CHAPTER 15

CROSS RATIOS

15.1 Cross Ratios

Directed Distances

Recall that we denote directed distances, or signed distances, with a bar over the distance, as in $\overline{AB}$, and that

$$\overline{AB} = -\overline{BA}.$$

Also recall the following facts:

1. Given points A, B, and C on a line, if $\overline{AB} = \overline{AC}$, then $B = C$.
2. Given points A, B, C, and X on a line, if $\overline{AB}/\overline{BC} = \overline{AX}/\overline{XC}$, then $B = X$.

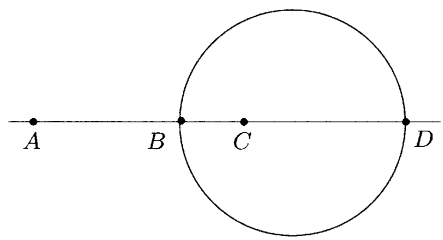
Classical Geometry: Euclidean, Transformational, Inversive and Projective, First Edition. By
I. E. Leonard, J. E. Lewis, A. C. F. Liu, G. W. Tokarsky

Directed Distances and Harmonic Conjugates

Recall that given points A, B, C, and D, then B and D are harmonic conjugates with respect to A and C if and only if

$$\frac{AB}{BC} = \frac{AD}{DC},$$

as in the figure below.



Here, we are using unsigned distances; for *signed distances*, B and D are harmonic conjugates with respect to A and C if and only if

$$\frac{\overline{AB}}{\overline{BC}} = -\frac{\overline{AD}}{\overline{DC}}.$$

Properties of Cross Ratios

Given a range of four points, A, B, C, and D, we define the quantity (AB, CD) by

$$(AB, CD) = \frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}}$$

and call it the ***cross ratio*** of the points A, B, C, and D, taken in that order.

Note.

1. The order refers to the order of the points in the notation, *not* the order of the points on the line.

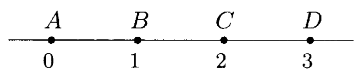
2. If $(AB, CD) = -1$, then A and B are harmonic conjugates with respect to C and D.

3. The value of (AB, CD) is independent of the direction of the line on which the range of points lie.

Example 15.1.1. *Find (AB, CD), (AC, BD), and (BA, DC) where A, B, C, and D are collinear with coordinates along the line given by 0, 1, 2, and 3, respectively.*

Solution. Working directly from the definition of cross ratios, we get

$$(AB, CD) = \frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = \frac{2/(-1)}{3/(-2)} = \frac{4}{3}.$$



Similarly,

$$(AC, BD) = -\frac{1}{3},$$

and

$$(BA, DC) = \frac{4}{3}.$$

□

Theorem 15.1.2. *If $(AB, CD) = k$, then:*

(1) If we interchange any pair of points and also interchange the other pair of points, then the resulting cross ratio has the same value k. Thus, (AB, CD), (BA, DC), (CD, AB), and (DC, BA) all have the value k.

(2) Interchanging only the first pair or only the last pair of points results in a cross ratio with the value $1/k$. Thus, $(BA, CD) = (AB, DC) = 1/k$.

(3) Interchanging only the middle pair or only the outer pair of points results in a cross ratio with the value $1 - k$. Thus, $(AC, BD) = (DB, CA) = 1 - k$.

Proof. (1) and (2) follow directly from the definition of the cross ratio. To prove (3), we will use the fact that for three collinear points X, Y, and Z, the directed distances are related by $\overline{XZ} = \overline{XY} + \overline{YZ}$, whether Y is between X and Z or not.

Interchanging the middle pair, we have

$$
\begin{aligned}
(AC, BD) &= \frac{\overline{AB}/\overline{BC}}{\overline{AD}/\overline{DC}} = \frac{\overline{AB}\cdot\overline{CD}}{\overline{AD}\cdot\overline{CB}} \\
&= \frac{(\overline{AC}+\overline{CB})(\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}} \\
&= \frac{\overline{AC}\cdot\overline{BD}}{\overline{AD}\cdot\overline{CB}} + \frac{\overline{AC}\cdot\overline{CB}+\overline{CB}(\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}} \\
&= -\frac{\overline{AC}\cdot\overline{DB}}{\overline{AD}\cdot\overline{CB}} + \frac{\overline{CB}(\overline{AC}+\overline{CB}+\overline{BD})}{\overline{AD}\cdot\overline{CB}} \\
&= -k+1.
\end{aligned}
$$

Interchanging the outer pair, from (1) we have

$$(DB, CA) = (CA, DB),$$

and interchanging the middle pair on the right-hand side, we have

$$(DB, CA) = 1 - (CD, AB),$$

and again by (1), we have

$$
\begin{aligned}
(DB, CA) &= 1 - (AB, CD) \\
&= 1 - k.
\end{aligned}
$$

□

Remark. Any permutation of the letters A, B, C, and D can be obtained by successively interchanging pairs using (1), (2), or (3) of Theorem 15.1.2.

