# CHAPTER 2
# Hilbert's Axioms

Our purpose in this chapter is to present (with minor modifications) a set of axioms for geometry proposed by Hilbert in 1899. These axioms are sufficient by modern standards of rigor to supply the foundation for Euclid's geometry. This will mean also axiomatizing those arguments where he used intuition, or said nothing. In particular, the axioms for betweenness, based on the work of Pasch in the 1880s, are the most striking innovation in this set of axioms.

Another choice has been to take the SAS theorem as an axiom, and thus bypass the method of superposition. It is possible to go the other route, and use motions of figures as a basic building block of geometry. This is what Hadamard does in his *Leçons de Géométrie Elémentaire* (1901–06), but the result is a step backward in logical clarity, because he never makes precise exactly what kind of motions he is allowing. See, however, Section 17 for a fuller discussion of rigid motions and SAS.

The first benefit of establishing the new system of axioms is, of course, to vindicate Euclid's *Elements*, and thus establish "Euclidean" geometry as a rigorous mathematical discipline. A second benefit is to pose carefully those problems that have bothered geometers for centuries, such as the question of the independence of the parallel postulate. Unless one has an exact understanding of precisely what is assumed and what is not, one risks going around in circles discussing these questions. In the development of our geometry with the new axioms, we will keep the parallel postulate separate and note carefully what depends on it and what does not.

Besides presenting the axioms, this chapter will also contain the first consequences of the axioms, including different proofs of some of Euclid's early propositions, until we have established enough so that Euclid's later results can be deduced without difficulty from the new foundations we have established. In Sections 10, 11, 12, we show how to recover all the results of Euclid, Books I–IV, except for the theory of area, whose proof is postponed until Chapter 5.

# 6 Axioms of Incidence

The axioms of incidence deal with points and lines and their intersections. The points and lines are undefined objects. We simply postulate a set, whose elements are called *points*, together with certain subsets, which we call *lines*. We do not say what the points are, nor which subsets form lines, but we do require that these undefined notions obey certain axioms:

**I1.** For any two distinct points $A, B$, there exists a unique line $l$ containing $A, B$.
**I2.** Every line contains at least two points.
**I3.** There exist three noncollinear points (that is, three points not all contained in a single line).

**Definition**
A set whose elements are called points, together with a set of subsets called lines, satisfying the axioms (I1), (I2), (I3), will be called an *incidence geometry*. If a point $P$ belongs to a line $l$, we will say that $P$ lies on $l$, or that $l$ passes through $P$.

From this modest beginning we cannot expect to get very interesting results, but just to illustrate the process, let us see how one can prove theorems based on these axioms.

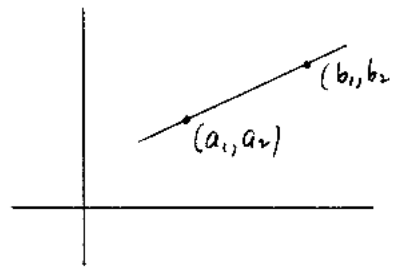
**Proposition 6.1**
*Two distinct lines can have at most one point in common.*

*Proof* Let $l, m$ be two lines, and suppose they both contain the points $A, B$, with $A \neq B$. According to axiom (I1), there is a unique line containing both $A$ and $B$, so $l$ must be equal to $m$.

Note that this fact, which was used by Euclid in the proof of (I.4) with the rather weak excuse that "two lines cannot enclose a space," follows here from the uniqueness part of axiom (I1). This should indicate the importance of stating explicitly the uniqueness of an object, which was rarely done in Euclid's *Elements*.

Now we have an axiom system, consisting of the undefined sets of points and lines, and the axioms (I1)–(I3). A *model* of that axiom system is a realization of the undefined terms in some particular context, such that the axioms are satisfied. You could also think of the model as an *example* of the *incidence geometry* defined above.

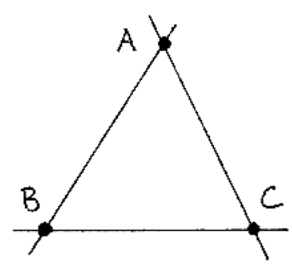
**Example 6.1.1** (The real Cartesian plane).
Here the set of points is the set $\mathbb{R}^2$ of ordered pairs of real numbers. The lines are those subsets of points $P = (x, y)$ that satisfy a linear equation $ax + by + c = 0$ in the variables $x, y$. To verify that the axioms hold, for (I1) think of the "two-point formula" from analytic geometry: Given two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$. They lie on the line




$$y - a_2 = \frac{b_2 - a_2}{b_1 - a_1}(x - a_1)$$

if $a_1 \neq b_1$; if $a_1 = b_1$, they lie on the line $x = a_1$. To verify (I2), take any linear equation involving $y$. Substitute two different values of $x$, and solve for $y$. This gives two points on the line. If the equation did not involve $y$, say $x = c$, take the points $(c, 0)$ and $(c, 1)$. To verify (I3), consider the points $(0,0), (0,1), (1,0)$. One sees easily that there is no linear equation with all three points as solutions.

**Example 6.1.2**
One can also make models out of finite sets. For example, let the set of points be a set of three elements $\{A, B, C\}$, and take for lines the subsets $\{A, B\}, \{A, C\}$, and $\{B, C\}$. We represent this symbolically by the diagram, where the dots represent the elements of the set, and the lines drawn on the page show which subsets are to be taken as lines.




This diagram should be understood as purely symbolic, however, and has nothing to do with a triangle in the ordinary Cartesian plane. The verification of the axioms in this case is trivial.

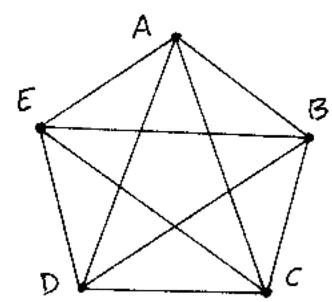
**Definition**
Two distinct lines are *parallel* if they have no points in common. We also say that any line is parallel to itself.

The parallel postulate, in its equivalent form given by Playfair, can be stated as a further axiom about incidence of lines. However, we do not include this axiom in the definition of incidence geometry. Thus we may speak of an incidence geometry that does or does not satisfy Playfair's axiom.

**P.** (Playfair's axiom, also called the parallel axiom). For each point $A$ and each line $l$, there is at most one line containing $A$ that is parallel to $l$.

Note that the real Cartesian plane (6.1.1) satisfies (P), as you know, and the three-point geometry (6.1.2) satisfies (P) vacuously, because there are no distinct parallel lines at all. Next we give an example of an incidence geometry that does not satisfy (P).

**Example 6.1.3**
Let our set consist of five points $A, B, C, D, E$, and let the lines be all subsets of two points. It is easy to see that this geometry satisfies (I1)–(I3). However, it does not satisfy (P), because, for example, $AB$ and $AC$ are two distinct lines through the point $A$ and parallel to the line $DE$.




Remember that the word *parallel* simply means that two lines have no points in common or are equal. It does not say anything about being in the same direction, or being equidistant from each other, or anything else.

We say that two models of an axiom system are *isomorphic* if there exists a 1-to-1 correspondence between their sets of points in such a way that a subset of the first set is a line if and only if the corresponding subset of the second set is a line. For short, we say "the correspondence takes lines into lines." So for example, we see that (6.1.1), (6.1.2), and (6.1.3) are nonisomorphic models of incidence geometry, for the simple reason that their sets of points have different cardinality: There are no 1-to-1 correspondences between any of these sets.

On the other hand, we can show that any model of incidence geometry having just three points is isomorphic to the model given in (6.1.2). Indeed, let $\{1, 2, 3\}$ be a geometry of three points. By (I3), there must be three noncollinear points. Since there are only three points here, we conclude that there is no line containing all three. But by (I1), each subset of two points must be contained in a line. Thus $\{1,2\},\{2,3\}$, and $\{1,3\}$ are lines. Now by (I2), every line contains at least two points, so these are all the possible lines. In other words, the lines are just all subsets of two elements. Since (6.1.2) also has this property, any 1-to-1 correspondence between the sets $\{A,B,C\}$ and $\{1,2,3\}$ will give an isomorphism.

By the way, this proof shows that the isomorphism just found is not unique. There are six choices. This leads to the notion of automorphism.

**Definition**
An *automorphism* of an incidence geometry is an isomorphism of the geometry with itself, that is, it is a 1-to-1 mapping of the set of points onto itself, preserving lines.

Note that the composition of two automorphisms is an automorphism, and so is the inverse of an automorphism. Thus the set of automorphisms forms a group. In the example above, any 1-to-1 mapping of the set of three elements onto itself gives an automorphism of the geometry, so we see that the group of automorphisms of this geometry is the symmetric group on three letters, $S_3$.

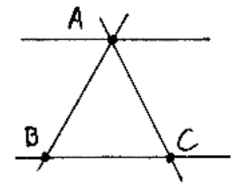
An important question about a set of axioms is whether the axioms are *independent* of each other. That is to say, that no one of them can be proved as a consequence of the others. For if one were a consequence of the others, then we would not need that one as an axiom. To try to prove directly that axiom $A$ is not a consequence of axioms $B,C,D,\ldots$ is usually futile. So instead, we search for a model in which axioms $B,C,D,\ldots$ hold but axiom $A$ does not hold. If such a model exists, then there can be no proof of $A$ as a consequence of $B,C,D,\ldots$, so we conclude that $A$ is independent of the others. This process must be repeated with each individual axiom, to show that each one is independent of all the others. With a long list of axioms this can become tedious and difficult, so we will forgo the process with our full list of axioms. But as an illustration of what is involved, let us show that the axioms (I1), (I2), (I3), and (P) are independent.

**Proposition 6.2**
*The axioms* (I1), (I2), (I3), (P) *are independent of each other.*

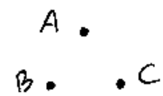
*Proof* We have already seen that (6.1.3) is a model satisfying (I1), (I2), (I3), and not (P). Hence (P) is independent of the others.

For a model satisfying (I1), (I2), (P), and not (I3), take a set of two points and the one line containing both of them. Note that (P) is satisfied trivially, because there are no points not on the line $l$.

For a model satisfying (I1), (I3), (P), and not (I2), take a set of three points $A, B, C$, and for lines take the subsets $\{A, B\}, \{A, C\}, \{B, C\}$, and $\{A\}$. The existence of the one-point line $\{A\}$ contradicts (I2). Yet (P) is still fulfilled, because that one-point line is then the unique line through $A$ parallel to $\{B, C\}$.




For a model satisfying (I2), (I3), (P) and not (I1), just take a set of three points and no lines at all.




While we are discussing axiom systems, there are a few more concepts we should mention. An axiom system is *consistent* if it will never lead to a contradiction. That is to say, if it is not possible to prove from the axioms a statement $A$ and also to prove its negation not $A$. This is obviously a highly desirable property of a system of axioms. We do not want to waste our time proving theorems from a system of axioms that one day may lead to a contradiction. Unfortunately, however, the logician Kurt Gödel has proved that for any reasonably rich set of axioms, it will be impossible to prove the consistency of that system. So we will have to settle for something less, which is *relative consistency*. As soon as you can find a model for your axiom system within some other mathematical theory $T$, it follows that if $T$ is consistent, then also your system of axioms is consistent. For any contradiction that might follow from your axioms would then also appear in the theory $T$, contradicting its consistency. So for example, if you believe in the consistency of the theory of real numbers, then you must accept the consistency of Hilbert's axiom system for geometry, because all of his axioms will hold in the real Cartesian plane. That is the best we can do about the question of consistency.

Another question about a system of axioms is whether it is *categorical*. This means, does it describe a unique mathematical object? Or in other words, is there a *unique* model (up to isomorphism) for the system of axioms? In fact, it will turn out that if we take the entire list of Hilbert's axioms, including the parallel axiom (P) and Dedekind's axiom (D), the system will be categorical, and the unique model will be the real Cartesian plane. (We will prove this result later (21.3).) Also, if we take all of Hilbert's axioms, together with (D) and the hyperbolic axiom (L) (see Section 40), we will have another categorical system, whose unique model is the non-Euclidean Poincaré model over the real numbers (Exercise 43.2).

However, from the point of view of this book, it is more interesting to have an axiom system that is not categorical, and then to investigate the different possible geometries that can arise. Therefore, we will almost never assume Dedekind's axiom (D), and we will only sometimes assume Archimedes' axiom (A), or the parallel axiom (P).

Finally, one can ask whether the axiom system is *complete*, which means, can every statement that is true in every model of the axiom system be proved as a consequence of the axioms? Again, Gödel has shown that any axiomatic system of reasonable richness cannot be complete. For a fuller discussion of these questions, see Chapter 51 of Kline (1972) on the foundations of mathematics.

# Exercises

6.1 Describe all possible incidence geometries on a set of four points, up to isomorphism. Which ones satisfy (P)?

6.2 The *Cartesian plane over a field F*. Let $F$ be any field. Take the set $F^2$ of ordered pairs of elements of the field $F$ to be the set of *points*. Define *lines* to be those subsets defined by linear equations, as in Example 6.1.1. Verify that the axioms (I1), (I2), (I3), and (P) hold in this model. (See Section 14 for more about Cartesian planes over fields.)

6.3 A *projective plane* is a set of points and subsets called lines that satisfy the following four axioms:

**P1.** Any two distinct points lie on a unique line.

**P2.** Any two lines meet in at least one point.

**P3.** Every line contains at least three points.

**P4.** There exist three noncollinear points.

Note that these axioms imply (I1)–(I3), so that any projective plane is also an incidence geometry. Show the following:

(a) Every projective plane has at least seven points, and there exists a model of a projective plane having exactly seven points.

(b) The projective plane of seven points is unique up to isomorphism.

(c) The axioms (P1), (P2), (P3), (P4) are independent.

6.4 Let $F$ be a field, and let $V = F^3$ be a three-dimensional vector space over $F$. Let $\Pi$ be the set of 1-dimensional subspaces of $V$. We will call the elements of $\Pi$ "points." So a "point" is a 1-dimensional subspace $P \subseteq V$. If $W \subseteq V$ is a 2-dimensional subspace of $V$, then the set of all "points" contained in $W$ will be called a "line." Show that the set $\Pi$ of "points" and the subsets of "lines" forms a projective plane (Exercise 6.3).

6.5 An *affine plane* is a set of points and subsets called lines satisfying (I1), (I2), (I3), and the following stronger form of Playfair's axiom.

**P′.** For every line $l$, and every point $A$, there exists a unique line $m$ containing $A$ and parallel to $l$.

(a) Show that any two lines in an affine plane have the same number of points (i.e., there exists a 1-to-1 correspondence between the points of the two lines).

(b) If an affine plane has a line with exactly $n$ points, then the total number of points in the plane is $n^2$.

(c) If $F$ is any field, show that the *Cartesian plane* over $F$ (Exercise 6.2) is a model of an affine plane.

(d) Show that there exist affine planes with 4, 9, 16, or 25 points. (The nonexistence of an affine plane with 36 points is a difficult result of Euler.)

6.6 In an incidence geometry, consider the relationship of parallelism, "$l$ is parallel to $m$," on the set of lines.

(a) Give an example to show that this need not be an equivalence relation.

(b) If we assume the parallel axiom (P), then parallelism is an equivalence relation.

(c) Conversely, if parallelism is an equivalence relation in a given incidence geometry, then (P) must hold in that geometry.

6.7 Let $\Pi$ be an affine plane (Exercise 6.5). A *pencil* of parallel lines is the set of all the lines parallel to a given line (including that line itself). We call each pencil of parallel lines an "ideal point," or a "point at infinity," and we say that an ideal point "lies on" each of the lines in the pencil. Now let $\Pi'$ be the enlarged set consisting of $\Pi$ together with all these new ideal points. A *line* of $\Pi'$ will be the subset consisting of a line of $\Pi$ plus its unique ideal point, or a new line, called the "line at infinity," consisting of all the ideal points.

(a) Show that this new set $\Pi'$ with subsets of lines as just defined forms a projective plane (Exercise 6.3).

(b) If $\Pi$ is the Cartesian plane over a field $F$ (Exercise 6.2), show that the associated projective plane $\Pi'$ is isomorphic to the projective plane constructed in Exercise 6.4.

6.8 If there are $n+1$ points on one line in a projective plane $\Pi$, then the total number of points in $\Pi$ is $n^2+n+1$.

6.9 Kirkman's schoolgirl problem (1850) is as follows: In a certain school there are 15 girls. It is desired to make a seven-day schedule such that each day the girls can walk in the garden in five groups of three, in such a way that each girl will be in the same group with each other girl just once in the week. How should the groups be formed each day?

