39.30 For an Archimedean example of a field as in Exercise 39.29, let F be the field of all those real numbers that can be expressed using rational numbers and a finite number of operations $+, -, \cdot, \div, a \mapsto \sqrt{1+a^2}$, and $a \mapsto \sqrt{a^2 - \sqrt{2}}$, provided that $a^2 - \sqrt{2} > 0$.

(a) F is a Pythagorean ordered field, $d = \sqrt{2}$ is in F, and F satisfies condition $(*d)$ of Exercise 39.26 for $d = \sqrt{2}$.

(b) Let $\varphi : \mathbb{Q}(\sqrt{2}) \to \mathbb{R}$ be the homomorphism that makes $\varphi(\sqrt{2}) = -\sqrt{2}$. Show inductively that φ extends to a homomorphism φ of F to $\mathbb{R}$.

(c) Since $\varphi(\sqrt{2}) < 0$, conclude that $\sqrt{2}$ cannot be a square in F.

39.31 Show that in the Poincaré model in the virtual circle $x^2 + y^2 = \sqrt{2}$ over the field F of Exercise 39.30, not every segment can be the side of an equilateral triangle, as follows.

(a) If $x \in F$ with $0 < x$ and $x^2 < \sqrt{2}$, let AB be the segment from $(0,0)$ to $(x,0)$ in the Poincaré model, and show that

$$\mu(AB) = \frac{\sqrt[4]{2} + x}{\sqrt[4]{2} - x}.$$

(b) If there is an equilateral triangle with side AB, let the angle at a vertex be α, and let $t = \tan(\alpha/2)$. Use Exercise 39.5 to show that

$$t = \sqrt{\frac{\sqrt{2} - x^2}{3\sqrt{2} + x^2}} = \frac{1}{3\sqrt{2} + x^2}\sqrt{6 - 2x^2\sqrt{2} - x^4}.$$

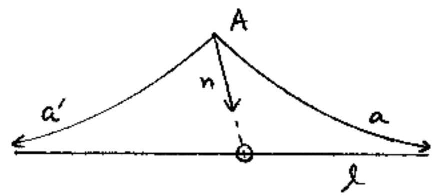
(c) Now take a suitable x, such as $x = \sqrt{3} - 1$, and use an argument similar to the previous exercise to show that the corresponding t is not in F. Hence the equilateral triangle with side AB does not exist. *Hint*: For these two exercises, it may be useful to review the techniques used in Exercises 16.10–16.14.

40 Hyperbolic Geometry

In the earlier sections of this chapter we have seen something of the development of neutral geometry and the study of the angle sum of a triangle using Archimedes' axiom. We have also seen the Poincaré model of a non-Euclidean geometry over a field. For the full development of the geometry of Bolyai and Lobachevsky, we need the limiting parallels. The existence of these limiting parallels, which we have seen in the Poincaré model (39.12), does not follow in the axiomatic treatment from what we have done so far (Exercises 39.24, 39.28). Therefore, following Hilbert, we will take the existence of the limiting parallels as an axiom. This axiom is quite strong. It will allow us to develop non-Euclidean geometry independently of Archimedes' axiom. It also allows the construction of an ordered field out of the geometry (Section 41), and a proof that the abstract geometry is isomorphic to the Poincaré model over this field (Section 43). Using coordinates from this field we can develop non-Euclidean analytic geometry and trigonometry (Section 42).

So at this point we start the axiomatic development of hyperbolic geometry, which is essentially the "classic" non-Euclidean geometry of Bolyai and Lobachevsky, freed from hypotheses of continuity. In particular, we will not use the circle–circle intersection axiom (E) nor Archimedes' axiom (A). Instead, we use Hilbert's axioms of incidence, betweenness, and congruence plus the following *hyperbolic axiom* (L):

L. For each line l and each point A not on l, there are two rays Aa and Aa' from A, not lying on the same line, and not meeting l, such that any ray An in the interior of the angle aAa' meets l.



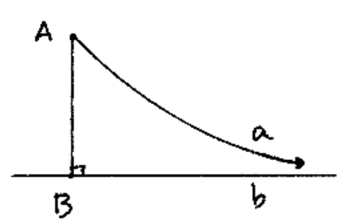
Note that (L) immediately implies that the geometry is non-Euclidean, because the two rays Aa and Aa' lie on distinct lines through A that will both be parallel to l.

Definition
A Hilbert plane satisfying (L) will be called a *hyperbolic* plane, or a *hyperbolic geometry*.

We will see shortly (40.3) that the angle sum of a triangle in a hyperbolic *plane* is less than 2RA, so this terminology is consistent with the term semi-hyperbolic introduced earlier (Section 34).

Recalling the definition of limiting parallel rays from Section 34, we see that if we pick any point B on l and let Bb, Bb' be the two rays from B lying on l, then Aa will be limiting parallel to Bb and Aa' limiting parallel to Bb'. Thus (L) implies that for any point A and any ray Bb, there exists a limiting parallel Aa to Bb. We define an *end* to be an equivalence class of limiting parallel rays (34.13).

