycle or hypercycle. (Contrast to Euclidean geometry, where only the first two possibilities occur.)

39.15 Show in the Poincaré model that it is in general not possible to trisect an angle (i.e., if α is an angle, the angle $\frac{1}{3}\alpha$ may not exist) (cf. Section 28).

39.16 Show in the Poincaré model, in contrast to the Euclidean case (Exercise 2.14), that it is in general not possible to trisect a line segment (i.e., the 3-division points may not exist).

39.17 In the Poincaré model, show that if two altitudes of a triangle meet in a point, then the third altitude also passes through that point. Here is a method. Let the triangle be ABC, and suppose that the altitudes from A and B meet. By a rigid motion of the Poincaré plane we move that meeting point to the center O of the defining circle Γ. Then those altitudes become Euclidean lines through O. We must show that the line OC is orthogonal to the side AB.



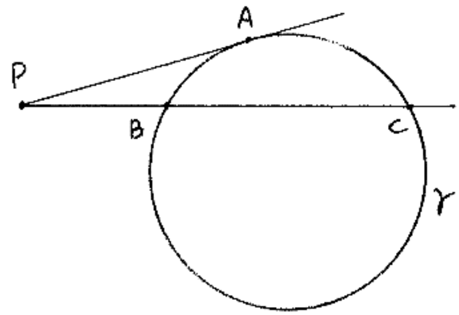
The P-lines AB, AC, BC are Euclidean circles orthogonal to Γ. Let D, E, F be the centers of these circles. Show that the altitudes of the P-triangle ABC are at the same time altitudes of the Euclidean triangle DEF. Then use the Euclidean theorem that the altitudes of a triangle meet (Proposition 5.6) to finish the proof.

Note: This is a curious method, whereby the Euclidean result is used to show (via Euclidean geometry) that the same result holds in the non-Euclidean Poincaré model. Since we now know that this result holds in both Euclidean and non-Euclidean geometry, it would be nice to have a single proof in neutral geometry that applies to both cases—cf. Exercise 40.14 and Theorem 43.15.

39.18 Show that the result of Exercise 1.15 is also valid in the Poincaré model, by moving the figure so that P becomes the center of Γ and using the Euclidean result already proved. Can you find a proof in neutral geometry that will cover both cases at once?

39.19 Prove a non-Euclidean analogue of (III.36) in the Poincaré model, as follows. Let P be a point outside a circle γ, let PA be a tangent to γ, and let PBC be a secant. Let $a = \mu(PA)$, $b = \mu(PB)$, and $c = \mu(PC)$. Then

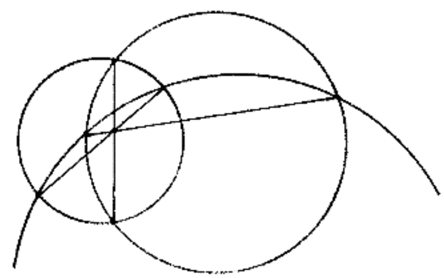
$$\left(\frac{a-1}{a+1}\right)^2 = \left(\frac{b-1}{b+1}\right)\left(\frac{c-1}{c+1}\right).$$



Hint: Move P to the center O of the Poincaré model, use the Euclidean (III.36)—cf. Proposition 20.9—and compute μ as in the proof of Proposition 39.13.

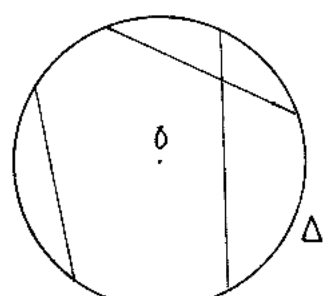
39.20 In the Poincaré model, if three circles each meet the others in two points, show that the three radical axes (Exercise 20.4) meet in a point.

(a) One method is to suppose that two of the radical axes meet in a point A. Move that point to O, and use the Euclidean result (Exercise 20.5).



(b) Another method is to use Exercise 39.19 to define the power of a point with respect to a circle, and imitate the proofs of Exercises 20.4, 20.5.

39.21 There is another model of a non-Euclidean geometry, due to Felix Klein, constructed as follows. In the Cartesian plane over a field F, fix a circle Δ. Then the K-points are the points inside Δ, and the K-lines are chords of Euclidean lines contained inside Δ. In this model the incidence axioms (I1)–(I3) and the betweenness axioms (B1)–(B4) are immediate, taking betweenness to be the same as in the Cartesian plane.