To make this into a geometry problem, think of the girls as points, think of the groups of three as lines, and think of each day as describing a set of five lines, which we call a pencil. Now consider a *Kirkman geometry*: a set, whose elements we call *points*, together with certain subsets we call *lines*, and certain sets of lines we call *pencils*, satisfying the following axioms:

**K1.** Two distinct points lie on a unique line.

**K2.** All lines contain the same number of points.

**K3.** There exist three noncollinear points.

**K4.** Each line is contained in a unique pencil.

**K5.** Each pencil consists of a set of parallel lines whose union is the whole set of points.

(a) Show that any affine plane gives a Kirkman geometry where we take the pencils to be the set of all lines parallel to a given line. (Hence by Exercise 6.5 there exist Kirkman geometries with 4, 9, 16, 25 points.)

(b) Show that any Kirkman geometry with 15 points gives a solution of the original schoolgirl problem.

(c) Find a solution for the original problem. (There are many inequivalent solutions to this problem.)

6.10 In a finite incidence geometry, the number of lines is greater than or equal to the number of points.

# 7 Axioms of Betweenness

In this section we present axioms to make precise the notions of betweenness (when one point is in between two others), on which is based the notion of sidedness (when a point is on one side of a line or the other), the concepts of inside and outside, and also the concepts of *order*, when one segment or angle is bigger than another. We have seen the importance of these concepts in reading Euclid's geometry, and we have also seen the dangers of using these concepts intuitively, without making their meaning precise. So these axioms form an important part of our new foundations for geometry. At the same time, these axioms and their consequences may seem difficult to understand for many readers, not because the mathematical concepts are technically difficult, but because the notions of order and separation are so deeply ingrained in our daily experience of life that it is difficult to let go of our intuitions and replace them with axioms. It is an exercise in forgetting what we already know from our inner nature, and then reconstituting it with an open mind as an external logical structure.

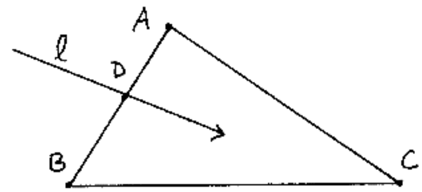
Throughout this section we presuppose axioms (I1)–(I3) of an incidence geometry. The geometrical notions of betweenness, separation, sidedness, and order will all be based on a single undefined relation, subject to four axioms. We postulate a *relation* between sets of three points $A, B, C$, called "$B$ is between $A$ and $C$." This relation is subject to the following axioms.

**B1.** If $B$ is between $A$ and $C$, (written $A * B * C$), then $A, B, C$ are three distinct points on a line, and also $C * B * A$.

**B2.** For any two distinct points $A, B$, there exists a point $C$ such that $A * B * C$.

**B3.** Given three distinct points on a line, one and only one of them is between the other two.

**B4.** (Pasch). Let $A, B, C$ be three noncollinear points, and let $l$ be a line not containing any of $A, B, C$. If $l$ contains a point $D$ lying between $A$ and $B$, then it must also contain either a point lying between $A$ and $C$ or a point lying between $B$ and $C$, but not both.




**Definition**
If $A$ and $B$ are distinct points, we define the *line segment* $\overline{AB}$ to be the set consisting of the points $A, B$ and all points lying between $A$ and $B$. We define a *triangle* to be the union of the three line segments $\overline{AB}, \overline{BC}$, and $\overline{AC}$ whenever $A, B, C$ are three noncollinear points. The points $A, B, C$ are the *vertices* of the triangle, and the segments $\overline{AB}, \overline{BC}, \overline{AC}$ are the *sides* of the triangle.

**Note:** The segments $\overline{AB}$ and $\overline{BA}$ are the same sets, because of axiom (B1). The *endpoints* $A, B$ of the segment $\overline{AB}$ are uniquely determined by the segment $\overline{AB}$ (Exercise 7.2). The vertices $A, B, C$, and the sides $\overline{AB}, \overline{AC}, \overline{BC}$ of a triangle $ABC$ are uniquely determined by the triangle (Exercise 7.3).

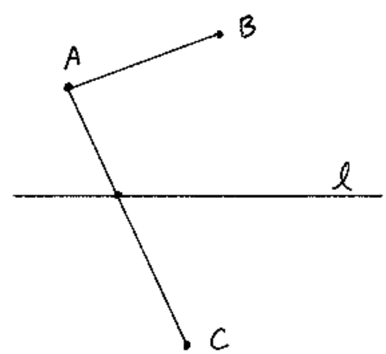
With this terminology, we can rephrase (B4) as follows: If a line $l$ that does not contain any of the vertices $A, B, C$ of a triangle meets one side $\overline{AB}$, then it must meet one of the other sides $\overline{AC}$ or $\overline{BC}$, but not both.

From these axioms together with the axioms of incidence (I1)–(I3) we will deduce results about the separation of the plane by a line, and the separation of a line by a point.

**Proposition 7.1** (Plane separation)
*Let $l$ be any line. Then the set of points not lying on $l$ can be divided into two nonempty subsets $S_1, S_2$ with the following properties:*

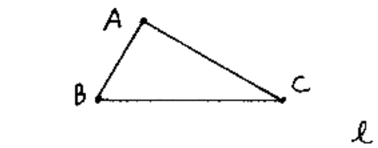
(a) *Two points $A, B$ not on $l$ belong to the same set ($S_1$ or $S_2$) if and only if the segment $\overline{AB}$ does not intersect $l$.*
(b) *Two points $A, C$ not on $l$ belong to the opposite sets (one in $S_1$, the other in $S_2$) if and only if the segment $\overline{AC}$ intersects $l$ in a point.*

*We will refer to the sets $S_1, S_2$ as the two* sides *of $l$, and we will say "$A$ and $B$ are on the* same side *of $l$," or "$A$ and $C$ are on* opposite sides *of $l$."*

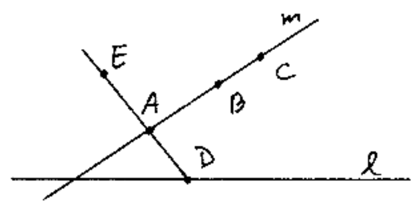


*Proof* We start by defining a relation $\sim$ among points not on $l$. We will say $A \sim B$ if either $A = B$ or if the segment $\overline{AB}$ does not meet $l$. Our first step is to show that $\sim$ is an equivalence relation. Clearly, $A \sim A$ by definition, and $A \sim B$ implies $B \sim A$ because the set $\overline{AB}$ does not depend on the order in which we write $A$ and $B$. The nontrivial step is to show the relation is transitive: If $A \sim B$ and $B \sim C$, we must show $A \sim C$.

*Case 1* Suppose $A, B, C$ are not collinear. Then we consider the triangle $ABC$. Since $A \sim B$, $l$ does not meet $\overline{AB}$. Since $B \sim C$, $l$ does not meet $\overline{BC}$. Now by Pasch's axiom (B4), it follows that $l$ does not meet $\overline{AC}$. Hence $A \sim C$.




*Case 2* Suppose $A, B, C$ lie on a line $m$. Since $A, B, C$ do not lie on $l$, the line $m$ is different from $l$. Therefore $l$ and $m$ can meet in at most one point (6.1). But by (I2) every line has at least two points. Therefore, there exists a point $D$ on $l, D$ not lying on $m$.




Now apply axiom (B2) to find a point $E$ such that $D * A * E$. Then $D, A, E$ are collinear (B1); hence $E$ is not on $l$, since $A$ is not on $l$, and the line $DAE$ already meets $l$ at the point $D$. Furthermore, the segment $\overline{AE}$ cannot meet $l$. For if it did, the intersection point would be the unique point in which the line $AE$ meets $l$, namely $D$. In that case $D$ would be between $A$ and $E$. But we constructed $E$ so that $D * A * E$, so by (B3), $D$ cannot lie between $A$ and $E$. Thus $\overline{AE} \cap l = \emptyset$, so $A \sim E$. Note also that $E$ does not lie on the line $m$, because if $E$ were on $m$, then the line $AE$ would be equal to $m$, so $D$ would lie on $m$, contrary to our choice of $D$. Therefore, $A, B, E$ are three noncollinear points. Then by Case 1 proved above, from $A \sim E$ and $A \sim B$ we conclude $B \sim E$. By Case 1 again, from $B \sim E$ and $B \sim C$ we conclude $C \sim E$. Applying Case 1 a third time to the three noncollinear points $A, C, E$, from $A \sim E$ and $C \sim E$ we conclude $A \sim C$ as required.

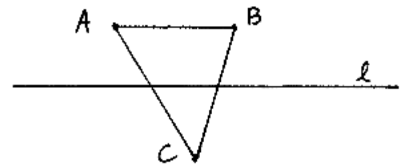
Thus we have proved that $\sim$ is an equivalence relation. An equivalence relation on a set divides that set into a disjoint union of equivalence classes, and these equivalence classes will satisfy property (a) by definition. To complete the proof it will be sufficient to show that there are exactly two equivalence classes $S_1, S_2$ for the relation $\sim$. Then to say that $\overline{AC}$ meets $l$, which is equivalent to $A \not\sim C$, will be the same as saying that $A, C$ belong to the opposite sets.

By (I3) there exists a point not on $l$, so there is at least one equivalence class $S_1$. Given $A \in S_1$, let $D$ be any point on $l$, and choose by (B2) a point $C$ such that

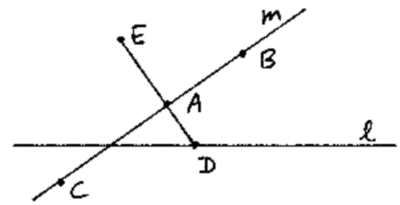
$A * D * C$. Then $A$ and $C$ do not satisfy $\sim$, so there must be at least two equivalence classes $S_1$ and $S_2$.

The last step is to show that there are at most two equivalence classes. To do this, we will show that if $A \not\sim C$ and $B \not\sim C$, then $A \sim B$.

*Case 1* If $A, B, C$ are not collinear, we consider the triangle $ABC$. From $A \not\sim C$ we conclude that $\overline{AC}$ meets $l$. From $B \not\sim C$ we conclude that $\overline{BC}$ meets $l$. Now by Pasch's axiom (B4) it follows that $\overline{AB}$ does not meet $l$. So $A \sim B$ as required.



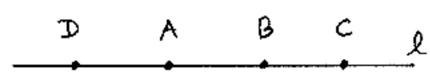

*Case 2* Suppose $A, B, C$ lie on a line $m$. As in Case 2 of the first part of the proof above, choose a point $D$ on $l$, not on $m$, and use (B2) to get a point $E$ with $D * A * E$. Then $A \sim E$ as we showed above.



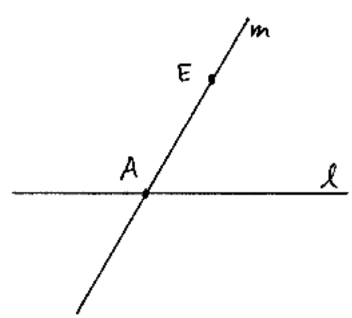

Now, $A \not\sim C$ by hypothesis, and $A \sim E$, so we conclude that $C \not\sim E$, since $\sim$ is an equivalence relation (if $C \sim E$, then $A \sim C$ by transitivity: contradiction). Looking at the three noncollinear points $B, C, E$, from $E \not\sim C$ and $B \not\sim C$ we conclude using Case 1 that $B \sim E$. But also $A \sim E$, so by transitivity, $A \sim B$ as required.

**Proposition 7.2** (Line separation)
*Let $A$ be a point on a line $l$. Then the set of points of $l$ not equal to $A$ can be divided into two nonempty subsets $S_1, S_2$, the two sides of $A$ on $l$, such that*

(a) *$B$, $C$ are on the same side of $A$ if and only if $A$ is not in the segment $\overline{BC}$;*
(b) *$B$, $D$ are on opposite sides of $A$ if and only if $A$ belongs to the segment $\overline{BD}$.*



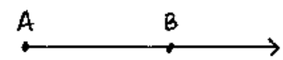

*Proof* Given the line $l$ and a point $A$ on $l$, we know from (I3) that there exists a point $E$ not on $l$. Let $m$ be the line containing $A$ and $E$. Apply (7.1) to the line $m$. If $m$ has two sides $S_1', S_2'$, we define $S_1$ and $S_2$ to be the intersections of $S_1'$ and $S_2'$ with $l$. Then properties (a) and (b) follow immediately from the previous proposition.

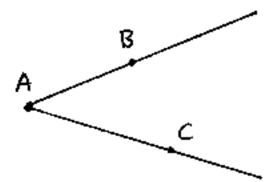


The only mildly nontrivial part is to show that $S_1$ and $S_2$ are nonempty. By (I2), there is a point $B$ on $l$ different from $A$. And by (B2) there exists a point $D$ such that $B * A * D$. Then $D$ will be on the opposite side of $A$ from $B$, and will lie on $l$, so both sides are nonempty.

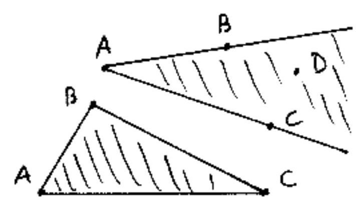
Now that we have some basic results on betweenness, we can define rays and angles.

**Definition**
Given two distinct points $A, B$, the *ray* $\overrightarrow{AB}$ is the set consisting of $A$, plus all points on the line $AB$ that are on the same side of $A$ as $B$. The point $A$ is the *origin*, or *vertex*, of the ray. An *angle* is the union of two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ originating at the same point, its *vertex*, and not lying on the same line. (Thus there is no "zero angle," and there is no "straight angle" (180°).) Note that the vertex of a ray or angle is uniquely determined by the ray or angle (proof similar to Exercises 7.2, 7.3).

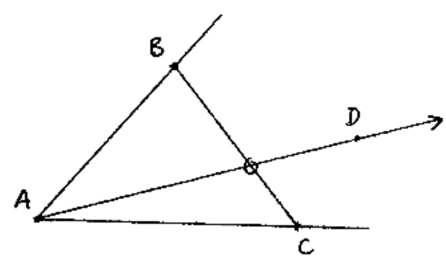


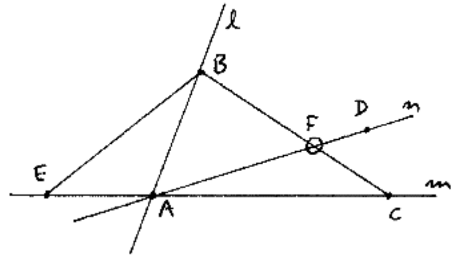

The *inside* (or *interior*) of an angle $\angle BAC$ consists of all points $D$ such that $D$ and $C$ are on the same side of the line $AB$, and $D$ and $B$ are on the same side of the line $AC$. If $ABC$ is a triangle, the *inside* (or *interior*) of the triangle $ABC$ is the set of points that are simultaneously in the insides of the three angles $\angle BAC, \angle ABC, \angle ACB$.




**Proposition 7.3** (Crossbar theorem)
*Let $\angle BAC$ be an angle, and let $D$ be a point in the interior of the angle. Then the ray $\overrightarrow{AD}$ must meet the segment $\overline{BC}$.*




*Proof* This is similar to Pasch's axiom (B4), except that we must consider a line $AD$ that passes through one vertex of the triangle $ABC$. We will prove it with Pasch's axiom and several applications of the plane separation theorem (7.1).




Let us label the lines $AB = l$, $AC = m$, $AD = n$. Let $E$ be a point on $m$ such that $E * A * C$ (B2). We will apply Pasch's axiom (B4) to the triangle $BCE$ and the line $n$. By construction $n$ meets the side $CE$ at $A$. Also, $n$ cannot contain $B$, because it meets the line $l$ at $A$. We will show that $n$ does not meet the segment $\overline{BE}$, so as to conclude by (B4) that it must meet the segment $\overline{BC}$.

So we consider the segment $\overline{BE}$. This segment meets the line $l$ only at $B$, so all points of the segment, except $B$, are on the same side of $l$. By construction, $C$ is on the opposite side of $l$ from $E$, so by (7.1) all points of $\overline{BE}$, except $B$, are on the opposite side of $l$ from $C$. On the other hand, since $D$ is in the interior of the angle $\angle BAC$, all the points of the ray $\overrightarrow{AD}$, except $A$, are on the same side of $l$ as $C$. Thus the segment $\overline{BE}$ does not meet the ray $\overrightarrow{AD}$.

A similar reasoning using the line $m$ shows that all points of the segment $\overline{BE}$, except $E$, lie on the same side of $m$ as $B$, while the points of the ray of $n$, opposite the ray $\overrightarrow{AD}$, lie on the other side of $m$. Hence the segment $\overline{BE}$ cannot meet the opposite ray to $\overrightarrow{AD}$. Together with the previous step, this shows that the segment $\overline{BE}$ does not meet the line $n$. We conclude by (B4) that $n$ meets the segment $\overline{BC}$ in a point $F$.

It remains only to show that $F$ is on the ray $\overrightarrow{AD}$ of the line $n$. Indeed, $B$ and $F$ are on the same side of $m$, and also $B$ and $D$ are on the same side of $m$, so (7.1) $D$ and $F$ are on the same side of $m$, and so $D$ and $F$ are on the same side of $A$ on the line $n$. In other words, $F$ lies on the ray $\overrightarrow{AD}$.