Definition
For any segment AB, let b be a line perpendicular to AB at b; choose one ray Bb on the line b, and let Aa be the limiting parallel ray to Bb, which exists by (L). Then we call $\alpha = \angle BAa$ the *angle of parallelism* of the segment AB, and we denote it by $\alpha(AB)$. (Lobachevsky uses the notation $\Pi(AB)$.)



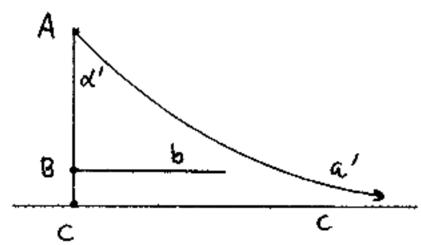
Note that the angle of parallelism in well-defined: If we reflect Aa in the line AB, then clearly it will be limiting parallel to the other ray on b, so that the angle α is independent of which ray we chose on the line b. Note also that the angle of parallelism α is necessarily *acute*, because the two limiting parallels from A to b do not lie on the same line, by (L).

Proposition 40.1
The angle of parallelism varies inversely with the segment:

(a) $AB < A'B' \Leftrightarrow \alpha(AB) > \alpha(A'B')$.
(b) $AB \cong A'B' \Leftrightarrow \alpha(AB) = \alpha(A'B')$.

Proof First suppose that $AB \cong A'B'$. Then by the (ASL) congruence theorem for limit triangles (Exercise 34.10) it follows that $\alpha(AB) = \alpha(A'B')$.

Next, suppose $AB < A'B'$. Mark off C on the ray $\overrightarrow{AB}$ such that $AC = A'B'$, draw the perpendicular c to AC at C, and let Aa' be the limiting parallel from A to Cc. Then $\alpha' = \angle CAa'$ is $\alpha(AC) = \alpha(A'B')$.



Let Bb be the ray perpendicular to AB at B on the same side of AC as a' and c. I claim that Bb meets a'. If not, then the ray Bb would be in the interior of the angles CAa' and ACc, meeting neither the ray a' nor c, and so it would be also limiting parallel to Aa' and Cc by (34.12.1). But this contradicts the fact that the angle of parallelism is always acute, since $Bb \parallel\!| Cc$ and the angles at B and C are both right angles.

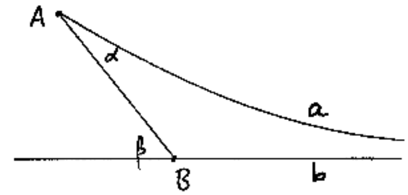
So Bb meets Aa', and this implies that the limiting parallel from A to Bb makes an angle α greater than α', i.e., $\alpha(AB) > \alpha(A'B')$.

Reversing the roles of AB and $A'B'$ we find that if $AB > A'B'$, then $\alpha(AB) < \alpha(A'B')$. Combining all three results now gives the desired reverse implications.

Remark 40.1.1
We will see later (40.7) that for every acute angle α, there exists a segment AB with $\alpha(AB) = \alpha$.

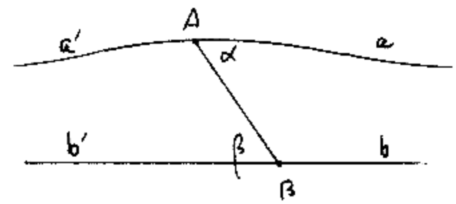
Our next goal is to establish some results about limiting parallel rays, limit triangles, and parallel lines that are not limiting parallel. We have already seen two congruence results (ASL) = (Exercise 34.10) and (ASAL) = (Exercise 34.9). We will prove some others, analogous to those for ordinary triangles in Euclid's *Elements*, Book I.

Proposition 40.2 (Exterior angle theorem)
If AB is a segment, with limiting parallel rays emanating from A and B, then the exterior angle β at B is greater than the interior angle α at A.



Proof Because the ray through A making an angle β with AB is parallel to l (I.27) we know at least that $\alpha \leq \beta$.

So suppose $\alpha = \beta$. Let a' and b' be the opposite rays to a and b. The supplementary angles at A and B will also be equal. Since AB is equal to itself, we can apply (ASAL) = (Exercise 34.9) to AB, a, b, and BA, b', a'. We conclude that a' is also limiting parallel to b'.

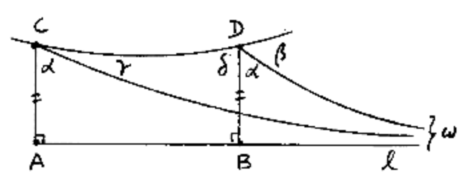


But this contradicts the axiom (L), which says the two limiting parallels from A to b do not lie on the same line. Therefore, $\alpha < \beta$, as required.

Corollary 40.3
In a hyperbolic plane, the sum of the angles of any triangle is less than two right angles.

Proof According to (34.6), for any triangle there is a Saccheri quadrilateral whose top two angles are equal to the angle sum of the triangle. So we have only to prove that the top two equal angles of any Saccheri quadrilateral are acute.

Let the Saccheri quadrilateral be $ABCD$, with base $AB = l$. Draw limiting parallels from C and D to l, with end ω by axiom (L). Then by (40.1) the angles of parallelism α are equal.