Example 15.1.3. *Given* $(AB, CD) = k$, *find* (DA, CB).

Solution. From (3), we have

$$(DA, CB) = 1 - (BA, CD),$$

while from (2), we have

$$
\begin{aligned}
(DA, CB) &= 1 - \frac{1}{(AB, CD)} \\
&= 1 - \frac{1}{k}.
\end{aligned}
$$

Alternatively, we have

$$(AB, CD) = k,$$

which implies that

$$(BA, CD) = 1/k,$$

which in turn implies that

$$(DA, CB) = 1 - 1/k.$$

□

Ideal Points

Suppose that *one* of the four points A, B, C, or D is an ideal point I. We use the convention that

$$\frac{\overline{XI}}{\overline{IY}} = -1 \qquad \text{and} \qquad \frac{\overline{XI}}{\overline{YI}} = +1.$$

For example, if $B = I$, then

$$(AI, CD) = \frac{\overline{AC}/\overline{CI}}{\overline{AD}/\overline{DI}} = \frac{\overline{AC}}{\overline{CI}} \cdot \frac{\overline{DI}}{\overline{AD}} = \frac{\overline{AC}}{\overline{AD}} \cdot \frac{\overline{DI}}{\overline{CI}} = \frac{\overline{AC}}{\overline{AD}}.$$

Now suppose we interchange the two middle symbols in (AI, CD). Then by direct computation we get

$$(AC, ID) = \frac{\overline{AI}/\overline{IC}}{\overline{AD}/\overline{DC}} = \frac{-1}{\overline{AD}/\overline{DC}} = -\frac{\overline{DC}}{\overline{AD}}.$$

However,

$$1 - \left(-\frac{\overline{DC}}{\overline{AD}}\right) = \frac{\overline{AD} + \overline{DC}}{\overline{AD}} = \frac{\overline{AC}}{\overline{AD}},$$

which shows that the theorem on permutation of symbols remains true if one of the points is an ideal point.

There are 24 different arrangements of the symbols A, B, C, and D, giving rise to 24 different cross ratios: (AB, CD), (BA, CD), (BC, AD), and so on. However, there are only six different values for the 24 cross ratios, namely,

$$k, \quad \frac{1}{k}, \quad 1 - k, \quad \frac{1}{1-k}, \quad \frac{k-1}{k}, \quad \text{and} \quad \frac{k}{k-1}.$$

To see why this is so, note that all permutations of the symbols A, B, C, and D can be obtained via a sequence of interchanges of the types described in Theorem 15.1.2 and that only the operations of the second and third types produce different values. Thus, if the cross ratio (WX, YZ) has the value k, then the only new values we can obtain by operation of the second or third type are $1/k$ and $1 - k$. The value $1/k$ in turn can produce values of k or $1 - 1/k = (k - 1)/k$. If we were to continue in this manner, we would see that only the six different values listed above can be obtained.

Theorem 15.1.4. *If $(AB, CD) = (AB, CX)$, then D and X are the same point.*

Proof. We have

$$(AB, CD) = (AB, CX),$$

so that

$$\frac{\overline{AC}/\overline{CB}}{\overline{AD}/\overline{DB}} = \frac{\overline{AC}/\overline{CB}}{\overline{AX}/\overline{XB}},$$

which implies that

$$\overline{AD}/\overline{DB} = \overline{AX}/\overline{XB}.$$

From the remarks at the beginning of the chapter on the ratios of directed distances, this implies that $D = X$.

□

Corollary 15.1.5. *If $(AB, CD) = (AB, XD)$, then $C = X$.*

Proof. We have

$$(AB, CD) = (AB, XD),$$

so that

$$(AB, DC) = (AB, DX),$$

and from the previous theorem, $C = X$.

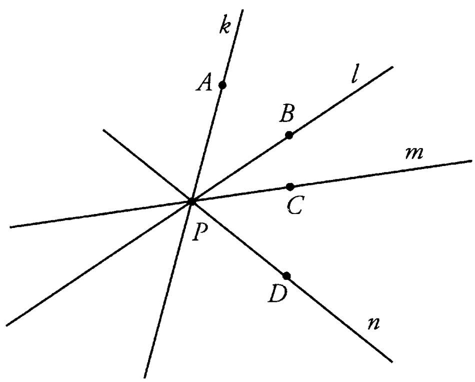
□

Cross Ratio of a Pencil of Lines

Let k, l, m, and n be a pencil of lines concurrent at a point P. This section will provide a definition of the cross ratio, denoted (kl, mn), of the pencil of lines.