**Example 7.3.1**
We will show that the real Cartesian plane (6.1.1), with the "usual" notion of betweenness, provides a model for the axioms (B1)–(B4).

First, we must make precise what we mean by the usual notion of betweenness. For three distinct real numbers $a, b, c \in \mathbb{R}$, let us define $a * b * c$ if either $a < b < c$ or $c < b < a$. Then it is easy to see that this defines a notion of betweenness on the real line $\mathbb{R}$ that satisfies (B1), (B2), and (B3).

If $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ are three points in $\mathbb{R}^2$, let us define $A * B * C$ to mean that $A, B, C$ are three distinct points on a line, and that either $a_1 * b_1 * c_1$ or $a_2 * b_2 * c_2$, or both. In fact, if either the $x$- or the $y$-coordinates satisfy this betweenness condition, and if the line is neither horizontal nor vertical, then the other coordinates will also satisfy it, because the points lie on a line, and linear operations (addition, multiplication) of real numbers either preserve or reverse inequalities. Thus linear operations preserve betweenness. So we can verify easily that this notion of betweenness in $\mathbb{R}^2$ satisfies (B1), (B2), and (B3).

For (B4), let $l$ be a line, and let $A, B, C$ be three noncollinear points not on $l$. The line $l$ is defined by some linear equation $ax + by + c = 0$. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be the linear function defined by $\varphi(x, y) = ax + by + c$. Since $\varphi$ is a linear function, $\varphi$ will preserve betweenness. For example, if $l$ meets the segment $\overline{AB}$, then 0 will lie between $\varphi(A)$ and $\varphi(B)$. In other words, one of $\varphi(A), \varphi(B)$ will be positive and the other negative. Suppose $\varphi(A) > 0$ and $\varphi(B) < 0$. Consider $\varphi(C)$. If $\varphi(C) > 0$, then $l$ will meet $\overline{BC}$ but not $\overline{AC}$. If $\varphi(C) < 0$, then $l$ will meet $\overline{AC}$ but not $\overline{BC}$. This proves (B4).

# Exercises

7.1 Using the axioms of incidence and betweenness and the line separation property, show that sets of four points $A, B, C, D$ on a line behave as we expect them to with respect to betweenness. Namely, show that

(a) $A * B * C$ and $B * C * D$ imply $A * B * D$ and $A * C * D$.

(b) $A * B * D$ and $B * C * D$ imply $A * B * C$ and $A * C * D$.

7.2 Given a segment $\overline{AB}$, show that there do not exist points $C, D \in \overline{AB}$ such that $C * A * D$. Hence show that the endpoints $A, B$ of the segment are uniquely determined by the segment.

7.3 Given a triangle $ABC$, show that the sides $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ and the vertices $A, B, C$ are uniquely determined by the triangle. *Hint*: Consider the different ways in which a line can intersect the triangle.

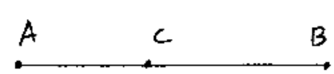
7.4 Using (I1)–(I3) and (B1)–(B4) and their consequences, show that every line has infinitely many distinct points.

7.5 Show that the line separation property (Proposition 7.2) is not a consequence of (B1), (B2), (B3), by constructing a model of betweenness for the set of points on a line, which satisfies (B1), (B2), (B3) but has only finitely many points. (Then by Exercise 7.4, line separation must fail in this model.) For example, in the ring $\{0, 1, 2, 3, 4\}$ of integers (mod 5), define $a * b * c$ if $b = \frac{1}{2}(a + c)$.

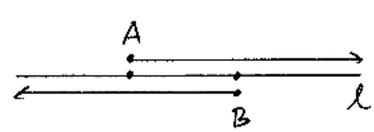
7.6 Prove directly from the axioms (I1)–(I3) and (B1)–(B4) that for any two distinct points $A, B$, there exists a point $C$ with $A * C * B$. (*Hint*: Use (B2) and (B4) to construct a line that will be forced to meet the segment $\overline{AB}$ but does not contain $A$ or $B$.)

7.7 Be careful not to assume without proof statements that may appear obvious. For example, prove the following:

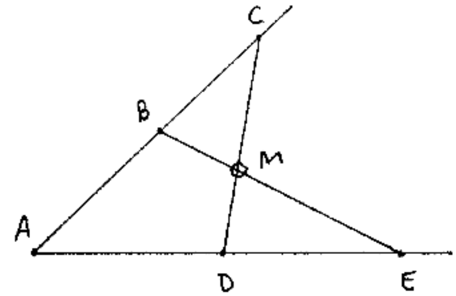
(a) Let $A, B, C$ be three points on a line with $C$ in between $A$ and $B$. Then show that $\overline{AC} \cup \overline{CB} = \overline{AB}$ and $\overline{AC} \cap \overline{CB} = \{C\}$.




(b) Suppose we are given two distinct points $A, B$ on a line $l$. Show that $\overrightarrow{AB} \cup \overrightarrow{BA} = l$ and $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

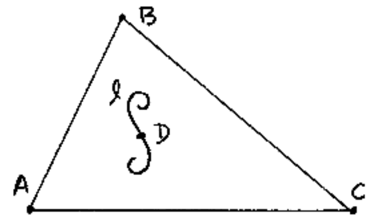


7.8 Assume $A * B * C$ on one line, and $A * D * E$ on another line. Show that the segment $\overline{BE}$ must meet the segment $\overline{CD}$ at a point $M$.




7.9 Show that the interior of a triangle is nonempty.

7.10 Suppose that a line $l$ contains a point $D$ that is in the inside of a triangle $ABC$. Then show that the line $l$ must meet (at least) one of the sides of the triangle.



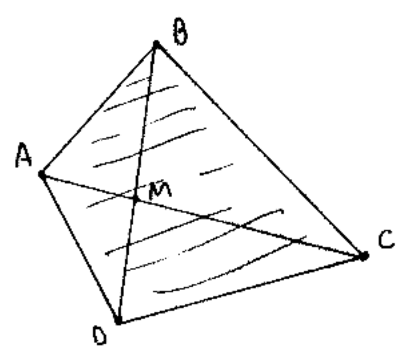

7.11 A set $U$ of points in the plane is a *convex set* if whenever $A, B$ are distinct points in $U$, then the segment $\overline{AB}$ is entirely contained in $U$. Show that the inside of a triangle is a convex set.

7.12 A subset $W$ of the plane is *segment-connected* if given any two points $A, B \in W$, there is a finite sequence of points $A = A_1, A_2, \ldots, A_n = B$ such that for each $i = 1, 2, \ldots, n-1$, the segment $\overline{A_i A_{i+1}}$ is entirely contained within $W$.

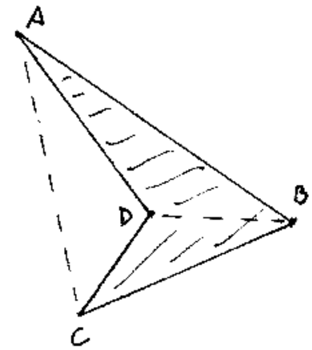
If $ABC$ is a triangle, show that the *exterior* of the triangle, that is, the set of all points of the plane lying neither on the triangle nor in its interior, is a segment-connected set.

7.13 Let $A, B, C, D$ be four points, no three collinear, and assume that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ have no intersections except at their endpoints. Then the union of these four segments is a *simple closed quadrilateral*. The segments $\overline{AC}$ and $\overline{BD}$ are the *diagonals* of the quadrilateral. There are two cases to consider.

*Case 1* $\overline{AC}$ and $\overline{BD}$ meet at a point $M$. In this case, show that for each pair of consecutive vertices (e.g., $A, B$), the remaining two vertices $(C, D)$ are on the same side of the line $AB$. Define the *interior* of the quadrilateral to be the set of points $X$ such that for each side (e.g., $\overline{AB}$), $X$ is on the same side of the line $AB$ as the remaining vertices $(C, D)$. Show that the interior is a convex set.



*Case 2* $\overline{AC}$ and $\overline{BD}$ do not meet. In this case, show that one of the diagonals ($\overline{AC}$ in the picture) has the property that the other two vertices $B, D$ are on the same side of the line $AC$, while the other diagonal $\overline{BD}$ has the property that $A$ and $C$ are on the opposite sides of the line $BD$. Define the *interior* of the quadrilateral to be the union of the interiors of the triangles $ABD$ and $CDB$ plus the interior of the segment $\overline{BD}$. Show in this case that the interior is a segment-connected set, but is not convex.




(For a generalization to $n$-sided figures, see Exercise 22.11.)

7.14 (*Linear ordering*) Given a finite set of distinct points on a line, it is possible to label them $A_1, A_2, \ldots, A_n$ in such a way that $A_i * A_j * A_k$ if and only if either $i < j < k$ or $k < j < i$.

7.15 Suppose that lines $a, b, c$ through the vertices $A, B, C$ of a triangle meet at three points inside the triangle. Label them

$$X = a \cdot c,$$

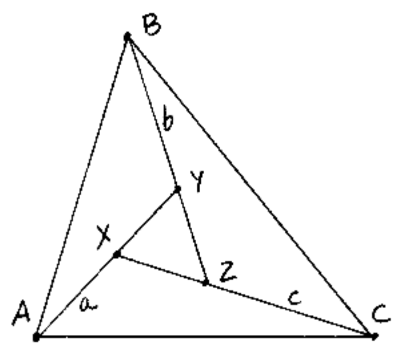
$$Y = a \cdot b,$$

$$Z = b \cdot c.$$

Show that one of the two following arrangements must occur:

(i) $A * X * Y$ and $B * Y * Z$ and $C * Z * X$ (shown in diagram), or

(ii) $A * Y * X$ and $B * Z * Y$ and $C * X * Z$.

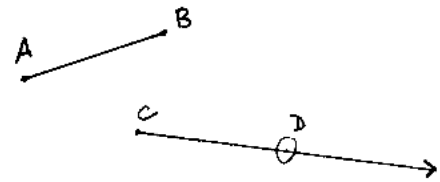



# 8 Axioms of Congruence for Line Segments

To the earlier undefined notions of point, line, and betweenness, and to the earlier axioms (I1)–(I3), (B1)–(B4), we now add an undefined notion of congruence for line segments, and further axioms (C1)–(C3) regarding this notion. This congruence is what Euclid called equality of segments. We postulate an undefined notion of *congruence*, which is a relation between two line segments $\overline{AB}$

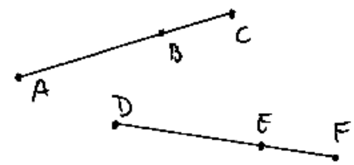
and $\overline{CD}$, written $AB \cong CD$. For simplicity we will drop the bars over $AB$ in the notation for a line segment, so long as no confusion can result. This undefined notion is subject to the following three axioms

**C1.** Given a line segment $AB$, and given a ray $r$ originating at a point $C$, there exists a unique point $D$ on the ray $r$ such that $AB \cong CD$.




**C2.** If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

**C3.** (Addition). Given three points $A, B$, $C$ on a line satisfying $A * B * C$, and three further points $D, E, F$ on a line satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.




Let us observe how these axioms are similar to Euclid's postulates and how they are different. First of all, while Euclid phrases some of his postulates in terms of constructions ("to draw a line through any two given points," and "to draw a circle with any given center and radius"), Hilbert's axioms are existential. (I1) says for any two distinct points there exists a unique line containing them. And here, in axiom (C1), it is the existence of the point $D$ (corresponding to Euclid's construction (I.3)) that is taken as an axiom. Hilbert does not make use of ruler and compass constructions. In their place he puts the axiom (C1) of the existence of line segments and later (C4) the existence of angles. If you like, you can think of (C1) and (C4) as being tools, a "transporter of segments" and a "transporter of angles," and consider some of Hilbert's theorems as constructions with these tools.

The second congruence axiom (C2) corresponds to Euclid's common notion that "things equal to the same thing are equal to each other." This is one part of the modern notion of an equivalence relation, so to be comfortable in using congruence, let us show that it is indeed an equivalence relation.

**Proposition 8.1**
*Congruence is an equivalence relation on the set of line segments.*

*Proof* To be an equivalence relation, congruence must satisfy three properties.

(1) *Reflexivity*: Every segment is congruent to itself. This is explicitly stated in (C2). And by the way, this corresponds to Euclid's fourth common notion that "things which coincide with each other are equal to each other."

# ΕΥΚΛΕΙΔΟΥ ΣΤΟΙ-ΧΕΙΟΝ ΠΡΩΤΟΝ.

## EVCLIDIS ELEMENTORVM GEO-*metricorum liber primus.*

Est hic liber primus totus ferè elementarius, non tantum ad reliquos sequentes huius Operis libros, sed etiam ad aliorum Geometrarum scripta intelligenda necessarius. Nam in hoc libro communium uocabulorum, quę subinde in geometria uersanti occurrunt, definitiones continentur. Pręceptiones deinde ducendi perpendicularem, quomodo item Trilaterę figurę, secundum latera uel angulos diuersæ, & Quadrilateræ, formari debeant. Figura item aliqua proposita, quomodo illa in alterius formæ figuram permutanda sit, præceptiones, ut diximus, traduntur. Cum igitur talia doceantur, & plura etiam alia, quàm hoc loco commemorare uoluimus, facile erit cuiuis, non solum quàm sit necessarius, sed etiam ad reliqua perdiscenda liber iste quàm utilis, perspicere.

### ΟΡΟΙ.

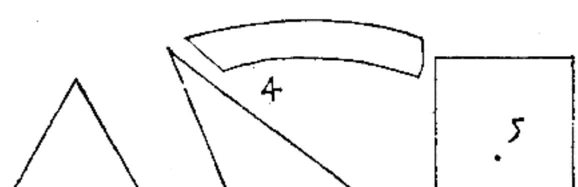
*Σημεῖόν ἐστιν, οὗ μέρος οὐθέν. Γραμμὴ δέ, μῆκος ἀπλατές. Γραμμῆς δὲ πέρατα, σημεῖα. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.*

### DEFINITIONES.

Punctum est, cuius pars nulla. 2. Linea uerò, longitudo latitudinis expers. Lineę autem termini puncta. 3. Recta linea est, quę ęquabiliter inter sua puncta iacet.

Prima defini.


*Ἐπιφάνειά ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει. Ἐπιφανείας δὲ πέρατα, γραμμαί. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.*




4. Superficies est, quę longitudinem & latitudinem tantum habet.

K 3 Super-

Plate IV. The beginning of the *Elements* in the edition of Scheubel (1550), showing the Greek text of the first few definitions.

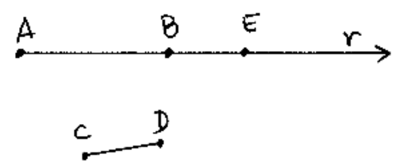
(2) *Symmetry*: If $AB \cong CD$, then $CD \cong AB$. This is a consequence of (C2): Given $AB \cong CD$, and writing $AB \cong AB$ by reflexivity, we conclude from (C2) that $CD \cong AB$.

(3) *Transitivity*: If $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$. This follows by first using symmetry to show $CD \cong AB$, and then applying (C2). Notice that Hilbert's formulation of (C2) was a clever way of including symmetry and transitivity in a single statement.

The third axiom (C3) is the counterpart of Euclid's second common notion, that "equals added to equals are equal." Let us amplify this by making a precise definition of the sum of two segments, and then showing that sums of congruent segments are congruent.

**Definition**

Let $AB$ and $CD$ be two given segments. Choose an ordering $A, B$ of the endpoints of $AB$. Let $r$ be the ray on the line $l = AB$ consisting of $B$ and all the points of $l$ on the other side of $B$ from $A$. Let $E$ be the unique point on the ray $r$ (whose existence is given by (C1)) such that $CD \cong BE$.




We then define the segment $AE$ to be the *sum* of the segments $AB$ and $CD$, depending on the order $A, B$, and we will write $AE = AB + CD$.

**Proposition 8.2** (Congruence of sums)

*Suppose we are given segments $AB \cong A'B'$ and $CD \cong C'D'$. Then $AB + CD \cong A'B' + C'D'$.*

*Proof* Let $E'$ be the point on the line $A'B'$ defining the sum $A'E' = A'B' + C'D'$. Then $A * B * E$ by construction of the sum $AB + CD$, because $E$ is on the ray from $B$ opposite $A$. Similarly, $A' * B' * E'$. We have $AB \cong A'B'$ by hypothesis. Furthermore, we have $CD \cong C'D'$ by hypothesis, and $CD \cong BE$ and $C'D' \cong B'E'$ by construction of $E$ and $E'$. From (8.1) we know that congruence is an equivalence relation, so $BE \cong B'E'$. Now by (C3) it follows that $AE \cong A'E'$ as required.