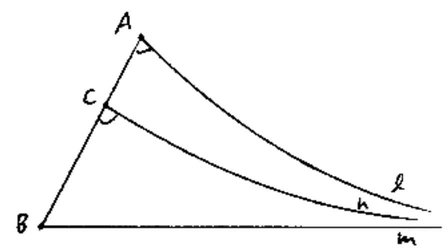
Looking at the limit triangle $CD\omega$, by the exterior angle theorem (40.2), $\beta > \gamma$. On the other hand, by (34.1), the top angles $\alpha + \gamma$ and δ of the Saccheri quadrilateral are equal. We conclude that $\alpha + \beta > \alpha + \gamma = \delta$, and so δ must be acute.

Remark 40.3.1
Note how different this proof is from the proof of the Saccheri–Legendre theorem (35.2), which reaches the same conclusion under different hypotheses.

There we made use of Archimedes' axiom and a countable limiting process. Here we do not need (A), but we use instead the powerful axiom (L) on the existence of limiting parallels. This result says that a hyperbolic plane is semi-hyperbolic, thus justifying the terminology introduced earlier (Section 34).

Proposition 40.4 (AAL)
Given two limit triangles $ABlm$ and $A'B'l'm'$, suppose that the angles at A and B are equal respectively to the angles at A' and B'. Then also the sides AB and $A'B'$ are equal.

Proof If not, let us suppose that $AB > A'B'$. Choose a point C on AB such that $CB = A'B'$, and draw a ray n at C, on the same side of AB as l and m, making an angle equal to the angle at A', which is also equal to the angle at A. Now comparing C, B, n, m to the limit triangle $A'B'l'm'$, it follows from (ASAL) = (Exercise 34.9) that n is limiting parallel to m.



Then by transitivity (34.11) it follows also that l is limiting parallel to n. But this contradicts the exterior angle theorem (40.2) because the angle at C, which is exterior to the limit triangle $ACln$, is equal to the angle at A.

We conclude that $AB = A'B'$, as required.

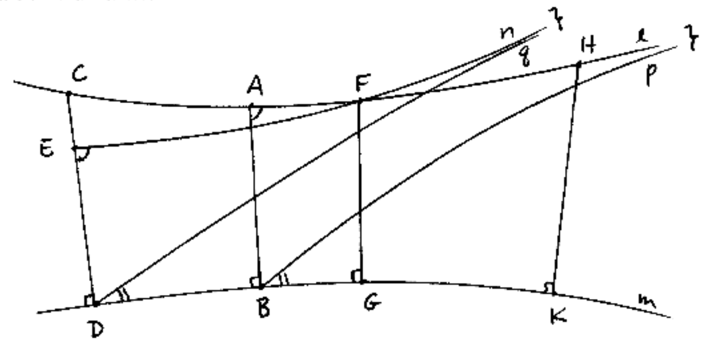
Remark 40.4.1
For some results about triangles with two or three "limit angles," see Exercises 40.2, 40.8.

Theorem 40.5
In a hyperbolic plane, if l and m are two parallel lines that are not limiting parallels, then there is a unique line in the plane that is perpendicular to both of them.

Proof Let l and m be two parallel lines that are not limiting parallels. Let AB and CD be two perpendiculars from points A, C on l to m. If $AB = CD$, then $DBCA$ is a Saccheri quadrilateral, and hence the line joining the midpoints of AC and BD will be perpendicular to both l and m, by (34.1).

If $AB \neq CD$, we may assume $CD > AB$, and we proceed as follows. Take E on CD such that $AB = ED$. Let n be a ray through E making the same angle with ED as l makes with AB. I claim that n will meet l in a point F. Indeed, let p be a limiting parallel from B to l. Since by hypothesis l and m are not limiting parallels, this ray does not lie on the line m. Let q be the ray through D making the same angle with m as p does at B. Then q is parallel to p by (I.28), but not a limiting parallel, by the exterior angle theorem (40.2). On the other hand, applying (ASAL) to $ABlp$ and $EDnq$, we find that q is limiting parallel to n. Therefore, n is not limiting parallel to p, and hence n must meet l at some point F. (In the figure we put F on the far side of A from C, but the proof works equally well if F is between A and C.)

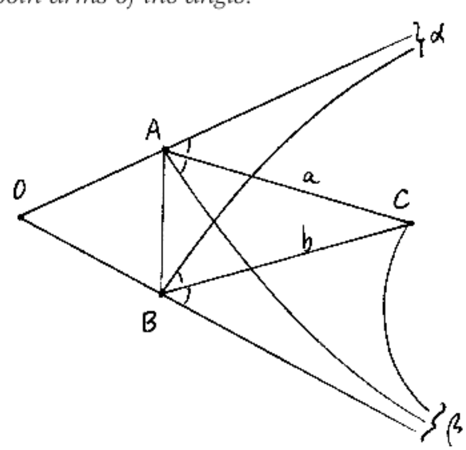
Now take H on l such that $AH = EF$, and take K on m such that $BK = DG$. Then comparing the quadrilaterals $EFDG$ and $AHBK$, two applications of (SAS) show that $FG = HK$ and HK is perpendicular to m. Thus $GKFH$ is a Saccheri quadrilateral, and the line joining the midpoints of FH and GK will be perpendicular to both l and m.