Point of Concurrency is an Ordinary Point

Suppose k, l, m, and n are concurrent at an ordinary point P, and let A, B, C, and D be points other than P on k, l, m, and n, respectively, as in the figure below.



We define $P(AB, CD)$ as follows:

$$P(AB,CD) = \frac{\sin \overline{APC} / \sin \overline{CPB}}{\sin \overline{APD} / \sin \overline{DPB}},$$

where the directed angle $\overline{XPY}$ is the angle from the ray $\overrightarrow{PX}$ to the ray $\overrightarrow{PY}$ and whose magnitude is between $0°$ and $180°$.

Note that if A and A' are points on k on the opposite sides of P, then the directed angles $\overline{APC}$ and $\overline{A'PC}$ are different, as are the directed angles $\overline{APD}$ and $\overline{A'PD}$. In fact,

$$\overline{APC} = \overline{A'PC} - 180 \qquad \text{and} \qquad \overline{APD} = \overline{A'PD} - 180.$$

Since $\sin(x - 180) = -\sin x$, it follows that if A and A' are on opposite sides of P on the line k, then

$$\frac{\sin \overline{APC} / \sin \overline{CPB}}{\sin \overline{APD} / \sin \overline{DPB}} = \frac{-\sin \overline{A'PC} / \sin \overline{CPB}}{-\sin \overline{A'PD} / \sin \overline{DPB}} = \frac{\sin \overline{A'PC} / \sin \overline{CPB}}{\sin \overline{A'PD} / \sin \overline{DPB}},$$

or, in other words,

$$P(AB,CD) = P(A'B,CD).$$

Thus, we have the following result:

Theorem 15.1.6. *For a pencil of lines k, l, m, and n that are concurrent at the ordinary point P, the definition of $P(AB, CD)$ is independent of the choice of the points A, B, C, and D, as long as none of them are the point P.*

This allows us to make the following definition:

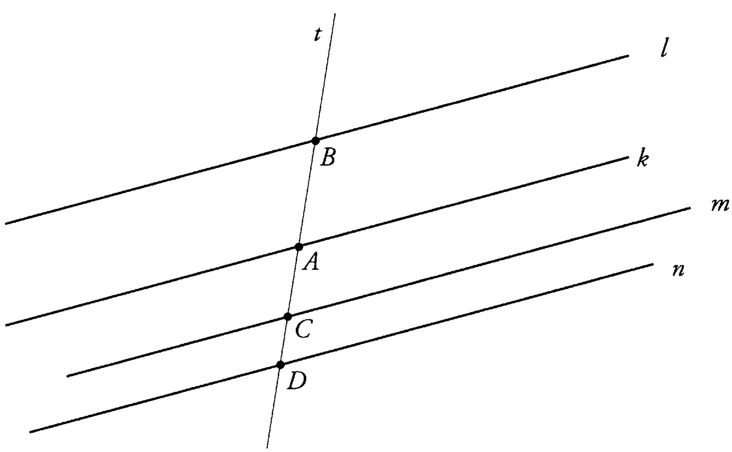
Definition. We define the ***cross ratio of a pencil of lines concurrent at an ordinary point*** P as

$$(kl, mn) = P(AB, CD),$$

where A, B, C, and D are points on k, l, m, and n other than P.

Point of Concurrency Is an Ideal Point

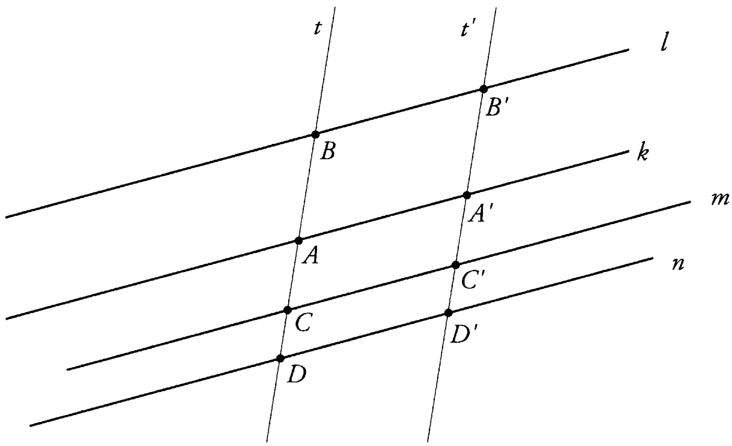
When the point of concurrency is an ideal point, the lines k, l, m, and n are parallel.



In this case, let t be any line intersecting k, l, m, and n at A, B, C, and D, respectively, and define (kl, mn) to be (AB, CD).