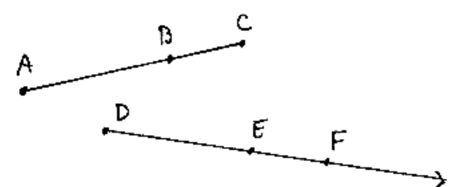
**Note:** Since the segment $AB$ is equal to the segment $BA$, it follows in particular that the sum of two segments is independent of the order $A, B$ chosen, up to congruence. Thus addition is well-defined on congruence equivalence classes of line segments. So we can speak of addition of line segments or congruent segments without any danger (cf. also Exercise 8.1, which shows that addition of line segments is associative and commutative, up to congruence). Later (Section

19) we will also define multiplication of segments and so create a field of segment arithmetic.

Euclid's third common notion is that "equals subtracted from equals are equal." Bearing in mind that subtraction does not always make sense, we can interpret this common notion as follows.

**Proposition 8.3**

*Given three points $A, B, C$ on a line such that $A * B * C$, and given points $E, F$ on a ray originating from a point $D$, suppose that $AB \cong DE$ and $AC \cong DF$. Then $E$ will be between $D$ and $F$, and $BC \cong EF$. (We regard $BC$ as the difference of $AC$ and $AB$.)*



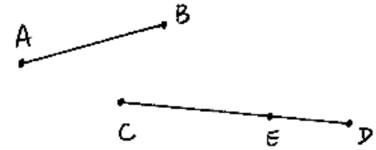

*Proof* Let $F'$ be the unique point on the ray originating at $E$, opposite to $D$, such that $BC \cong EF'$. Then from $AB \cong DE$ and $BC \cong EF'$ we conclude by (C3) that $AC \cong DF'$. But $F$ and $F'$ are on the same ray from $D$ (check!) and also $AC \cong DF$, so by (C2) and the uniqueness part of (C1), we conclude that $F = F'$. It follows that $D * E * F$ and $BC \cong EF$, as required.

Note the role played by the uniqueness part of (C1) in the above proof. We can regard this uniqueness as corresponding to Euclid's fifth common notion, "the whole is greater than the part." Indeed, this statement could be interpreted as meaning, if $A * B * C$, then $AB$ cannot be congruent to $AC$. And indeed, this follows from (C1), because $B$ and $C$ are on the same ray from $A$, and if $AB \cong AC$, then $B$ and $C$ would have to be equal by (C1).

So we see that Euclid's common notions, at least in the case of congruence of line segments, can be deduced as consequences of the new axioms (C1)–(C3). Another notion used by Euclid without definition is the notion of inequality of line segments. Let us see how we can define the notions of greater and lesser also using our axioms.

**Definition**

Let $AB$ and $CD$ be given line segments. We will say that $AB$ is *less than* $CD$, written $AB < CD$, if there exists a point $E$ in between $C$ and $D$ such that $AB \cong CE$. In this case we say also that $CD$ is *greater than* $AB$, written $CD > AB$.



In the next proposition, we will see that this notion of less than is compatible with congruence, and gives an order relation on congruence equivalence classes of line segments.

**Proposition 8.4**

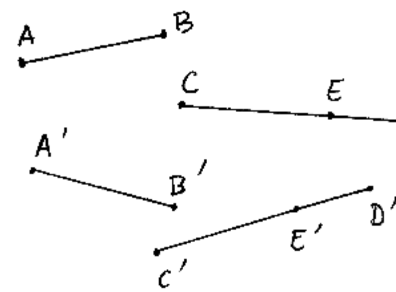
(a) *Given line segments* $AB \cong A'B'$ *and* $CD \cong C'D'$, *then* $AB < CD$ *if and only if* $A'B' < C'D'$.

(b) *The relation* $<$ *gives an order relation on line segments up to congruence, in the following sense:*

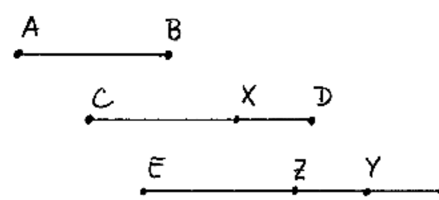
(i) *If* $AB < CD$, *and* $CD < EF$, *then* $AB < EF$.

(ii) *Given two line segments* $AB$, $CD$, *one and only one of the three following conditions holds:* $AB < CD$, $AB \cong CD$, $AB > CD$.

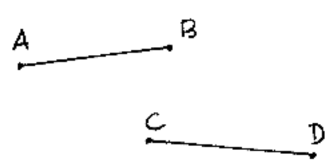
*Proof* (a) Given $AB \cong A'B'$ and $CD \cong C'D'$, suppose that $AB < CD$. Then there is a point $E$ such that $AB \cong CE$ and $C * E * D$. Let $E'$ be the unique point on the ray $\overrightarrow{C'D'}$ such that $CE \cong C'E'$. It follows from (8.3) that $C' * E' * D'$. Furthermore, by transitivity of congruence, $A'B' \cong C'E'$, so $A'B' < C'D'$ as required. The "if and only if" statement follows by applying the same argument starting with $A'B' < C'D'$.



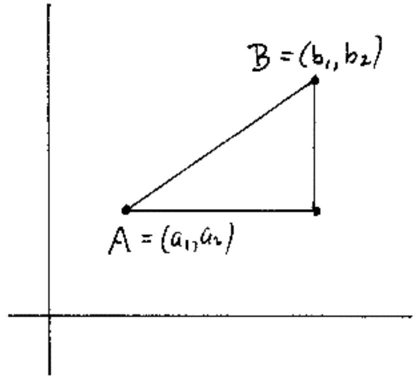

(b) (i) Suppose we are given $AB < CD$ and $CD < EF$. Then by definition, there is a point $X \in CD$ such that $AB \cong CX$, and there is a point $Y \in EF$ such that $CD \cong EY$. Let $Z \in \overrightarrow{EF}$ be such that $CX \cong EZ$. Then by (8.3) we have $E * Z * Y$. It follows that $E * Z * F$ (Exercise 7.1) and that $AB \cong EZ$. Hence $AB < EF$ as required.




(ii) Given line segments $AB$ and $CD$, let $E$ be the unique point on the ray $\overrightarrow{CD}$ for which $AB \cong CE$. Then either $D = E$ or $C * E * D$ or $C * D * E$. We cannot have $D * C * E$ because $D$ and $E$ are on the same side of $C$. These conditions are equivalent to $AB \cong CD$, or $AB < CD$, or $AB > CD$, respectively, and one and only one of them must hold.



**Example 8.4.1**
Let us define congruence for line segments in the real Cartesian plane $\mathbb{R}^2$, so that it becomes a model for the axioms (I1)–(I3), (B1)–(B4), and (C1)–(C3) that we have introduced so far. We have already seen how to define lines and betweenness (7.3.1). Given two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$, we define the *distance* $d(A, B)$ by

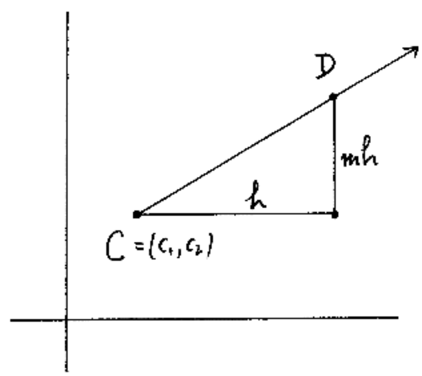



$$d(A,B) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

This is sometimes called the *Euclidean distance* or the *Euclidean metric* on $\mathbb{R}^2$. Note that $d(A, B) \geq 0$, and $d(A, B) = 0$ only if $A = B$.

Now we can give an interpretation of the undefined notion of congruence in this model by *defining* $AB \cong CD$ if $d(A, B) = d(C, D)$. Let us verify that the axioms (C1), (C2), (C3) are satisfied.

For (C1), we suppose that we are given a segment $AB$, and let $d = d(A, B)$. We also suppose that we are given a point $C = (c_1, c_2)$ and a ray emanating from $C$. For simplicity we will assume that the ray has slope $m > 0$ and that it is going in the direction of increasing $x$-coordinate (we leave the other cases to the reader). Then any point $D$ on this ray has coordinates $D = (c_1 + h, c_2 + mh)$ for some $h \geq 0$. The corresponding distance is




$$d(C, D) = h\sqrt{1 + m^2}.$$

To find a point $D$ with $AB \cong CD$ is then equivalent to solving the equation (in a variable $h > 0$)

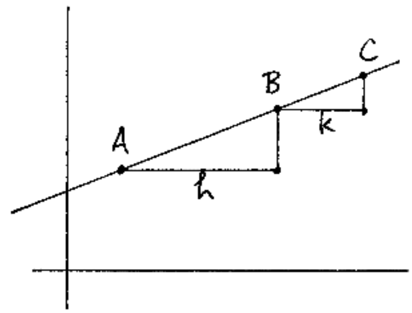
$$h\sqrt{1 + m^2} = d,$$

where $m$ and $d > 0$ are given. Clearly, there is a unique solution $h \in \mathbb{R}$, $h > 0$, for given $d, m$. This proves (C1).

The second axiom (C2) is trivial from the definition of congruence using a distance function.

To prove (C3), it will be sufficient to prove that the distance function is additive for points in a line: If $A * B * C$, then

$$d(A,B) + d(B,C) = d(A,C).$$




Suppose the line is $y = mx + b$, and $A = (a_1, a_2)$ is the point with smallest $x$-coordinate.

Then there are $h, k > 0$ such that

$$B = (a_1 + h, a_2 + mh),$$
$$C = (a_1 + h + k, a_2 + m(h + k)).$$

In this case

$$d(A,B) = h\sqrt{1+m^2},$$
$$d(B,C) = k\sqrt{1+m^2},$$
$$d(A,C) = (h+k)\sqrt{1+m^2},$$

so the additivity of the distance function follows.

We will sometimes call this model, the real Cartesian plane with congruence of segments defined by the Euclidean distance function, the *standard model* of our axiom system.

# Exercises

The following exercises (unless otherwise specified) take place in a geometry with axioms (I1)–(I3), (B1)–(B4), (C1)–(C3).

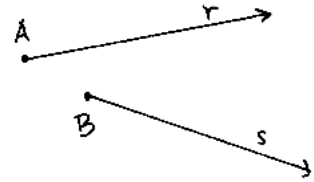
8.1 (a) Show that addition of line segments is associative: Given segments $AB, CD, EF$, and taking $A, B$ in order, then $(AB + CD) + EF = AB + (CD + EF)$. (This means that we obtain the same segment as the sum, not just congruent segments.)

(b) Show that addition of line segments is commutative up to congruence: Given segments $AB, CD$, then $AB + CD \cong CD + AB$.

8.2 Show that "halves of equals are equal" in the following sense: if $AB \cong CD$, and if $E$ is a *midpoint* of $AB$ in the sense that $A * E * B$ and $AE \cong EB$, and if $F$ is a midpoint of $CD$, then $AE \cong CF$. (Note that we have not yet said anything about the existence of a midpoint: That will come later (Section 10).) Conclude that a midpoint of $AB$, if it exists, is unique.

8.3 Show that addition preserves inequalities: If $AB < CD$ and if $EF$ is any other segment, then $AB + EF < CD + EF$.

8.4 Let $r$ be a ray originating at a point $A$, and let $s$ be a ray originating at a point $B$. Show that there is a 1-to-1 mapping $\varphi : r \to s$ of the set $r$ onto the set $s$ that preserves congruence and betweenness. In other words, if for any $X \in r$ we let $X' = \varphi(X) \in s$, then for any $X, Y, Z \in r$, $XY \cong X'Y'$, and $X * Y * Z \Leftrightarrow X' * Y' * Z'$.




8.5 Given two distinct points $O, A$, we define the *circle* with *center* $O$ and *radius* $OA$ to be the set $\Gamma$ of all points $B$ such that $OA \cong OB$.

(a) Show that any line through $O$ meets the circle in exactly two points.

(b) Show that a circle contains infinitely many points.

(Warning: It is not obvious from this definition whether the center $O$ is uniquely determined by the set of points $\Gamma$ that form the circle. We will prove that later (Proposition 11.1).)

8.6 Consider the *rational Cartesian plane* $\mathbb{Q}^2$ whose points are ordered pairs of rational numbers, where lines are defined by linear equations with rational coefficients and betweenness and congruence are defined as in the standard model (Examples 7.3.1 and 8.4.1). Verify that (I1)–(I3) and (B1)–(B4) are satisfied in this model. Then show that (C2) and (C3) hold in this model, but (C1) fails.

8.7 Consider the real Cartesian plane $\mathbb{R}^2$, with lines and betweenness as before (Example 7.3.1), but define a different notion of congruence of line segments using the distance function given by the sum of the absolute values:

$$d(A, B) = |a_1 - b_1| + |a_2 - b_2|,$$

where $A = (a_1, a_2)$ and $B = (b_1, b_2)$. Some people call this "taxicab geometry" because it is similar to the distance by taxi from one point to anther in a city where all streets run east–west or north–south. Show that the axioms (C1), (C2), (C3) hold, so that this is another model of the axioms introduced so far. What does the circle with center $(0, 0)$ and radius 1 look like in this model?

8.8 Again consider the real Cartesian plane $\mathbb{R}^2$, and define a third notion of congruence for line segments using the sup of absolute values for the distance function:

$$d(A, B) = \sup\{|a_1 - b_1|, |a_2 - b_2|\}.$$

Show that (C1), (C2), (C3) are also satisfied in this model. What does the circle with center $(0, 0)$ and radius 1 look like in this case?

8.9 Following our general principles, we say that two models $M, M'$ of our geometry are *isomorphic* if there exists a 1-to-1 mapping $\varphi : M \to M'$ of the set of points of $M$ onto the set of points of $M'$, written $\varphi(A) = A'$, that sends lines into lines, preserves betweenness, i.e., $A * B * C$ in $M \Leftrightarrow A' * B' * C'$ in $M'$, and preserves congruence of line segments, i.e., $AB \cong CD$ in $M \Leftrightarrow A'B' \cong C'D'$ in $M'$.

Show that the models of Exercise 8.7 and Exercise 8.8 above are isomorphic to each other, but they are not isomorphic to the standard model (Example 8.4.1). Note: To show that the two models of Exercise 8.7 and Exercise 8.8 are isomorphic, you do not need to make the distance functions correspond. It is only the notion of congruence of line segments that must be preserved. To show that two models are not isomorphic, one method is to find some statement that is true in one model but not true in the other model.

8.10 Nothing in our axioms relates the size of a segment on one line to the size of a congruent segment on another line. So we can make a weird model as follows. Take the real Cartesian plane $\mathbb{R}^2$ with the usual notions of lines and betweenness. Using the Euclidean distance function $d(A,B)$, define a new distance function

$$d'(A,B) = \begin{cases} d(A,B) & \text{if the segment } AB \text{ is either horizontal or vertical,} \\ 2d(A,B) & \text{otherwise.} \end{cases}$$

Define congruence of segments $AB \cong CD$ if $d'(A,B) = d'(C,D)$.
Show that (C1), (C2), (C3) are all satisfied in this model. What does a circle with center $(0,0)$ and radius 1 look like?

8.11 The *triangle inequality* is the statement that if $A, B, C$ are three distinct points, then $AC \leq AB + BC$.

(a) The triangle inequality always holds for collinear points.

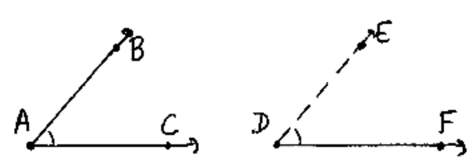
(b) The triangle inequality holds for any three points in the standard model (Example 8.4.1) and also in taxicab geometry (Exercise 8.7).

(c) The triangle inequality does not hold in the model of Exercise 8.10. Thus the triangle inequality is not a consequence of the axioms of incidence, betweenness, and congruence of line segments (C1)–(C3). (However, we will see in Section 10 that the triangle inequality, in the form of Euclid (I.20), is a consequence of the full set of axioms of a Hilbert plane.)

# 9 Axioms of congruence for Angles

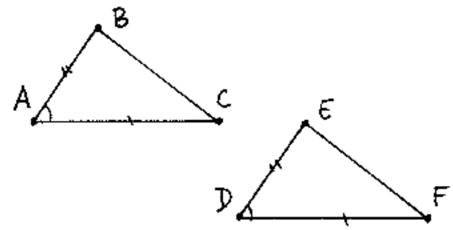
Recall that we have defined an *angle* to be the union of two rays originating at the same point, and not lying on the same line. We postulate an undefined notion of *congruence* for angles, written $\cong$, that is subject to the following three axioms:

**C4.** Given an angle $\angle BAC$ and given a ray $\overrightarrow{DF}$, there exists a unique ray $\overrightarrow{DE}$, on a given side of the line $DF$, such that $\angle BAC \cong \angle EDF$.




**C5.** For any three angles $\alpha, \beta, \gamma$, if $\alpha \cong \beta$ and $\alpha \cong \gamma$, then $\beta \cong \gamma$. Every angle is congruent to itself.

**C6.** (SAS) Given triangles $ABC$ and $DEF$, suppose that $AB \cong DE$ and $AC \cong DF$, and $\angle BAC \cong \angle EDF$. Then the two triangles are congruent, namely, $BC \cong EF$, $\angle ABC \cong \angle DEF$ and $\angle ACB \cong \angle DFE$.