It remains to show the uniqueness of the line perpendicular to both l and m. Suppose to the contrary that AB and CD were two common perpendiculars to l and m. Then $ABCD$ would be a rectangle, which is impossible—cf. (40.3) and (34.7).

Proposition 40.6
Given an angle in the hyperbolic plane, there is a unique line (called the enclosing line *of the angle) that is limiting parallel to both arms of the angle.*

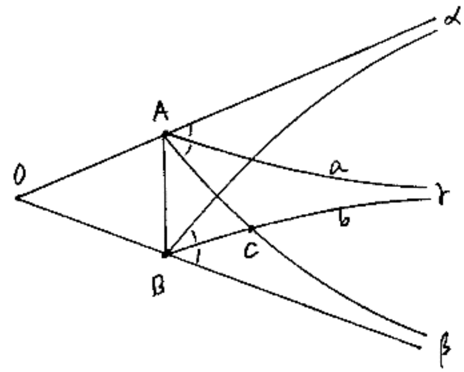
Proof Let O be the vertex of the angle, and choose points A, B on the two arms of the angle, at equal distance from O. It will be convenient at this point to introduce a new notation. We denote by α the end of the ray OA, that is, the equivalence class of all rays limiting parallel to OA. Then we may draw the line $B\alpha$, meaning, let $B\alpha$ be the ray through B limiting parallel to OA. We may also speak of the limit triangle $AB\alpha$, consisting of the segment AB plus the two limiting parallel rays $A\alpha$ and $B\alpha$.



To continue our proof, let α be the end of OA, and let β be the end of OB. Draw $B\alpha$ and $A\beta$. Let a be the ray bisecting the angle $\alpha A\beta$, and let b be the ray bisecting the angle $\alpha B\beta$. Note by symmetry (!) that the bisected angles at A and B are equal. We distinguish three cases.

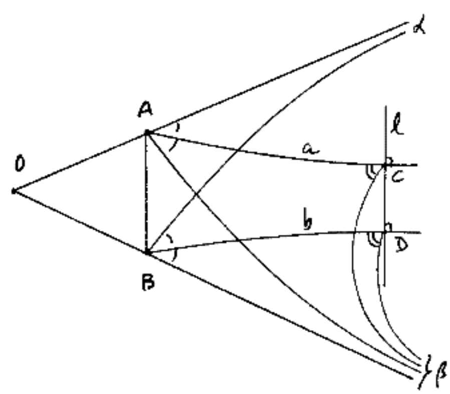
Case 1 The lines a and b meet at a point C. By symmetry (!) $AC = BC$. Draw the line $C\beta$. Then by (ASL) = (Exercise 34.10) applied to the limit triangle $AC\beta$ and $BC\beta$, the angles at C of these two triangles are equal. But this is clearly not so, so this case cannot occur. (See diagram on previous page.)

Case 2 The rays a and b are limiting parallel with an end γ. In this case the ray $B\gamma$ is in the interior of the angle $AB\beta$, so it meets $A\beta$ in a point C. By (AAL) = (40.4) applied to the limit triangles $AC\gamma$ and $BC\beta$, the sides AC and BC are equal. Therefore, by (I.5) the angles BAC and ABC are equal. But this is not so, because the angle BAC is also equal to the angle $AB\alpha$, which is properly contained in the angle ABC. So this case cannot occur either.



Case 3 The only remaining possibility is that a and b are parallel but not limiting parallels. Then by (40.5) there is a common perpendicular line l, meeting a at C and b at D. I claim that l is the required enclosing line, i.e., l has the ends α and β.

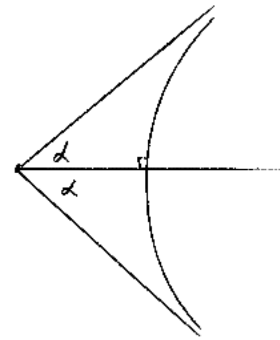
By symmetry it is enough to show that l has end β. If not, draw the lines $C\beta$ and $D\beta$, which will be distinct from l. We compare the limit triangles $AC\beta$ and $BD\beta$. The angles at A and B are equal, by construction. The sides AC and BD are equal by symmetry (!), so by (ASL) the angles at C and D are equal. It follows that $C\beta$ and $D\beta$ make equal angles with l at C and D, which contradicts the exterior angle theorem (40.2). We conclude that l has ends α and β, as required.



The uniqueness of the enclosing line is clear, because by (L) we cannot have two distinct lines that are limiting parallel at both ends.

Corollary 40.7
For any acute angle α, there exists a line that is limiting parallel to one arm of the angle and orthogonal to the other arm of the angle. In particular, there is a segment whose angle of parallelism is equal to α.

Proof Given the acute angle α, we double it, and consider the enclosing line (40.6) of the angle 2α. This will be orthogonal to the angle bisector of 2α, which is one arm of the original angle α. Thus α becomes the angle of parallelism of the segment cut off on that arm of the angle.