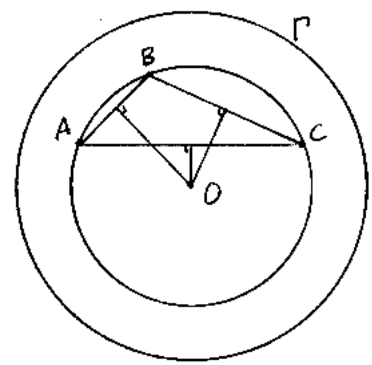


However, the model is not conformal (i.e., angles are not the same as Euclidean angles), so the definition and properties of congruence for line segments and for angles are more complicated. Rather than doing this directly, we will show in this exercise how to obtain the Klein model from the Poincaré model.

Let Δ be a circle of radius 1 centered at the origin, and in the Cartesian 3-space, place a sphere of radius 1 on the plane, with its south pole at the origin (cf. Exercise 37.1). Let Γ be the circle of radius 2 centered at the origin. For each K-point inside Δ, project it straight up to obtain a point of the southern hemisphere of the sphere, and then use the stereographic projection (Exercise 37.1) from the north pole to obtain a P-point inside Γ.

Show that this transformation gives a 1-to-1 correspondence between the points of the K-plane inside Δ with the points of the P-plane inside Γ, which sends K-lines to P-lines and vice versa. Then we can transport the notions of congruence for P-segments and P-angles to the K-plane, so that the K-plane becomes a model of a non-Euclidean Hilbert plane, isomorphic to the Poincaré model.

39.22 If ABC is a triangle having a circumscribed circle, prove that the medians of ABC meet in a point, as follows. Use the Klein model (Exercise 39.21) and place the center of the circumscribed circle at the center O of the circle Δ. Then the perpendicular bisectors of the sides of ABC become diameters of the circle Δ. Conclude that the *K*-midpoints of the sides of the triangle are equal to the Euclidean midpoints, and then use the Euclidean theorem about medians in the ambient plane.



39.23 In the Cartesian plane over the field F, let Γ be a circle of radius r centered at the origin, and let Γ' be a concentric circle of radius r'. Consider the map φ from the set of points inside Γ to the set of points inside Γ' given by

$$\begin{cases} x' = kx, \\ y' = ky, \end{cases}$$

where $k = r'/r$. Show that φ gives an isomorphism of the Poincaré model made with Γ to the Poincaré model made with Γ', which preserves the multiplicative distance function of Lemma 39.10. Conclude that if Γ and Γ' are *any* two circles in the Cartesian plane over F, the associated Poincaré models are isomorphic Hilbert planes.

39.24 Let F be a non-Archimedean Euclidean field such as the one described in Proposition 18.4. Let Π be the Poincaré model over F and let Π_0 be the subset of points that are at finitely bounded multiplicative distance μ from some fixed point O. Show that Π_0 is a non-Euclidean Hilbert plane with properties (a) and (b) below.

(a) The angle sum of any triangle is less than 2RA, so it is semihyperbolic.

(b) Limiting parallel rays on distinct lines do not exist.

(c) Let Π_1 be the subset of those points of Π whose distance from O is infinitesimal. Show that Π_1 is another Hilbert plane satisfying (a) and (b) above.

(d) Show that Π_0 and Π_1 are not isomorphic Hilbert planes.

Compare Exercises 18.3–18.6.

39.25 In this and the following exercises we investigate the Poincaré model over a field that need not be Euclidean. Let F be a Pythagorean ordered field, let $d \in F$, and let Γ be the circle $x^2 + y^2 = d$, which may be a virtual circle if $\sqrt{d} \notin F$ (Exercise 37.17). We define the *Poincaré model* in Γ as in the text. The *interior* of Γ is the set of points (x, y) with $x^2 + y^2 < d$. These are the P-points. The P-lines are segments of circles γ orthogonal to Γ (which means stable under circular inversion in Γ (Exercises 37.16, 37.17)) as before.

(a) Show that the incidence axioms (I1)–(I3) holds, as in Proposition 39.1.

(b) If γ is a P-line, the intersection points P, Q of γ with Γ may not exist, but the line PQ is still well-defined: It is the perpendicular to OO' at the inverse of O', where O' is the center of γ. So we can define betweenness as before. Show that betweenness satisfies axioms (B1)–(B3) as in the text.