Note that Hilbert takes the existence of an angle congruent to a given one (C4) as an axiom, while Euclid proves this by a ruler and compass construction (I.23). Since Hilbert does not make use of the compass, we may regard this axiom as a tool, the "transporter of angles," that acts as a substitute for the compass.

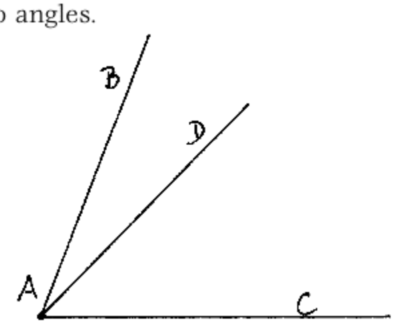
As with (C2), we can use (C5) to show that congruence is an equivalence relation.

**Proposition 9.1**
*Congruence of angles is an equivalence relation.*

*Proof* The proof is identical to the proof of (8.1), using (C5) in place of (C2).

As in the case of congruence of line segments, we would like to make sense of Euclid's common notions in the context of congruence of angles. This proposition (9.1) is the analogue of the first common notion, that "things equal to the same thing are equal to each other." The second common notion, that "equals added to equals are equal," becomes problematic in the case of angles, because in general we cannot define the sum of two angles.

If $\angle BAC$ is an angle, and if a ray $\overrightarrow{AD}$ lies in the interior of the angle $\angle BAC$, then we will say that the angle $\angle BAC$ is the *sum* of the angles $\angle DAC$ and $\angle BAD$.




However, if we start with the two given angles, there may not be an angle that is their sum in this sense. For one thing, they may add up to a straight line, or "two right angles" as Euclid says, but this is not an angle. Or their sum may be greater than 180°, in which case we get an angle, but the two original angles will not be in the interior of the new angle. So we must be careful how we state results having to do with sums of angles.

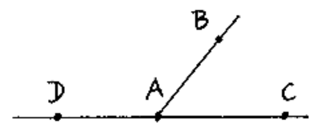
Note that we do not have an axiom about congruence of sums of angles analogous to the axiom (C3) about addition of line segments. That is because we can prove the corresponding result for angles. But in order to do so, we will need (C6).

Hilbert's use of (C6) = (SAS) as an axiom is a recognition of the insufficiency of Euclid's proof of that result (I.4) using the method of superposition. To justify the method of superposition by introducing axioms allowing motion of figures in the plane would be foreign to Euclid's approach to geometry, so it seems prudent to take (C6) as an axiom. However, we will show later (17.5) that the (SAS) axiom is essentially equivalent to the existence of a sufficiently large group of rigid motions of the plane. The axiom (C6) is necessary, since it is independent of the other axioms (Exercise 9.3). This axiom is essentially what tells us that our plane is homogeneous: Geometry is the same at different places in the plane.

Now let us show how to deal with sums of angles and inequalities among angles based on these axioms.

**Definition**

If $\angle BAC$ is an angle, and if $D$ is a point on the line $AC$ on the other side of $A$ from $C$, then the angles $\angle BAC$ and $\angle BAD$ are *supplementary*.



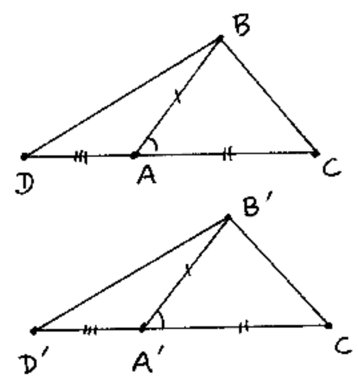

**Proposition 9.2**

*If $\angle BAC$ and $\angle BAD$ are supplementary angles, and if $\angle B'A'C'$ and $\angle B'A'D'$ are supplementary angles, and if $\angle BAC \cong \angle B'A'C'$, then also $\angle BAD \cong \angle B'A'D'$.*

*Proof* Replacing $B', C', D'$ by other points on the same rays, we may assume that $AB \cong A'B'$, $AC \cong A'C'$, and $AD \cong A'D'$. Draw the lines $BC$, $BD$, $B'C'$, and $B'D'$.

First we consider the triangles $ABC$ and $A'B'C'$. By hypothesis we have $AB \cong A'B'$ and $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$. So by (C6) we conclude that the triangles are congruent. In particular, $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$.

Next we consider the triangles $BCD$ and $B'C'D'$. Since $AC \cong A'C'$ and $AD \cong A'D'$, and $C * A * D$ and $C' * A' * D'$, we conclude from (C3) that $CD \cong C'D'$. Using $BC \cong B'C'$ and $\angle BCA \cong \angle B'C'A'$ proved above, we can apply (C6) again to see that the triangles $BCD$ and $B'C'D'$ are congruent. In particular, $BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$.

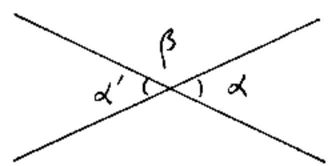


Now we consider the triangles $BDA$ and $B'D'A'$. From the previous step we have $BD \cong B'D'$ and $\angle BDA \cong \angle B'D'A'$. But by hypothesis we have $DA \cong D'A'$. So a third application of (C6) shows that the triangles $BDA$ and $B'D'A'$ are congruent. In particular, $\angle BAD \cong \angle B'A'D'$, which was to be proved.

**Note:** We may think of this result as a replacement for (I.13), which says that the angles made by a ray standing on a line are either right angles or are equal to two right angles. We cannot use Euclid's statement directly, because in our terminology, the sum of two right angles is not an angle. However, in applications, Euclid's (I.13) can be replaced by (9.2). So for example, we have the following corollary.

**Corollary 9.3**
*Vertical angles are congruent.*

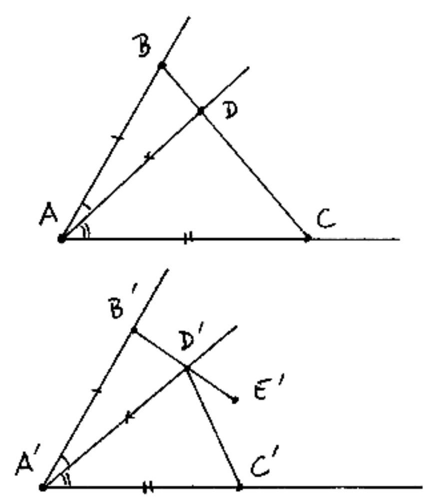
*Proof* Recall that *vertical* angles are defined by the opposite rays on the same two lines. The vertical angles $\alpha$ and $\alpha'$ are each supplementary to $\beta$, and $\beta$ is congruent to itself, so by the proposition, $\alpha$ and $\alpha'$ are congruent.




**Proposition 9.4** (Addition of angles)
*Suppose $\angle BAC$ is an angle, and the ray $\overrightarrow{AD}$ is in the interior of the angle $\angle BAC$. Suppose $\angle D'A'C' \cong \angle DAC$, and $\angle B'A'D' \cong \angle BAD$, and the rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ are on opposite sides of the line $A'D'$. Then the rays $\overrightarrow{A'B'}$ and $\overrightarrow{A'C'}$ form an angle, and $\angle B'A'C' \cong \angle BAC$, and the ray $\overrightarrow{A'D'}$ is in the interior of the angle $\angle B'A'C'$. For short, we say "sums of congruent angles are congruent."*

*Proof* Draw the line $BC$. Then the ray $\overrightarrow{AD}$ must meet the segment $\overline{BC}$, by the crossbar theorem (7.3). Replacing the original $D$ by this intersection point, we may assume that $B, D, C$ lie on a line and $B * D * C$. On the other hand, replacing $B', C', D'$ by other points on the same rays, we may assume that $AB \cong A'B'$, and $AC \cong A'C'$, and $AD \cong A'D'$. We also have $\angle BAD \cong \angle B'A'D'$ and $\angle DAC \cong \angle D'A'C'$ by hypothesis.

By (C6) we conclude that the triangles $\Delta BAD$ and $\Delta B'A'D'$ are congruent. In particular, $BD \cong B'D'$ and $\angle BDA \cong B'D'A'$.



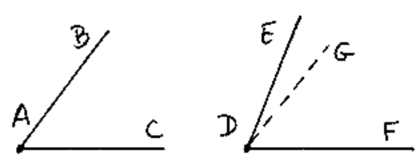
Again by (C6) we conclude that the triangles $\Delta DAC$ and $\Delta D'A'C'$ are congruent. In particular, $DC \cong D'C'$ and $\angle ADC \cong \angle A'D'C'$.

Let $E'$ be a point on the line $B'D'$ with $B' * D' * E'$. Then $\angle A'D'E'$ is supplementary to $\angle A'D'B'$, which is congruent to $\angle ADB$. So by (9.2) and transitivity of congruence, we find that $\angle A'D'E' \cong \angle A'D'C'$. Since these angles are on the same side of the line $A'D'$, we conclude from the uniqueness part of (C4) that they are the same angle. In other words, the three points $B', D'$, and $C'$ lie on a line.

Then from (C3) we conclude that $BC \cong B'C'$. Since $\angle ABD \cong \angle A'B'D'$ by the first congruence of triangles used in the earlier part of the proof, we can apply (C6) once more to the triangles $ABC$ and $A'B'C'$. The congruence of these triangles implies $\angle BAC \cong \angle B'A'C'$ as required. Since $B', D'$, and $C'$ are collinear and $D'A'C'$ is an angle, it follows that $A', B', C'$ are not collinear, so $B'A'C'$ is an angle. Since $B'$ and $C'$ are on opposite sides of the line $A'D'$, it follows that $B' * D' * C'$, and so the ray $\overrightarrow{A'D'}$ is in the interior of the angle $\angle B'A'C'$, as required.

Next, we will define a notion of inequality for angles analogous to the inequality for line segments in Section 8.

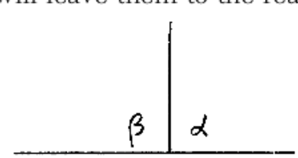
**Definition**
Suppose we are given angles $\angle BAC$ and $\angle EDF$. We say that $\angle BAC$ is *less than* $\angle EDF$, written $\angle BAC < \angle EDF$, if there exists a ray $\overrightarrow{DG}$ in the interior of the angle $\angle EDF$ such that $\angle BAC \cong \angle GDF$. In this case we will also say that $\angle EDF$ is *greater than* $\angle BAC$.




**Proposition 9.5**
(a) *If $\alpha \cong \alpha'$ and $\beta \cong \beta'$, then $\alpha < \beta \Leftrightarrow \alpha' < \beta'$.*
(b) *Inequality gives an order relation on angles, up to congruence. In other words:*
(i) *If $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$.*
(ii) *For any two angles $\alpha$ and $\beta$, one and only one of the following holds: $\alpha < \beta$; $\alpha \cong \beta$; $\alpha > \beta$.*

*Proof* The proofs of these statements are essentially the same as the corresponding statements for line segments (8.4), so we will leave them to the reader.

**Definition**
A *right angle* is an angle $\alpha$ that is congruent to one of its supplementary angles $\beta$.

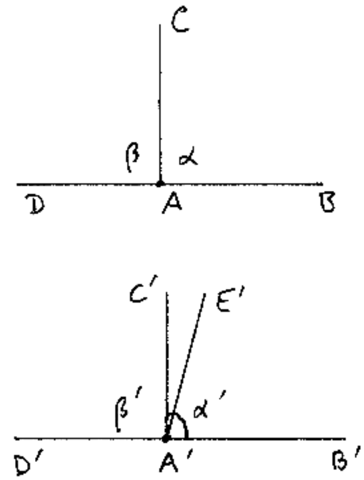


**Note:** In this definition, it does not matter which supplementary angle to $\alpha$ we consider, because the two supplementary angles to $\alpha$ are vertical angles, hence congruent by (9.3). Two lines are *orthogonal* if they meet at a point and one, hence all four, of the angles they make is a right angle.

**Proposition 9.6**
*Any two right angles are congruent to each other.*

*Proof* Suppose that $\alpha = \angle CAB$ and $\alpha' = \angle C'A'B'$ are right angles. Then they will be congruent to their supplementary angles $\beta, \beta'$, by definition. Suppose $\alpha$ and $\alpha'$ are not congruent. Then by (9.5) either $\alpha < \alpha'$ or $\alpha' < \alpha$. Suppose, for example, $\alpha < \alpha'$. Then by definition of inequality there is a ray $\overrightarrow{A'E'}$ in the interior of angle $\alpha'$ such that $\alpha \cong \angle E'A'B'$.

It follows (check!) that the ray $\overrightarrow{A'C'}$ is in the interior of $\angle E'A'D'$, so that $\beta' < \angle E'A'D'$. But $\angle E'A'D'$ is supplementary to $\angle E'A'B'$, which is congruent to $\alpha$, so by (9.2), $\angle E'A'D' \cong \beta$. Therefore, $\beta' < \beta$. But $\alpha \cong \beta$ and $\alpha' \cong \beta'$, so we conclude that $\alpha' < \alpha$, which is a contradiction.



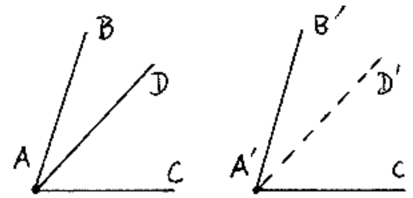

**Note:** Thus the congruence of all right angles can be proved and does not need to be taken as an axiom as Euclid did (Postulate 4). The idea of this proof already appears in Proclus.

**Example 9.6.1**
We will show later that the real Cartesian plane $\mathbb{R}^2$ provides a model of all the axioms listed so far. You are probably willing to believe this, but the precise definition of what we mean by congruence of angles in this model, and the proof that axioms (C4)–(C6) hold, requires some work. We will postpone this work until we make a systematic study of Cartesian planes over arbitrary fields, and then we will show more generally that the Cartesian plane over any ordered field satisfying a certain algebraic condition gives a model of Hilbert's axioms (17.3).

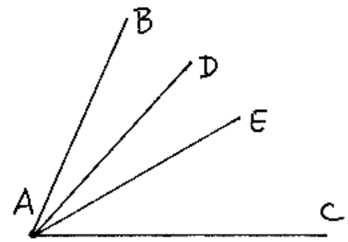
The other most important model of Hilbert's axioms is the non-Euclidean Poincaré model, which we will discuss in Section 39.

# Exercises

9.1 (Difference of angles). Suppose we are given congruent angles $\angle BAC \cong \angle B'A'C'$. Suppose also that we are given a ray $\overrightarrow{AD}$ in the interior of $\angle BAC$. Then there exists a ray $\overrightarrow{A'D'}$ in the interior of $\angle B'A'C'$ such that $\angle DAC \cong \angle D'A'C'$ and $\angle BAD \cong \angle B'A'D'$. This statement corresponds to Euclid's Common Notion 3: "Equals subtracted from equals are equal," where "equal" in this case means congruence of angles.

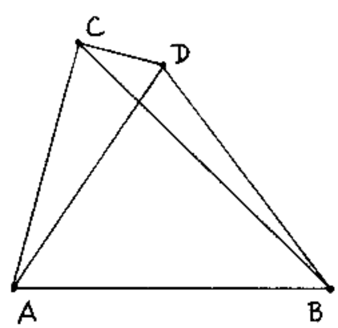



9.2 Suppose the ray $\overrightarrow{AD}$ is in the interior of the angle $\angle BAC$, and the ray $\overrightarrow{AE}$ is in the interior of the angle $\angle DAC$. Show that $\overrightarrow{AE}$ is also in the interior of $\angle BAC$.




9.3 Consider the real Cartesian plane where congruence of line segments is given by the absolute value distance function (Exercise 8.7). Using the usual congruence of angles that you know from analytic geometry (Section 16), show that (C4) and (C5) hold in this model, but that (C6) fails. (Give a counterexample.)

9.4 Provide the missing betweenness arguments to complete Euclid's proof of (I.7) in the case he considers. Namely, assuming that the ray $\overrightarrow{AD}$ is in the interior of the angle $\angle CAB$, and assuming that $D$ is outside the triangle $ABC$, prove that $\overrightarrow{CB}$ is in the interior of the angle $\angle ACD$ and $\overrightarrow{DA}$ is in the interior of the angle $\angle CDB$.




# 10 Hilbert Planes

We have now introduced the minimum basic notions and axioms on which to found our study of geometry.

**Definition**
A *Hilbert plane* is a given set (of *points*) together with certain subsets called *lines*, and undefined notions of *betweenness*, *congruence* for line segments, and *congruence* for angles (as explained in the preceding sections) that satisfy the axioms (I1)–(I3), (B1)–(B4), and (C1)–(C6). (We do not include the parallel axiom (P).)