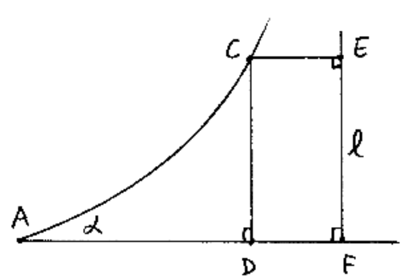


Remark 40.7.1
Combining with (40.1), we see that there is a one-to-one correspondence between the set of congruence equivalence classes of line segments and the set of congruence equivalence classes of acute angles, given by associating a segment AB to its angle of parallelism α. In particular, there is a uniquely determined standard or absolute segment size corresponding to one-half of a right angle.

Be careful, however, because this correspondence does not send sums of segments into sums of angles. There is a more complex relationship that we will see later (Exercise 42.7).

Proposition 40.8
In a hyperbolic plane, Aristotle's axiom holds, namely, given an angle α and a segment AB, there exists a point C on one arm of the angle such that the perpendicular CD from C to the other arm of the angle is greater than AB.

Proof Given the angle α at A, let l be a line limiting parallel to one arm of α and meeting the other arm at right angles at a point F (40.7). Take E on l such that $EF = AB$. Draw a perpendicular to l at E, and let it meet the other arm of the angle α at C (cf. Exercise 34.12). Drop a perpendicular CD from C to AF.



Volgen des Ersten Bůch Euclidis Proposition/Das ist/Fürgåben/oder Schlußreden.

I. Die erst Proposition/ Lehr

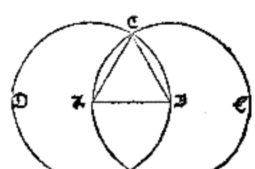
Auff ein gerade linj so einer bekhantten lenge ist / ainen gleichseittigen Triangel stellen.

Warnung an den Leser.

FReündtlicher lieber leser/dieweil die Demonstrationes (das ist grundtlich vnnd onwider sprechliche beweysungen deß yenigen/so Euclides in seinen Proposition als warhafftig fürgibt) nit von jme dem Euclide selbs/sondern von andern hoch gelerten khůnstreichen menneren/als Theone/Hypsicle/Campano/rc. hinzůgesetzt worden: Zů dem bemelte Demonstration/ ettwa schwårlich von vngelertten mögen vernommen vnnd begriffen werden/vnnd dann ein einfeltiger Teütscher liebhaber diser khünsten woll begnůgt vnnd zefriden ist/So er die sach versteht/ ob er schon vrsach vnnd den grund desselben nitt allmal erkhent: Hab ich solche Demonstration zů zeitten außgelassen/vnnd (welchs ich dem leser nützer vnd angenemmer vermain) an statt derselben/den gebrauch vnnd nutz solcher Proposition/wa ich das füglich sein vermaint angezaigt/vnnd mit Exemplen vnnd der ziffer zimlich erklertt.

Theoremata. Vnd seittenmal die Schlußreden oder Proposition Euclidis zwaierlai seind Dann etliche schlecht ein aigenschafft anzaigen/welche zůbeweisen andere ain grund ist (Als da seind in disem ersten bůch die 4.5.6 propositiones/rc.) Etlich aber lehren ettwas auß solchem grund machen/alß dise Erste ist/vnd *Problemata* darnach die 9.10.11/rc. Hab ich allmal mit fleiß anzaigt/wie sollichs zůmachen sei so Euclides leeret/vnnd zů vnderschaid allweg zů der zaal der Proposition geschriben/ Lehr. Auch alß dann gewonlich den grund sollicher handlung ainfeltiger weiß mit angezaigt. Auch hab ich zů zeitten in den figuren/bůchstaben oder ziffern gebraucht/zů zeitten nicht/nach dem ich hab mögen erachten füglich vnd verstendtlich sein woll der Leser für gůtl haben. Waß forthin in disem bůch von linien geredt/versthe alles von rechten gestracken linien.

Figur vnd Erklårung der ersten Proposition.



Wiewol dise proposition leichtlich mag verstanden werden/auß beigesetzter figur/will ich sich je doch (diewil sy die erst) weitleffig erklerē. Die für geben linj darauff ich den triangell soll machen/ ist bezaichnet mit denn bůchstaben a b/sollicher linj lenge begreiff ich mitt einem zirckel/vnnd setz den ainen fůß in den puncten a/vnd reiß mit dem andern den zirckel b c d/darnach setz den ainen fůß in den puncten b/vnnd reiß den zirckel a c e/dise zwen zirckell werden on zweiffel gleich sein/dann sy baid mit onuerruckhten zirckel/ in ainer weittin beschribenn/Vnnd bey den puncten c / gehn sy durcheinander/vnd machen ain creütz. demnach zeuch von den puncten c/gegen dem a ain rechte linj/vnnd dergleichen aine gegen dem b/ so hastu den

tria ngel

Plate XIV. Beginning of the first German translation of Euclid's *Elements*, by Wilhelm Holtzman, known as Xylander (1562). He says the simple German reader may have some difficulty understanding the proofs, so he will just give explanations instead.