39.26 With hypotheses as in Exercise 39.25, now suppose that F satisfies the additional condition $(*d)$: For any $a \in F$, if $a^2 - d > 0$, then $\sqrt{a^2 - d} \in F$.

(a) Show that the circle–circle intersection property (E) holds for circles γ, δ orthogonal to Γ. *Hint*: Write the equations of γ, δ, and show that the square root needed to find their intersection exists because of condition $(*d)$.

(b) Conclude that axiom (B4) also holds in this model.

39.27 Continuing with the situation of the two previous exercises, if γ is a P-line, the points of intersection P, Q with Γ do not exist, but at least they have coordinates in the field $F(\sqrt{d})$. Hence we can compute the cross-ratio (AB, PQ) in that field, and define congruence of angles and segments as in the text.

(a) Using condition $(*d)$, show that for any point A' outside Γ, there exists a circle γ with center A' and orthogonal to Γ.

(b) Verify that Propositions 39.4, 39.5, 39.6 hold in this model, so it is a Hilbert plane. We call it the Poincaré model in the (virtual) circle $x^2 + y^2 = d$. You will need part (a) of the proof of Proposition 39.5.

39.28 In the model of Exercise 39.27, if $\sqrt{d} \notin F$, show that there are no limiting parallel rays on distinct lines, but that any two parallel lines have a common orthogonal.

39.29 For an example of a field F satisfying the conditions of Exercises 39.25–39.28, let K be a Pythagorean ordered field, for example the field of constructible real numbers; let $F = K((z))$ be the field of Laurent series over K (Exercise 18.9); and let $d = z$. Verify that $d > 0$, $\sqrt{d} \notin F$, and that F satisfies condition $(*d)$.

39.30 For an Archimedean example of a field as in Exercise 39.29, let F be the field of all those real numbers that can be expressed using rational numbers and a finite number of operations $+, -, \cdot, \div, a \mapsto \sqrt{1+a^2}$, and $a \mapsto \sqrt{a^2 - \sqrt{2}}$, provided that $a^2 - \sqrt{2} > 0$.

(a) F is a Pythagorean ordered field, $d = \sqrt{2}$ is in F, and F satisfies condition $(*d)$ of Exercise 39.26 for $d = \sqrt{2}$.

(b) Let $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ be the homomorphism that makes $\varphi(\sqrt{2}) = -\sqrt{2}$. Show inductively that φ extends to a homomorphism φ of F to $\mathbb{R}$.

(c) Since $\varphi(\sqrt{2}) < 0$, conclude that $\sqrt{2}$ cannot be a square in F.

39.31 Show that in the Poincaré model in the virtual circle $x^2 + y^2 = \sqrt{2}$ over the field F of Exercise 39.30, not every segment can be the side of an equilateral triangle, as follows.

(a) If $x \in F$ with $0 < x$ and $x^2 < \sqrt{2}$, let AB be the segment from $(0, 0)$ to $(x, 0)$ in the Poincaré model, and show that

$$\mu(AB) = \frac{\sqrt[4]{2} + x}{\sqrt[4]{2} - x}.$$

(b) If there is an equilateral triangle with side AB, let the angle at a vertex be α, and let $t = \tan(\alpha/2)$. Use Exercise 39.5 to show that

$$t = \sqrt{\frac{\sqrt{2} - x^2}{3\sqrt{2} + x^2}} = \frac{1}{3\sqrt{2} + x^2}\sqrt{6 - 2x^2\sqrt{2} - x^4}.$$

(c) Now take a suitable x, such as $x = \sqrt{3} - 1$, and use an argument similar to the previous exercise to show that the corresponding t is not in F. Hence the equilateral triangle with side AB does not exist. *Hint*: For these two exercises, it may be useful to review the techniques used in Exercises 16.10–16.14.

40 Hyperbolic Geometry

In the earlier sections of this chapter we have seen something of the development of neutral geometry and the study of the angle sum of a triangle using Archimedes' axiom. We have also seen the Poincaré model of a non-Euclidean geometry over a field. For the full development of the geometry of Bolyai and Lobachevsky, we need the limiting parallels. The existence of these limiting parallels, which we have seen in the Poincaré model (39.12), does not follow in the axiomatic treatment from what we have done so far (Exercises 39.24, 39.28). Therefore, following Hilbert, we will take the existence of the limiting parallels as an axiom. This axiom is quite strong. It will allow us to develop non-Euclidean geometry independently of Archimedes' axiom. It also allows the construction of an ordered field out of the geometry (Section 41), and a proof that the abstract