We could go on immediately and introduce the parallel axiom and axioms of intersection of lines and circles, so as to recover all of Euclid's *Elements*, but it seems worthwhile to pause at this point and see how much of the geometry we can develop with this minimal set of axioms. The main reason for doing this is that the axioms of a Hilbert plane form the basis for non-Euclidean as well as Euclidean geometry. In fact, some people call the Hilbert plane *neutral geometry*, because it neither affirms nor denies the parallel axiom.

In this section we will see how much of Euclid's Book I we can recover in a Hilbert plane. With two notable exceptions, we can recover everything that does not make use of the parallel postulate.

Let us work in a given Hilbert plane. Euclid's definitions, postulates, and common notions have been replaced by the undefined notions, definitions, and axioms that we have discussed so far (excluding Playfair's axiom). We will now discuss the propositions of Euclid, Book I.

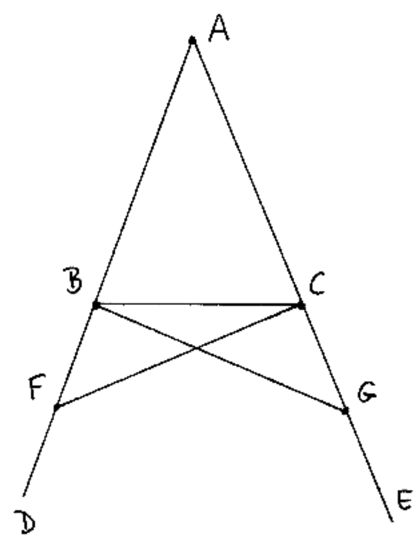
The first proposition (I.1) is our first exception! Without some additional axiom, it is not clear that the two circles in Euclid's construction will actually meet. In fact, the existence of an equilateral triangle on a given segment does not follow from the axioms of a Hilbert plane (Exercise 39.31). We will partially fill this gap by showing (10.2) that there do exist isosceles triangles on a given segment.

Euclid's Propositions (I.2) and (I.3) about transporting line segments are effectively replaced by axiom (C1). Proposition (I.4), (SAS), has been replaced by axiom (C6).

Proposition (I.5) and its proof are ok as they stand. In other words, every step of Euclid's proof can be justified in a straightforward manner within the framework of a Hilbert plane. To illustrate this process of reinterpreting one of Euclid's proofs within our new axiom system, let us look at Euclid's proof step by step.

*Proof of (I.5)* Let $ABC$ be the given isosceles triangle, with $AB \cong AC$ (congruent line segments). We must prove that the base angles $\angle ABC$ and $\angle ACB$ are congruent. "In $BD$ take any point $F$." This is possible by axiom (B2). "On $AE$ cut off $AG$ equal to $AF$." This is possible by (C1). Now $AC \cong AB$ and $AF \cong AG$, and the enclosed angle $\angle BAC$ is the same, so the triangles $\Delta AFC$ and $\Delta AGB$ are congruent by a direct application of (C6). So $FC \cong GB$ and $\angle AFC \cong \angle AGB$ and $\angle ACF \cong \angle ABG$.

Since "equals subtracted from equals are equal," referring in this case to congruence of line segments, we conclude from (8.3) that $BF \cong CG$. Then by another application of (C6), the triangles $\Delta FBC$ and $\Delta GCB$ are congruent. It follows that $\angle CBG \cong \angle BCF$. Now by subtraction of congruent angles (Exercise 9.1), the base angles $\angle ABC$ and $\angle ACB$ are congruent, as required. (We omit the proof of the second assertion, which follows similarly.)




At certain steps in this proof we need to know something about betweenness, which can also be formally proved from our axioms. For example, in order to subtract the line segment $AB$ from $AF$, we need to know that $B$ is between $A$ and $F$. This follows from our choice of $F$. At the last step, subtracting angles, we need to know that the ray $\overrightarrow{BC}$ is in the interior of the angle $\angle ABG$. This follows from the fact that $C$ is between $A$ and $G$.

So in the following, when we say that Euclid's proof is ok as is, we mean that each step can be justified in a natural way, without having to invent additional steps of proof, from Hilbert's axioms and the preliminary results we established in the previous sections.

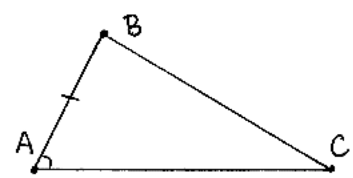
Looking at (I.6), the converse of (I.5), everything is ok except for one doubtful step at the end. Euclid says, "the triangle $DBC$ is equal to the triangle $ACB$, the less to the greater; which is absurd." It is not clear what this means, since we have not defined a notion of inequality for triangles. However, a very slight change will give a satisfactory proof. Namely, from the congruence of the triangles $\Delta DBC \cong \Delta ACB$, it follows that $\angle DCB \cong \angle ABC$. But also $\angle ABC \cong \angle ACB$ by hypothesis. So $\angle DCB \cong \angle ACB$, "the less to the greater," as Euclid would say. For us, this is a contradiction of the uniqueness part of axiom (C4), since there can be only one angle on the same side of the ray $\overrightarrow{CB}$ congruent to the angle $\angle ACB$. We conclude that the rays $\overrightarrow{CA}$ and $\overrightarrow{CD}$ are equal, so $A = D$, and the triangle is isosceles, as required.

Proposition (I.7), as we have mentioned before, needs some additional justification regarding the relative positions of the lines, which can be supplied from our axioms of betweenness (Exercise 9.4).

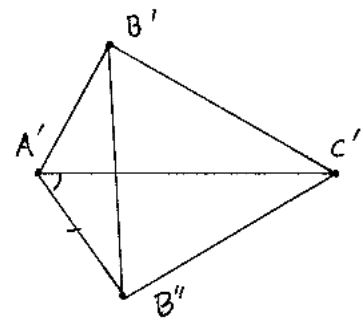
For (I.8), (SSS), we will need a new proof, since Euclid's method of superposition cannot be justified from our axioms. The following proof is due to Hilbert.

**Proposition 10.1** (SSS)
*If two triangles $ABC$ and $A'B'C'$ have their respective sides equal, namely $AB \cong A'B'$, $AC \cong A'C'$, and $BC \cong B'C'$, then the two triangles are congruent.*

*Proof* Using (C4) and (C1), construct an angle $\angle C'A'B''$ on the other side of the ray $\overrightarrow{A'C'}$ from $B'$ that is congruent to $\angle BAC$, and make $A'B''$ congruent to $AB$. Then $AB \cong A'B''$ by construction, $AC \cong A'C'$ by hypothesis, and $\angle BAC \cong \angle B''A'C'$ by construction, so by (C6), the triangle $\Delta ABC$ is congruent to the triangle $\Delta A'B''C'$. It follows that $BC \cong B''C'$.




Draw the line $B'B''$. Now $A'B' \cong AB \cong A'B''$, so by transitivity, $A'B' \cong A'B''$. Thus the triangle $A'B'B''$ is isosceles, and so by (I.5) its base angles $\angle A'B'B''$ and $\angle A'B''B'$ are congruent. Similarly, $B'C' \cong B''C'$, so the triangle $C'B'B''$ is isosceles, and its base angles $\angle B''B'C'$ and $\angle B'B''C'$ are congruent. By addition of congruent angles (9.4) it follows that $\angle A'B'C' \cong \angle A'B''C'$.



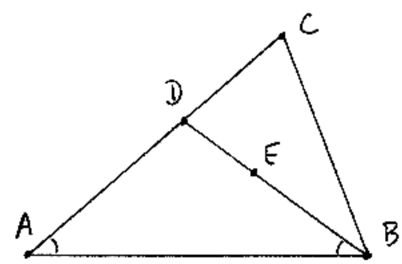

This latter triangle was shown congruent to $\Delta ABC$, so $\angle A'B''C' \cong \angle ABC$. Now by transitivity of congruence, $\angle ABC \cong \angle A'B'C'$, so we can apply (C6) again to conclude that the two triangles are congruent.

**Note:** This proof and the accompanying figure are for the case where $A'$ and $C'$ are on opposite sides of the line $B'B''$. The case where they are on the same side is analogous, and the case where one of $A'$ or $C'$ lies on the line $B'B''$ is easier, and left to the reader.

Starting with the next proposition (I.9) we have a series of constructions with ruler and compass. We cannot carry out these constructions in a Hilbert plane, because we have not yet added axioms to ensure that lines and circles will meet when they ought to (cf. Section 11). However, we can reinterpret these propositions as existence theorems, and these we can prove from Hilbert's axioms. Since we do not have the equilateral triangles that Euclid constructed in (I.1), we will prove the existence of isosceles triangles, and we will use them as a substitute for equilateral triangles in the following existence proofs.

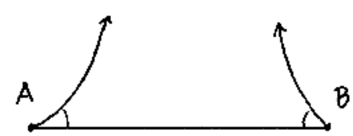
**Proposition 10.2** (Existence of isosceles triangles)
*Given a line segment AB, there exists an isosceles triangle with base AB.*

*Proof* Let $AB$ be the given line segment. Let $C$ be any point not on the line $AB$ (axiom (I3)). Consider the triangle $\Delta ABC$. If the angles at $A$ and $B$ are equal, then $\Delta ABC$ is isosceles (I.6). If not, then one angle is less than the other. Suppose $\angle CAB < \angle CBA$. Then there is a ray $\overrightarrow{BE}$ in the interior of the angle $\angle CBA$ such that $\angle CAB \cong EBA$.




By the crossbar theorem (7.3) this ray must meet the opposite side $AC$ in a point $D$. Now the base angles of the triangle $DAB$ are equal, so by (I.6) it is isosceles.

**Note:** It would not suffice to construct equal angles at the two ends of the interval, because without the parallel axiom, even if the angles are small, there is no guarantee that the two rays would meet.




Now let us return to Euclid. We interpret (I.9) as asserting the existence of an angle bisector. We use the same method as Euclid, except that we use (10.2) to give the existence of an isosceles triangle $\Delta DEF$ where Euclid used an equilateral triangle. We may assume that this isosceles triangle is constructed on the opposite side of $DE$ from $A$. Then Euclid's proof, using (SSS), shows that $\angle DAF \cong \angle EAF$. It is not obvious from the construction that the ray $\overrightarrow{AF}$ is in the interior of the angle $\angle DAE$, but it does follow from the conclusion: For if $\overrightarrow{AF}$ were not in the interior of the angle, then $\overrightarrow{AD}$ and $\overrightarrow{AE}$ would be on the same side of $\overrightarrow{AF}$, and in that case the congruence of the angles $\angle DAF \cong \angle EAF$ would contradict the uniqueness in axiom (C4).

For (I.10) to bisect a given line segment, we again use (10.2) to construct an isosceles triangle instead of an equilateral triangle. The rest of Euclid's proof then works to show that a midpoint of the segment exists.

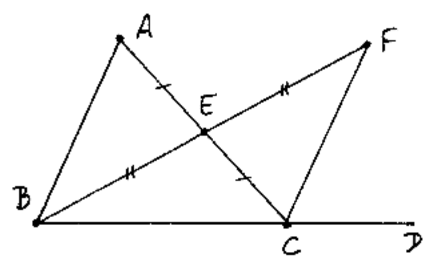
For (I.11) we can also use (10.2) to construct a line perpendicular to a line at a point. By the way, this also proves the existence of right angles, which is not obvious a priori.

For (I.12), to drop a perpendicular from a point $C$ to a line not containing $C$, Euclid's method using the compass does not work in a Hilbert plane. We need a new existence proof (see Exercise 10.4).

Proposition (I.13) has been replaced by the result on congruence of supplementary angles (9.2), and (I.14) is an easy consequence (Exercise 10.7). The congruence of vertical angles (I.15) has already been mentioned above (9.3). The theorem on exterior angles (I.16) is sufficiently important that we will reproduce Euclid's proof here, with the extra justifications necessary to make it work.

**Proposition 10.3** (Exterior angle theorem (I.16))
*In any triangle, the exterior angle is greater than either of the opposite interior angles.*

*Proof* Let $ABC$ be the given triangle. We will show that the exterior angle $\angle ACD$ is greater than the opposite interior angle at $A$. Let $E$ be the midpoint of $AC$ (I.10), and extend $BE$ to $F$ so that $BE \cong EF$ (axiom (C1)). Draw the line $CF$. Now the vertical angles at $E$ are equal (I.15), so by SAS (C6), the triangles $\Delta ABE$ and $\Delta CFE$ are congruent. Hence $\angle A \cong \angle ECF$.




To finish the proof, that is, to show that $\angle ECF$ is less than $\angle ACD$, we need to know that the ray $\overrightarrow{CF}$ is in the interior of the angle $\angle ACD$. This we can prove based on our axioms of betweenness. Since $D$ is on the side $BC$ of the triangle extended, $B$ and $D$ are on opposite sides of the line $AC$. Also, by construction of $F$, we have $B$ and $F$ on opposite sides of $AC$. So from the plane separation property (7.1) it follows that $D$ and $F$ are on the same side of the line $AC$.

Now consider sides of the line $BC$. Since $B * E * F$, it follows that $E$ and $F$ are on the same side of $BC$. Since $A * E * C$, it follows that $A$ and $E$ are on the same side of $AC$. By transitivity (7.1) it follows that $A$ and $F$ are on the same side of the line $BC = CD$. So by definition, $F$ is in the interior of the angle $\angle ACD$, and hence the ray $\overrightarrow{CF}$ is also. Therefore, by definition of inequality for angles, $\angle BAC$ is less than $\angle ACD$, as required.

Propositions (I.17)–(I.21) are all ok as is, except that we should reinterpret the statement of (I.17). Instead of saying "any two angles of a triangle are less than two right angles," which does not make sense in our system, since "two right angles" is not an angle, we simply say; if $\alpha$ and $\beta$ are any two angles of a triangle, then $\alpha$ is less than the supplementary angle of $\beta$.

Proposition (I.22) is our other exception. Without knowing that two circles intersect when they ought to, we cannot prove the existence of the triangle required in this proposition. In fact, we will see later (Exercise 16.11) that there are Hilbert planes in which a triangle with certain given sides satisfying the hypotheses of this proposition does not exist!

The next proposition (I.23), which Euclid proved using (I.22), is replaced by Hilbert's axiom (C4), the "transporter of angles."

The remaining results that Euclid proved without using the parallel postulate are ok as is in the Hilbert plane: (I.24), (I.25), (I.26) = (ASA) and (AAS), (I.27) "alternate interior angles equal implies parallel," and even the existence of parallel lines (I.31).

Summing up, we have the following theorem.

**Theorem 10.4**
*All of Euclid's propositions* (I.1) *through* (I.28), *except* (I.1) *and* (I.22), *can be proved in an arbitrary Hilbert plane, as explained above.*

### Constructions with Hilbert's Tools

Euclid used ruler and compass constructions to prove the existence of various objects in his geometry, such as the midpoint of a given line segment. We used Hilbert's axioms to prove corresponding existence results in a Hilbert plane. However, we can reinterpret these existence results as constructions if we imagine tools corresponding to certain of Hilbert's axioms. Thus (I1), the existence of a line through two points, corresponds to the ruler. For axiom (C1), imagine a tool, such as a compass with two sharp points (also called a pair of dividers), that acts as a transporter of segments. For axiom (C4), imagine a new tool, the transporter of angles, that can reproduce a given angle at a new point. It could be made of two rulers joined with a stiff but movable hinge.

We call these three tools, the ruler, the dividers, and the transporter of angles, *Hilbert's tools*. We also allow ourselves to pick points (using (I3) and (B2)) as required.

Now we can regard (10.2) as a construction of an isosceles triangle using Hilbert's tools. Counting steps, with one step for each use of a tool, we have the construction as follows:

Given a line segment $AB$. Pick $C$ not on the line $AB$.

1. Draw line $AC$.
2. Draw line $BC$. Suppose $\angle CAB$ is less than $\angle CBA$.
3. Transport $\angle CAB$ to $\angle ABE$, get point $D$.

Then $ABD$ is the required isosceles triangle.

# Exercises

10.1 Construct with Hilbert's tools the angle bisector of a given angle (par = 4).

10.2 Construct with Hilbert's tools the midpoint of a given segment (par = 4).

10.3 Construct with Hilbert's tools a line perpendicular to a given line $l$ at a given point $A \in l$ (par = 5).

10.4 Construct with Hilbert's tools a line perpendicular to a given line $l$ from a point $A$ not on $l$ (par = 4).

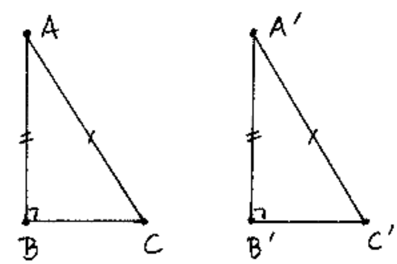
10.5 Construct with Hilbert's tools a line parallel to a given line $l$, and passing through a given point $A$ not on $l$ (par = 2).

10.6 Write out a careful proof of Euclid (I.18), justifying every step in the context of a Hilbert plane, and paying especial attention to questions of betweenness and inequalities.