Consider the quadrilateral $DFCE$. Because of (40.3), the angle at C must be acute. Therefore, $CD > EF = AB$ (34.2), as required.

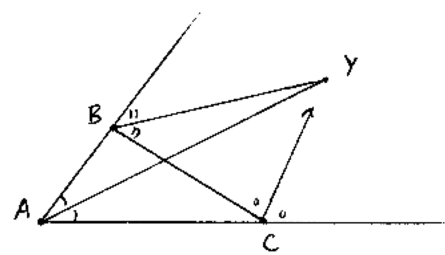
Remark 40.8.1
In fact, a stronger result is true, namely, given α and AB as above, one can find C such that $CD = AB$. The proof uses hyperbolic trigonometry (Exercise 42.8).

Now, as an illustration of the techniques of this section, we will give the hyperbolic version of a familiar Euclidean theorem on the angle bisectors of a triangle. The fact that the (internal) angle bisectors of a triangle meet in a point is true in neutral geometry, hence both in Euclidean and hyperbolic geometry, as we have seen before (Exercise 11.6). The following result has to do with the external angle bisectors of a triangle.

Proposition 40.9
In a hyperbolic plane, let ABC be a triangle, and consider the (internal) angle bisector at A and the external angle bisectors at B and C.

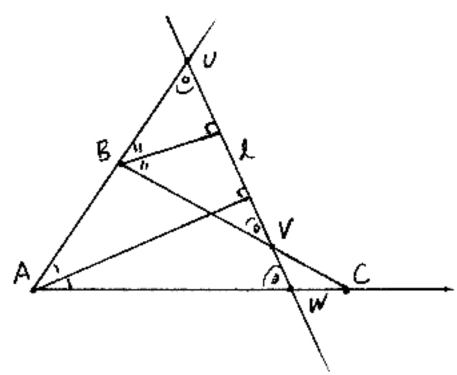
(a) *If two of these angle bisectors meet in a point, so does the third.*
(b) *If two of these angle bisectors have a common perpendicular line l, then the third is also perpendicular to l.*
(c) *If two of these angle bisectors are limiting parallels, so is the third, at the same end.*

Proof (a) If two of them meet in a point Y, then Y is equidistant from all three sides of the triangle; hence it lies on the third angle bisector. The proof in this case is the same as the Euclidean case (IV.4).



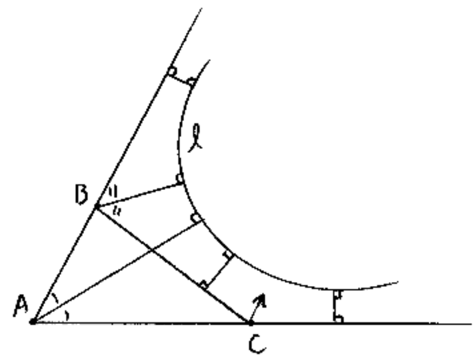
(b) Suppose that the angle bisectors at A and B have a common perpendicular line l.

We first claim that l cannot meet any side of the triangle. If it meets one side, then by reflecting in the two angle bisectors, it will meet the other two sides, and it will meet all three at the same angle. Two out of three of these intersections (in the diagram V, W) will have the angles in corresponding positions, so that by (I.28) the lines BC and AC will be parallel. This contradicts their meeting at the point C. Thus l cannot meet any side of the triangle.



Secondly, we note that l cannot be a limiting parallel to any side of the triangle. If it were, then by reflecting in the angle bisectors, it would be limiting parallel to the other two sides, and so would have three ends, which is absurd.

So l neither meets nor is limiting parallel to any side of the triangle; hence by (40.5) it has a common perpendicular with each side of the triangle. Using the lemma below, the first and second of these common perpendiculars are equal. Similarly, the first and third are equal, because the angle bisectors at A and B are orthogonal to l. Therefore, the second and third are equal, and using the lemma in the other direction, we see that the angle bisector at C is also perpendicular to l.

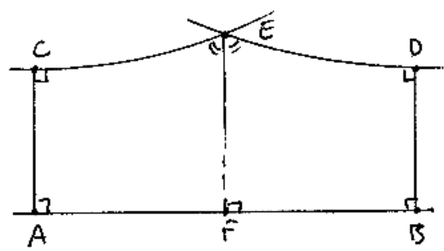


(c) This third case of the proposition follows by elimination. Suppose two angle bisectors are limiting parallel. If the third is not, then it either meets one of the others or has a common perpendicular, which puts us in case (a) or (b), contradicting the first two being limiting parallel.

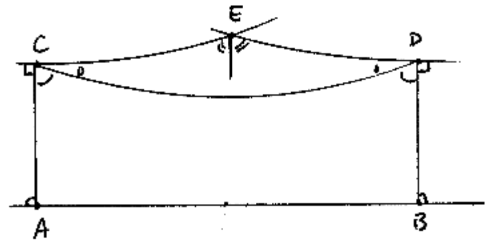
Lemma 40.10

Consider a five-sided figure ABCDE with right angles at A, B, C, D. Then AC = BD if and only if the angle bisector at E meets the opposite side at a point F at right angles.

Proof First suppose that the angle bisector at E meets AB at a point F, making a right angle there. Then reflection in the line EF sends the line AB into itself and interchanges the lines CE and DE. So the segments AC and BD are interchanged, because they are the unique common perpendiculars (40.5) between the lines AB and CE and AB and DE. Hence $AC = DB$.



Conversely, suppose $AC = DB$. Draw the line CD. Then $ABCD$ is a Saccheri quadrilateral, and the angles at C and D are equal (34.1). It follows that the base angles of the triangle CDE are equal. Hence it is an isosceles triangle, and the angle bisector at E will meet CD at its midpoint at right angles. Now it follows from (34.1) that this line continued will meet AB at its midpoint F, at right angles.

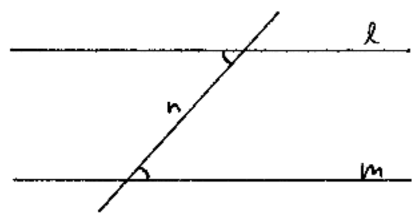


Remark 40.10.1
To make a more unified statement of (40.9) we will define an *ideal point* P^* to be an equivalence class of lines, all of which have a common perpendicular line p. We will say that P^* lies on a line l, if $l \perp p$. We define a *generalized point* to be either a usual point, or an end, or an ideal point. Using this language, we can say that the three angle bisectors of (40.9) meet in a common generalized point.

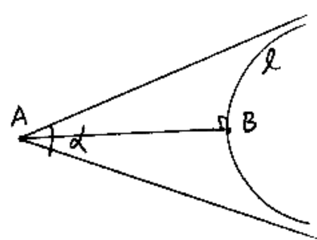
Exercises

The following exercises all take place in a hyperbolic plane, that is, a Hilbert plane satisfying (L).

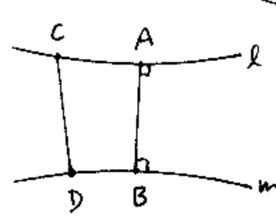
40.1 If two lines l, m have a transversal n that makes equal alternate interior angles, then l, m are parallel but not limiting parallel. Furthermore, in that case there is a unique point P such that every transversal that makes equal alternate interior angles to l and m passes through P.



40.2 (ALL) Suppose we are given equal angles at two points A and A', and let l and l' be their enclosing lines. Show that the perpendicular AB from A to l is equal to the perpendicular $A'B'$ from A' to l'.



40.3 If l and m are two parallel, but not limiting parallel, lines, show that their common perpendicular AB is the shortest distance between the two lines. Namely, show for any other points $C \in l$ and $D \in m$ that $CD > AB$.

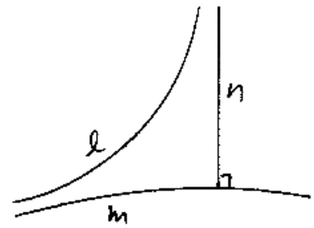


40.4 Show that ends of lines behave somewhat like points, as follows.

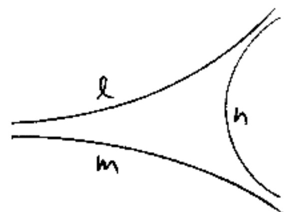
(a) Given a point P and an end α, there exists a unique line l passing through P and having end α.

(b) Given two distinct ends α, β, there exists a unique line l having ends α and β.

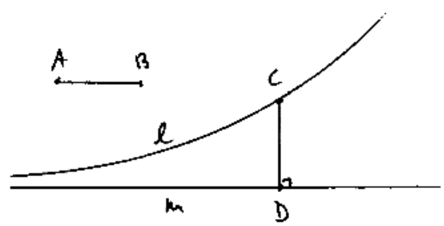
40.5 Given two lines l and m, limiting parallel at one end, show that there exists a line n, limiting parallel to (the other end of) l, and orthogonal to m.



40.6 Given two lines l, m, limiting parallel at one end, show that there exists a third line n, limiting parallel to the other ends of both l and m.

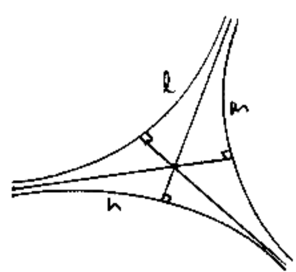


40.7 Given two lines l, m, limiting parallel at one end, and given a segment AB, no matter how large or how small, there exists a point C on l such that the perpendicular CD to m is equal to AB. *Hint*: Take m' perpendicular to AB through B and let l' be the limiting parallel to m' through A. Apply Exercise 40.5 to both the pair l, m and the pair l', m', and compare.



40.8 (LLL). Let l, m, n be three lines, each limiting parallel to the other two at opposite ends.

(a) Show that the three midlines (Exercise 34.11) to the three pairs of limiting parallel rays are orthogonal to the opposite sides of the trilimit triangle l, m, n, and all meet in a single point A, which is equidistant from l, m, n.