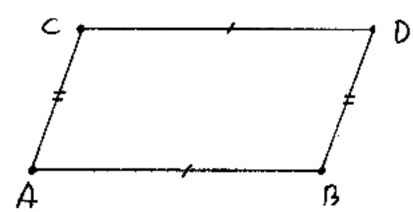
10.7 Rewrite the statement (I.14) so that it makes sense in a Hilbert plane, and then give a careful proof.

10.8 Write a careful proof of (I.20) in a Hilbert plane.

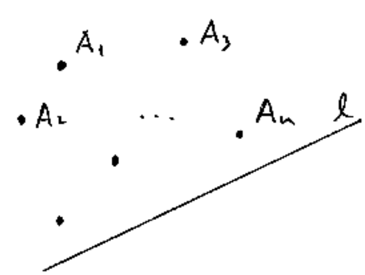
10.9 Show that the right-angle–side–side congruence theorem (RASS) holds in a Hilbert plane: If $ABC$ and $A'B'C'$ are triangles with right angles at $B$ and $B'$, and if $AB \cong A'B'$ and $AC \cong A'C'$, then the triangles are congruent.




10.10 In a Hilbert plane, suppose that we are given a quadrilateral $ABCD$ with $AB = CD$ and $AC = BD$. Prove that $CE$ is parallel to $AB$ (without using the parallel axiom (P)). *Hint*: Join the midpoints of $AB$ and $CD$; then use (I.27).



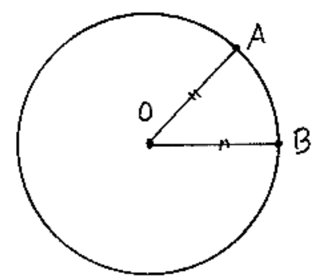

10.11 Given a finite set of points $A_1, \ldots, A_n$ in a Hilbert plane, prove that there exists a line $l$ for which all the points are on the same side of $l$.



# 11 Intersections of Lines and Circles

In this section we will discuss the intersections of lines and circles in the Hilbert plane, and we will introduce the further axiom (E), which will guarantee that lines and circles will intersect when they "ought" to. With this axiom we can justify Euclid's ruler and compass constructions in Book I and Book III. We work in a Hilbert plane (Section 10) without assuming the parallel axiom (P). Because of (10.4) we can use Euclid's results (I.2)–(I.28) (except (I.22)) in our proofs.

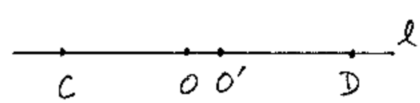
**Definition**
Given distinct points $O$, $A$, the *circle* $\Gamma$ with center $O$ and radius $OA$ is the set of all points $B$ such that $OA \cong OB$. The point $O$ is the *center* of the circle. The segment $OA$ is a *radius*.




From this definition it is clear that a circle always has points. The point $A$ is on the circle. Moreover, if $l$ is any line through $O$, then by axiom (C1) there will be exactly two points on the line $l$, one on each side of $O$, lying on the circle. However, it is not obvious from the definition that the center is uniquely determined by the set of points of the circle.

**Proposition 11.1**
*Let $\Gamma$ be a circle with center $O$ and radius $OA$, and let $\Gamma'$ be a circle with center $O'$ and radius $O'A'$. Suppose $\Gamma = \Gamma'$ as point sets. Then $O = O'$. In other words the center of a circle is uniquely determined.*

*Proof* Suppose $O \neq O'$. Then we consider the line $l$ through $O$ and $O'$. Since it passes through the center $O$ of $\Gamma$, it must meet $\Gamma$ in two points $C, D$, satisfying $C * O * D$ and $OC \cong OD$.




Since $\Gamma = \Gamma'$, the points $C, D$ are also on $\Gamma'$, so we have $O'C \cong O'D$ and $C * O' * D$. We do not know which of $O$ or $O'$ is closer to $C$, but the two cases are symmetric, so let us assume $C * O * O'$. In this case we must have $O * O' * D$ by the properties of betweenness(!). Then $OC < O'C \cong O'D < OD$, which is impossible, since $OC \cong OD$. Hence $O = O'$.

Now that we know that the center of a circle is uniquely determined, it makes sense to define the inside and the outside of a circle.

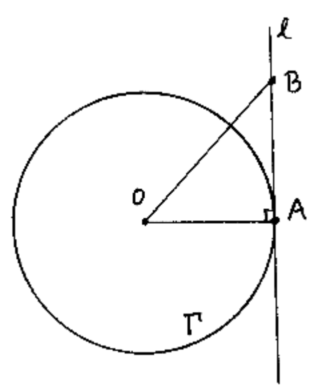
**Definition**
Let $\Gamma$ be a circle with center $O$ and radius $OA$. A point $B$ is *inside* $\Gamma$ (or in the *interior* of $\Gamma$) if $B = O$ or if $OB < OA$. A point $C$ is *outside* $\Gamma$ (or *exterior* to $\Gamma$) if $OA < OC$.

**Definition**
We say that a line $l$ is *tangent* to a circle $\Gamma$ if $l$ and $\Gamma$ meet in just one point $A$. We say that a circle $\Gamma$ is *tangent* to another circle $\Delta$ if $\Gamma$ and $\Delta$ have just one point in common.

This definition of tangent circles is a little different from Euclid's: His definition of two circles touching is that they meet in a point but do not cut each other. Since it is not clear what he means by "cut," we prefer the definition above, and we will prove that these notions of tangency have the usual properties.

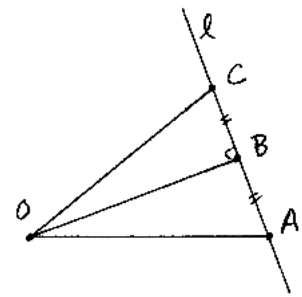
**Proposition 11.2**
*Let $\Gamma$ be a circle with center $O$ and radius $OA$. The line perpendicular to the radius $OA$ at the point $A$ is tangent to the circle, and (except for the point $A$) lies entirely outside the circle. Conversely, if a line $l$ is tangent to $\Gamma$ at $A$, then it is perpendicular to $OA$. In particular, for any point $A$ of a circle, there exists a unique tangent line to the circle at that point.*

*Proof* First, let $l$ be the line perpendicular to $OA$ at $A$. Let $B$ be any other point on the line $l$. Then in the triangle $OAB$, the exterior angle at $A$ is a right angle, so the angles at $O$ and at $B$ are less than a right angle (I.16). It follows (I.19) that $OB > OA$, so $B$ is outside the circle. Thus $l$ meets $\Gamma$ only at the point $A$, so it is a tangent line.



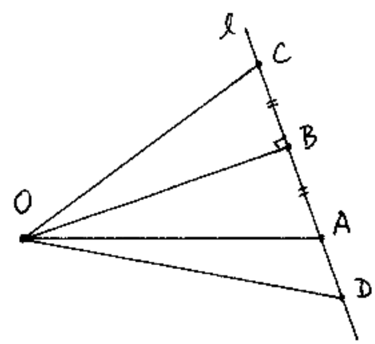

Now suppose that $l$ is a line tangent to $\Gamma$ at $A$. We must show that $l$ is perpendicular to $OA$. It cannot be equal to $OA$, because that line meets $\Gamma$ in another point opposite $A$. So consider the line from $O$, perpendicular to $l$, meeting $l$ at $B$. If $B \neq A$, take a point $C$ on the other side of $B$ from $A$, so that $AB \cong BC$

(axiom (C1)). The $\Delta OBA \cong \Delta OBC$ by SAS, so we have $OA \cong OC$, and hence $C$ is also on $\Gamma$. Since $C \neq A$, this is a contradiction. We conclude that $B = A$, and so $l$ is perpendicular to $OA$.



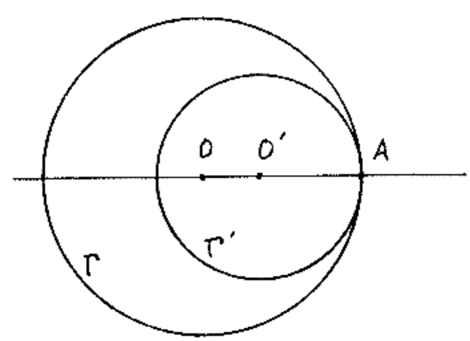

**Corollary 11.3**
*If a line $l$ contains a point $A$ of a circle $\Gamma$, but is not tangent to $\Gamma$, then it meets $\Gamma$ in exactly two points.*

*Proof* If $l$ is not tangent to $\Gamma$ at $A$, then it is not perpendicular to $OA$, in which case, as we saw in the previous proof, it meets $\Gamma$ in another point $C$. We must show that $l$ cannot contain any further points of $\Gamma$. For if $D$ were another point of $l$ on $\Gamma$, then $OD \cong OA, OB$ is congruent to itself, so by (RASS) (Exercise 10.9) we would have $\Delta ODB \cong \Delta OAB$. Then $AB \cong BD$, so by axiom (C1) $D$ must be equal to $A$ or $C$.

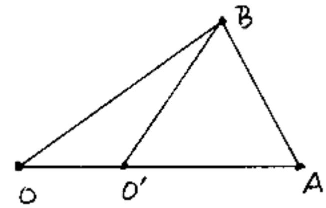



**Proposition 11.4**
*Let $O, O', A$ be three distinct collinear points. Then the circle $\Gamma$ with center $O$ and radius $OA$ is tangent to the circle $\Gamma'$ with center $O'$ and radius $O'A$. Conversely, if two circles $\Gamma$, $\Gamma'$ are tangent at a point $A$, then their centers $O, O'$ are collinear with $A$.*

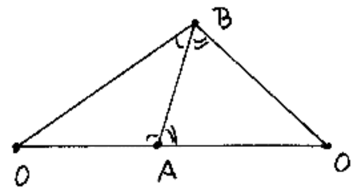
*Proof* Let $O, O', A$ be collinear. We must show that the circles $\Gamma$ and $\Gamma'$ have no further points in common besides $A$. The argument of (11.1) shows that there is no other point on the line $OO'$ that lies on both $\Gamma$ and $\Gamma'$. So suppose there is a point $B$ not on $OO'$ lying on both $\Gamma$ and $\Gamma'$. We divide into two cases depending on the relative position of $O, O'$, and $A$.



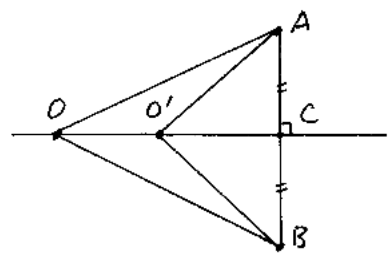
*Case 1* $O * O' * A$. Since $OA = OB$, $\angle OAB \cong \angle OBA$. Also, since $O'A = O'B$, $\angle O'AB \cong \angle O'BA$, using (I.5). It follows that $\angle OBA \cong \angle O'BA$, which contradicts axiom (C4). (This argument also applies if $O' * O * A$.)




*Case 2* $O * A * O'$. Again using (I.5) we find that $\angle OAB \cong \angle OBA$ and $\angle O'AB \cong \angle O'BA$. But the two angles at $A$ are supplementary, so it follows that the two angles at $B$ are supplementary (9.2). But then $O$, $B$, and $O'$ would be collinear (I.14), which is a contradiction.




Conversely, suppose that $\Gamma$ and $\Gamma'$ are tangent at $A$, and suppose that $O, O', A$ are not collinear. Then we let $AC$ be perpendicular to the line $OO'$, and choose $B$ on the line $AC$ on the other side of $OO'$ with $AC \cong BC$. It follows by congruent triangles that $OA \cong OB$ and $O'A \cong O'B$, so $B$ also lies on $\Gamma$ and $\Gamma'$, contradicting the hypothesis $\Gamma$ tangent to $\Gamma'$. We conclude that $O, O', A$ are collinear.

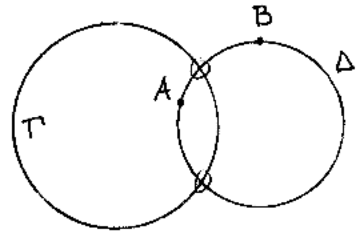



**Corollary 11.5**
*If two circles meet at a point A but are not tangent, then they have exactly two points in common.*

*Proof* We have seen above that if they are not tangent, then $O, O', A$ are not collinear, and they meet in an additional point $B$. We must show there are no further intersection points. If $D$ is a third point on $\Gamma$ and $\Gamma'$, then $OD \cong OA$ and $O'D \cong O'A$, so by (I.7), $D$ must be equal to $A$ or $B$.

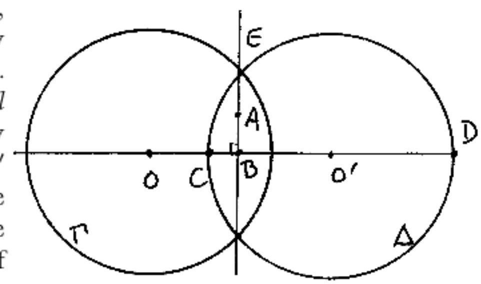
In the above discussion of lines and circles meeting, we have seen that a line and a circle, or two circles, can be tangent (meeting in just one point), or if they meet but are not tangent, they will meet in exactly two points. There is nothing here to guarantee that a line and a circle, or two circles, will actually meet if they are in a position such that they "ought" to meet according to the usual intuition. For this we need an additional axiom (and we will see later (17.3) that this axiom is independent of the axioms of a Hilbert plane).

**E.** (Circle–circle intersection property). Given two circles $\Gamma, \Delta$, if $\Delta$ contains at least one point inside $\Gamma$, and $\Delta$ contains at least one point outside $\Gamma$, then $\Gamma$ and $\Delta$ will meet. (*Note*: It follows from Exercise 11.3 and (11.5) that they will then meet in exactly two points.)




**Proposition 11.6** (Line–circle intersection property LCI)
*In a Hilbert plane with the extra axiom* (E), *if a line $l$ contains a point $A$ inside a circle* $\Gamma$, *then $l$ will meet* $\Gamma$ (*necessarily in two points, because of* (11.2) *and* (11.3)).

*Proof* Suppose we are given the line $L$ with a point $A$ inside the circle $\Gamma$. Our strategy is to construct another circle $\Delta$, show that $\Delta$ meets $\Gamma$, and then show that the intersection point also lies on $l$. Let $OB$ be the perpendicular from $O$ to $l$ (if $O$ is on the line $l$, we already know that $l$ meets $\Gamma$ by (C1)). Find a point $O'$ on the other side of $l$ from $O$, on the line $OB$, with $O'B \cong OB$. Let $\Delta$ be the circle with center $O'$ and radius $r =$ radius of $\Gamma$. (Here we denote by $r$ the congruence equivalence class of any radius of the circle $\Gamma$.)




Now the line $OO'$ meets $\Delta$ in two points $C, D$, labeled such that $O, C$ are on the same side of $O'$, and $D$ on the opposite side.

By hypothesis, $A$ is a point on $l$, inside $\Gamma$. Hence $OA < r$. In the right triangle $OAB$, using (I.19) we see that $OB < OA$, so $OB < r$. It follows that $O'B < r = O'C$, so $O'$ and $C$ are on opposite sides of $l$. Hence $O, C$ are on the same side of $l$. We wish to show that $C$ is inside $\Gamma$. There are two cases.

*Case 1* If $O * C * B$, then $OC < OB < r$, so $C$ is inside $\Gamma$.

*Case 2* If $C * O * B$, then also $C * O * O'$, so $OC < O'C = r$, and again we see that $C$ is inside $\Gamma$.

On the other hand, the point $D$ satisfies $O * O' * D$, so $OD > O'D = r$, so $D$ is outside $\Gamma$.

## PROP. XXII. B. I.

Some Authors blame Euclid becauſe he does not demonſtrate that the two circles made uſe of in the conſtruction of this Problem ſhall cut one another. but this is very plain from the determination he has given, viz. that any two of the ſtraight lines DF, FG, GH muſt be greater than the third. for who is ſo dull, tho' only beginning to learn the Elements, as not to perceive that the circle deſcribed from the centre F, at the diſtance FD, muſt meet FH betwixt F and H, becauſe FD is leſſer than FH; and that, for the like reaſon, the circle deſcribed from the centre G, at the diſtance GH or GM muſt meet DG betwixt D and G; and that theſe circles muſt meet one another, becauſe FD and GH are together greater than FG? and this determination is eaſier to be underſtood than that which Mr. Thomas Simpſon derives from it, and puts inſtead of Euclid's, in the 49. page of his Elements of Geometry, that he may ſupply the omiſſion he blames Euclid for; which determination is, that any of the three ſtraight lines muſt be leſſer than the ſum, but greater than the difference of the other two. from this he ſhews the circles muſt meet one another, in one caſe; and ſays that it may be proved after the ſame manner in any other caſe. but the ſtraight line GM which he bids take from GF may be greater than it, as in the figure here annexed, in which caſe his demonſtration muſt be changed into another.