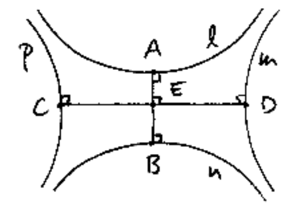


(b) If l', m', n' is another such trilimit triangle, with corresponding point A', show that the distance from A to the three sides l, m, n is equal to the distance from A' to the three sides l', m', n' of the second triangle.

(c) Given any point P on one side of the trilimit triangle, show that the perpendiculars PQ, PR from P to the other two sides make a right angle at P.

40.9 Given two angles α, β, with $\alpha + \beta < 2\text{RA}$, show that there exists a limit triangle with angles α, β.

40.10 A *limit quadrilateral* is a figure consisting of four lines l, m, n, p, with each limit parallel at opposite ends to the next, in cyclic order.



(a) If $lmnp$ is a limit quadrilateral, show that opposite sides are parallel but not limit parallel.

(b) Show that the common orthogonals AB of l and n and CD of m and p meet at right angles at a point E.

(c) Show that there exists a limit quadrilateral with AB equal to any prescribed segment.

(d) Two such limit quadrilaterals can be moved one to the other by a rigid motion of the plane if and only if the segment AB of the first is equal to one of the segments $A'B'$ or $C'D'$ of the second.

40.11 Show that ideal points (40.10.1) behave somewhat like regular points, as follows.

(a) Given a (regular) point P and an ideal point Q^*, there is a unique line containing them both.

(b) Given an end α and an ideal point Q^*, and assuming that α is not an end of the defining line q of Q^*, then there is a unique line containing Q^* with end α.

(c) Any two distinct lines have a unique generalized point in common.

40.12 You may have noticed while doing Exercise 40.11 that two ideal points do not necessarily lie on a line. So we define a *generalized line* to be either.

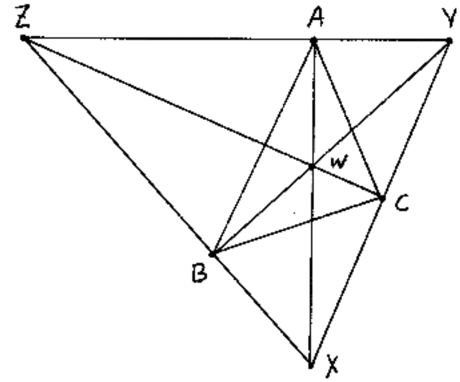
(1) a regular line, together with its two ends and ideal points, or

(2) a *limit line*, which consists of an end α, together with all ideal points P^* whose defining line p contains α, or

(3) an *ideal line*, which consists of all ideal points P^* whose defining line p contains a fixed (regular) point L.

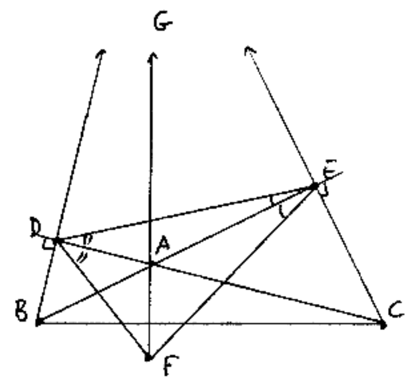
Show that the set of all generalized points of the hyperbolic plane, together with the subsets of generalized lines, forms a *projective plane* (Exercise 6.3). In particular, any two generalized points lie on a unique generalized line, and any two generalized lines meet in a unique generalized point.

40.13 Let ABC be any triangle. Show that the external angle bisectors at A, B, C form a "generalized triangle," i.e., a set of three lines meeting in generalized points X, Y, Z. Show that the internal angle bisectors of ABC, which meet at a point W, are the altitudes of the new triangle XYZ.



40.14 Reverse the argument of Exercise 40.13 to prove that in any triangle ABC, the three altitudes will meet in a generalized point.

Hint: Let BD and CE be two altitudes. Reflect the line DE in AB and in AC to get two new lines, which meet at a generalized point F. Show that B is equidistant from the three sides of the (generalized) triangle DEF, and from this conclude that F is a real point (not an end or an ideal point). Now apply Exercise 40.13 to the triangle DEF. Conclude that BD, AF, CE meet in a generalized point G, and that F lies on BC, and AF is orthogonal to BC, so in fact, AF is the third altitude of the original triangle.



Note that if we assume that two altitudes of the triangle meet in a (regular) point, then the entire proof can be carried out in a Hilbert plane with no further hypothesis, i.e., in neutral geometry.

40.15 Extend the theorem on (internal) angle bisectors of a triangle as follows. Consider a generalized triangle, consisting of three nonconcurrent lines (meaning they have no generalized point in common). Let the vertices be generalized points A, B, C.

(a) Define the analogue of an angle bisector for two lines meeting at an end or an ideal point.

(b) Show that the three (internal) angle bisectors of the generalized triangle ABC always meet at a (regular) point W.

(c) Show that W is the center of an *inscribed circle* that is tangent to the three sides of the triangle.

40.16 Prove the results of Exercises 35.8, 35.9 in a hyperbolic plane, without using Archimedes' axiom.