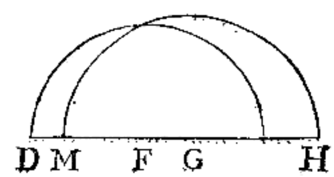
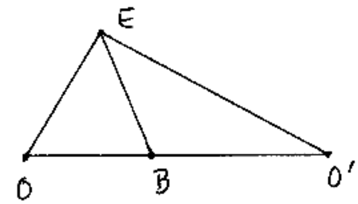



Plate V. Simson's commentary on (I.22) from his English translation of Euclid (1756).

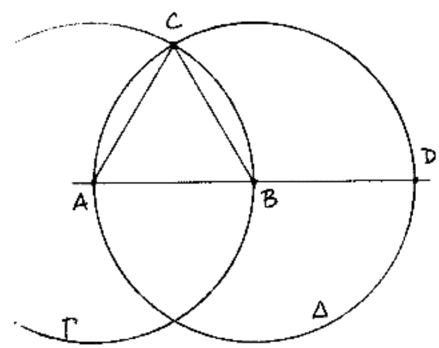
Now we can apply the axiom (E) to conclude that $\Gamma$ meets $\Delta$ at a point $E$. We must show that $E$ lies on $l$. We know that $OE \cong r \cong O'E$ and $OB \cong O'B$ by construction, and $BE$ is equal to itself, so by (SSS) $\Delta OEB \cong \Delta O'EB$. It follows that the angles at $B$ are equal, so they are right angles, so $BE$ is equal to the line $l$, and so $E$ lies on $l$ and $\Gamma$, as required.




**Remark 11.6.1**

We will see later (16.2) that in the Cartesian plane over a field, the circle–circle intersection property is equivalent to the line–circle intersection property. In an arbitrary Hilbert plane, the equivalence of these two statements follows from the classification theorem of Pejas (cf. Section 43), but I do not know any direct proof.

Using the new axiom (E) we can now justify Euclid's first construction (I.1), the equilateral triangle. Given the segment $AB$, let $\Gamma$ be the circle with center $A$ and radius $AB$. Let $\Delta$ be the circle with center $B$ and radius $BA$. Then $A$ is on the circle $\Delta$, and it is inside $\Gamma$ because it is the center of $\Gamma$. The line $AB$ meets $\Delta$ in another point $D$, such that $A * B * D$. Hence $AD > AB$, so $D$ is outside $\Gamma$.




Thus $\Delta$ contains a point inside $\Gamma$ and a point outside $\Gamma$, so it must meet $\Gamma$ in a point $C$. From here, Euclid's proof shows that $\Delta ABC$ is an equilateral triangle.

In a similar way one can justify Euclid's other ruler and compass constructions in Book I. Several of them depend only on using the equilateral triangle constructed in (I.1). For (I.12) and (I.22) see Exercise 11.4 and Exercise 11.5. Thus we have the following theorem.

**Theorem 11.7**

*Euclid's constructions* (I.1) *and* (I.22) *are valid in a Hilbert plane with the extra axiom* (E).

We can also justify the results of Euclid, Book III, up through (III.19) (note that (III.20) and beyond need the parallel axiom). The statements (III.10), (III.11), (III.12) about circles meeting and (III.16), (III.18), (III.19) about tangent lines can be replaced by the propositions of this section. (We omit the controversial last phrase of (III.16) about the angle of the semicircle, also called a horned angle or angle of contingency, because in our treatment we consider only angles defined by rays lying on straight lines.) In (III.14) Euclid uses (I.47) to prove (RASS), but that is not necessary: One can prove it with only the axioms of a Hilbert plane (Exercise 10.9). For (III.17), to draw a tangent to a circle from a point outside the circle, we need the line–circle intersection property (11.6) and hence the axiom (E). (Note that the other popular construction of the tangent line using (III.31) requires the parallel axiom!) The other results of Book III, up to (III.19) (except (III.17)), are valid in any Hilbert plane, provided that we assume the existence of the intersection points of lines and circles used in the statement and proofs, and their proofs are ok as is, except as noted.

**Theorem 11.8**
*Euclid's propositions* (III.1) *through* (III.19) *are valid in any Hilbert plane, except that for the constructions* (III.1) *and* (III.17) *we need also the additional axiom* (E).

# Exercises

11.1 (a) The interior of a circle $\Gamma$ is a *convex* set: Namely, if $B, C$ are in the interior of $\Gamma$, and if $D$ is a point such that $B * D * C$, then $D$ is also in the interior of $\Gamma$.

(b) Assuming the parallel axiom (P), show that if $B, C$ are two points outside a circle $\Gamma$, then there exists a third point $D$ such that the segments $BD$ and $DC$ are entirely outside $\Gamma$. (This implies that the exterior of $\Gamma$ is a *segment-connected* set. See also Exercise 12.6.)

11.2 Two circles $\Gamma, \Gamma'$ that meet at a point $A$ are tangent if and only if the tangent line to $\Gamma$ at $A$ is equal to the tangent line to $\Gamma'$ at $A$.

11.3 If two circles $\Gamma$ and $\Delta$ are tangent to each other at a point $A$, show that (except for the point $A$) $\Delta$ lies either entirely inside $\Gamma$ or entirely outside $\Gamma$.

11.4 Use the line–circle intersection property (Proposition 11.6) to give a careful justification of Euclid's construction (I.12) of a line from a point perpendicular to a given line.

11.5 Given three line segments such that any two taken together are greater than the third, use (E) to justify Euclid's construction (I.22) of a triangle with sides congruent to the three given segments.

11.6 Show that Euclid's construction of the circle inscribed in a triangle (IV.4) is valid in any Hilbert plane. Be sure to explain why two angle bisectors of a triangle must meet in a point. Conclude that all three angle bisectors of a triangle meet in the same point.

11.7 Using (E), show that Euclid's construction of a hexagon inscribed in a circle (IV.15) makes sense. Without using (P) or results depending on it, which sides can you show are equal to each other?

## 12 Euclidean Planes

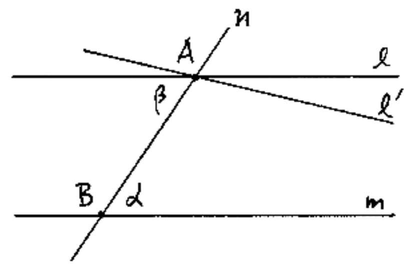
Let us look back at this point and see how well Hilbert's axioms have fulfilled their goal of providing a new solid base for developing Euclid's geometry. The major problems we found with Euclid's method have been settled: Questions of relative position of figures have been clarified by the axioms of betweenness; the problematic use of the method of superposition has been replaced by the device of taking SAS as an axiom; the existence of points needed in ruler and compass constructions is guaranteed by the circle–circle intersection property stated as axiom (E). Also, in the process of rewriting the foundations of geometry we have formulated a new notion, the *Hilbert plane*, which provides a minimum context in which to develop the beginnings of a geometry, free from the parallel axiom. Hilbert planes serve as a basis both for Euclidean geometry, and also later, for the non-Euclidean geometries.

In this section we will complete the work of earlier sections by showing how the addition of the parallel axiom allows us to recover almost all of the first four books of Euclid's *Elements*. We will also mention two more axioms, those of Archimedes and of Dedekind, which will be used in some parts of later chapters.

**Definition**
A *Euclidean plane* is a Hilbert plane satisfying the additional axioms (E), the circle–circle intersection property, and (P), Playfair's axiom, also called the parallel axiom. In other words, a Euclidean plane is a set of points with subsets called lines, and undefined notions of betweenness and congruence satisfying the axioms (I1)–(I3), (B1)–(B4), (C1)–(C6), (E), and (P). The Euclidean plane represents our modern formulation of the axiomatic basis for developing the geometry of Euclid's *Elements*.

We have already seen in Section 10 and Section 11 how to recover those results of Euclid's Books I and III that do not depend on the parallel axiom. The first use of the parallel axiom is in (I.29). Since we have replaced Euclid's fifth postulate by Playfair's axiom, we need to modify Euclid's proofs of a few early results in the theory of parallels.

So for example, to prove (I.29) we proceed as follows. Given two parallel lines $l$, $m$, and a transversal line $n$, we must show that the alternate interior angles $\alpha$ and $\beta$ are equal. If not, construct a line $l'$ through $A$ making an angle $\alpha$ with $n$ (axiom (C4)). By (I.27), $l'$ will be parallel to $m$. But then $l$ and $l'$ are two lines through $A$ parallel to $m$, so by (P), we must have $l = l'$, hence $\alpha = \beta$.




Proposition (I.30) is essentially equivalent to (P). The existence of parallel lines (I.31) follows from (C4) and (I.27) as mentioned before, so now we can reinterpret (I.31) in the stronger form that given a point $A$ not on a line $l$, there exists a unique parallel to $l$ passing through $A$. The remaining propositions using (P), namely (I.32)–(I.34), follow without difficulty. In particular, we have the famous (I.32), that "the sum of the angles of a triangle is equal to two right angles," though if we want to be scrupulous, we would have to say that sum is not defined, and rephrase the theorem by saying that the sum of any two angles of a triangle is supplementary to the third angle.

**Theorem 12.1**
*Euclid's theory of parallels, that is, propositions* (I.29)–(I.34), *hold in any Hilbert plane with* (P), *hence in any Euclidean plane.*

Starting with (I.35), and continuing to the end of Book I and through Book II, is Euclid's theory of area. Since Euclid does not define what he means by this new equality, we must presume that he takes it as another undefined notion, which we call *equal content*, just as the notion of congruence for line segments and angles were taken as undefined notions. Since Euclid freely applies the common notions to this concept, we may say that he has taken the common notions applied to equal content as further axioms, for example, "figures having equal content to a third figure have equal content to each other," or "halves of figures of equal content have equal content."

Hilbert showed that it is not necessary to regard the notion of equal content as an undefined notion subject to further axioms. He shows instead that it is possible to *define* the notion of equal content for figures (by cutting them up, rearranging, and adding and subtracting), and then *prove* the properties suggested by Euclid's common notions. To be more precise, we have the following theorem.

**Theorem 12.2** (Theory of area)
*In a Hilbert plane with* (P) *there is an equivalence relation called* equal content *for rectilineal figures that has the following properties*:

(1) *Congruent figures have equal content.*
(2) *Sums of figures with equal content have equal content.*
(3) *Differences of figures with equal content have equal content.*
(4) *Halves of figures with equal content have equal content.*
(5) *The whole is greater than the part.*
(6) *If two squares have equal content, their sides are congruent.*

We will prove this theorem in Chapter 5, (22.5), (23.1), (23.2). For the present you can either accept this result as something to prove later, or (as Euclid implicitly did) you can regard equal content of figures as another undefined notion, subject to the axioms that it is an equivalence relation and has these properties (1)–(6). For further discussion and more details about the exact meaning of a figure, the notions of sum and difference, etc., see Section 22 and Section 23.

Using this theory of area, the remaining results (I.35)–(I.48) of Book I follow without difficulty. Note in particular the Pythagorean theorem (I.47), which says that the sum of the squares on the legs of a right triangle have equal content with the square on the hypotenuse. Also, the results of Book II, (II.1)–(II.14), phrased as results about equal content, all follow easily. Proposition (II.11), how to cut a line segment in extreme and mean ratio, is used later in the construction of the regular pentagon. Only (II.14), to construct a square with content equal to a given rectilineal figure, uses the axiom (E).

**Theorem 12.3**
*In a Hilbert plane with* (P), *using the theory of area* (12.2), *Euclid's propositions* (I.35)–(I.48) *and* (II.1)–(II.14) *can all be proved as he does, using the extra axiom* (E) *only for* (II.14). *In particular, all these results hold in a Euclidean plane.*

In Book III, the first use of the parallel axiom is in (III.20), that the angle at the center of a circle subtending a given arc is twice the angle on the circumference subtending the same arc. This result uses (I.32), that the exterior angle of a triangle is equal to the sum of the two opposite interior angles, and thus depends on the parallel axiom (P). The following propositions (III.21), (III.22), and then (III.31)–(III.34) follow with no further difficulties. For the propositions (III.23)–(III.30) we need a notion of "equal" segments of circles, a congruence notion that has not been defined by Euclid, though we can infer from the proof of (III.24) that it means being able to place one segment on the other by a rigid motion. Indeed, if we take this as a definition of congruence, then the proofs of these results are all ok (Exercise 17.13). The final propositions (III.35)–(III.37) make use of the theory of area for their statements, and depend on the earlier area results from Books I and II.

**Theorem 12.4**
*In Book* III, *Euclid's propositions* (III.20)–(III.37) *hold in any Euclidean plane. The last three* (III.35)–(III.37) *make use of the theory of area* (12.2).

Most of the results of Book IV require the parallel axiom (P), some need circle–circle intersection (E), and some, notably (IV.10), (IV.11), require (P), (E), and the theory of area. Thus we may regard the construction of the regular pentagon as the crowning result of the first four books of the *Elements*, making use of all the results developed so far.

**Theorem 12.5**
*All the propositions* (IV.1)–(IV.16) *of Euclid's Book* IV *hold in a Euclidean plane.*

We end this section with a discussion of two further axioms that are not needed for Books I–IV, but will be used later. The first is *Archimedes' axiom.*

**A.** Given line segments $AB$ and $CD$, there is a natural number $n$ such that $n$ copies of $AB$ added together will be greater than $CD$.

This axiom is used implicitly in the theory of proportion developed in Book V, for example in Definition 4, where Euclid says that quantities have a ratio when one can be multiplied to exceed the other. It appears explicitly in (X.1), in a form reminiscent of the $\varepsilon$-arguments of calculus: Given two quantities $AB$ and $CD$, if we remove from $AB$ more than its half, and again from the remainder remove more than its half, and continue in this fashion, then eventually we will have a quantity less than $CD$. In modern texts this would appear as the statement "given any $\varepsilon > 0$, there is an integer $n$ sufficiently large that $1/2^n < \varepsilon$." Euclid applies this "method of exhaustion" to the study of the volume of three-dimensional figures in Book XII. When he cannot compare solids by cutting into a finite number of pieces and reassembling, he uses a limiting process where the solid is represented as a union of a sequence of subsolids so that the remainder can be made as small as you like. See Sections 26, 27 for Euclid's theory of volume.

Archimedes' axiom is independent of all the axioms of a Hilbert plane or a Euclidean plane, so we will see examples of *Archimedean* geometries that satisfy (A) and *non-Archimedean* geometries that do not (Section 18).

The other axiom we would like to consider is *Dedekind's axiom*, based on Dedekind's definition in the late nineteenth century of the real numbers:

**D.** Suppose the points of a line $l$ are divided into two nonempty subsets $S$, $T$ in such a way that no point of $S$ is between two points of $T$, and no point of $T$ is between two points of $S$. Then there exists a unique point $P$ such that for any $A \in S$ and any $B \in T$, either $A = P$ or $B = P$ or the point $P$ is between $A$ and $B$.

This axiom is very strong. It implies (A) and (E), and a Euclidean plane with (D) is forced to be isomorphic to the Cartesian plane over the real numbers. (See Exercise 12.2, Exercise 12.3, (15.5), and (21.3).) So if you want a categorical axiom system, just add (D) to the axioms of a Euclidean plane. From the point of view of this book, however, there are two reasons to avoid using Dedekind's axiom. First of all, it belongs to the modern development of the real numbers and notions of continuity, which is not in the spirit of Euclid's geometry. Second, it is too strong. By essentially introducing the real numbers into our geometry, it masks many of the more subtle distinctions and obscures questions such as constructibility that we will discuss in Chapter 6. So we include this axiom only to acknowledge that it is there, but with no intention of using it.

## Exercises

12.1 Show that in a Hilbert plane with (P), the perpendicular bisectors of the sides of a triangle will meet in a point, and thus justify Euclid's construction of the circumscribed circle of a triangle (IV.5). *Note*: In a non-Euclidean geometry, there may be triangles having no circumscribed circle: cf. Exercise 18.4, Exercise 39.14, and Proposition 41.1.

12.2 Show that in a Hilbert plane Dedekind's axiom (D) implies Archimedes' axiom (A). *Hint*: Given segments $AB$ and $CD$, let $T$ be the set of all points $E$ on the ray $\overrightarrow{CD}$ for which there is no integer $n$ with $n \cdot AB > CE$. Let $S$ be the set of points of the line $CD$ not in $T$, and apply (D).

12.3 Show that in a Hilbert plane (D) implies (E). *Hint*: Follow the discussion in Heath (1926), vol. I, p. 238.

12.4 For the construction and proof of (IV.2), to inscribe a triangle equiangular with a given triangle in a given circle (assume also that you are given the center of the circle), is axiom (E) necessary? Is (P) necessary?

12.5 Same question for (IV.6), to inscribe a square in a given circle.

12.6 In a Hilbert plane with (A), show that the exterior of a circle is a segment-connected set (cf. Exercise 11.1). Without assuming either (P) or (A), this may be false (Exercise 43.17).

> To each book are appended explanatory notes, in which especial care has been taken to guard the student against the common mistake of confounding ideas of number with those of magnitude.
>
> – Preface to Potts' Euclid,
> London (1845)