To check that this definition is independent of the choice of the line t, let t' be another line intersecting k, l, m, and n at A', B', C', and D'. There are two cases to consider:

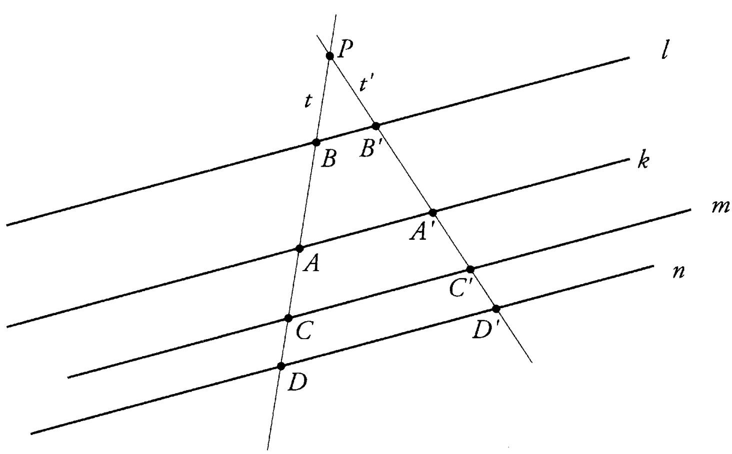
Case (i). t and t' are parallel.



In this case, obviously $AC = A'C'$, $CB = C'B'$, etc., since they are opposite sides of a parallelogram, so that

$$(AB, CD) = (A'B', C'D').$$

Case (ii). t and t' meet at an ordinary point P.



In this case, by similar triangles,

$$\frac{\overline{A'C'}}{\overline{C'B'}} = \frac{\overline{AC}}{\overline{CB}},$$

with similar results for the other ratios. From this it follows that

$$(AB, CD) = (A'B', C'D').$$

Theorem 15.1.7. *Suppose that k, l, m, and n form a pencil of lines concurrent at the ordinary point P. If a transversal cuts the lines k, l, m, and n at the points A, B, C, and D, respectively, then*

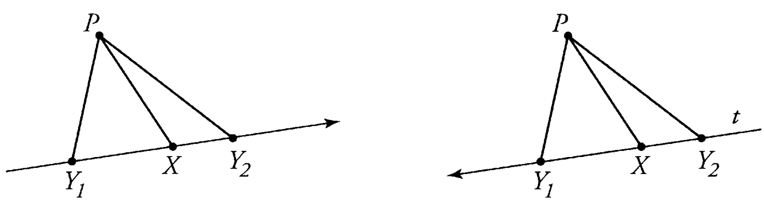
$$P(AB, CD) = (AB, CD)\,.$$

Proof. In order to prove the theorem, we will show two things:

(1) The signs (signum) of $P(AB, CD)$ and (AB, CD) are the same.

(2) The magnitudes of $P(AB, CD)$ and (AB, CD) are the same.

Proof of (1).



To check that (1) is true, note that given a directed line t and a point P not on t, then either

$$\operatorname{sgn}(\sin \overline{XPY}) = \operatorname{sgn}(\overline{XY})$$

for all pairs of points X and Y on t or else

$$\operatorname{sgn}(\sin \overline{XPY}) = -\operatorname{sgn}(\overline{XY})$$

for all pairs of points X and Y on t.

For points A, B, C, and D on t, the value of (AB, CD) is independent of the direction of t (see statement (3) in the note following the definition of (AB, CD) at the beginning of this chapter). In particular, we can choose the direction of t so that for all pairs of points X and Y,

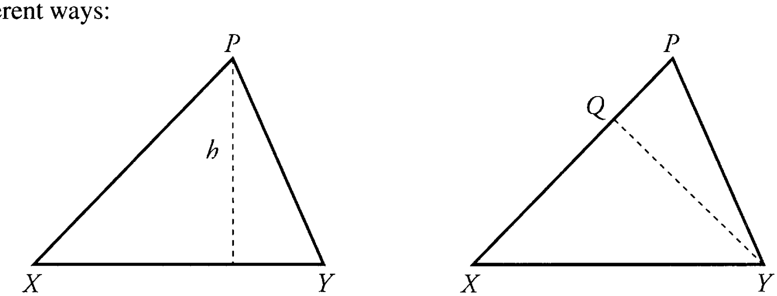
$$\operatorname{sgn}(\sin \overline{XPY}) = \operatorname{sgn}(\overline{XY}).$$

Thus,

$$\begin{aligned}
\operatorname{sgn}(P(AB,CD)) &= \operatorname{sgn}\left(\frac{\sin\overline{APC}/\sin\overline{CPB}}{\sin\overline{APD}/\sin\overline{DPB}}\right) \\
&= \frac{\operatorname{sgn}(\sin\overline{APC})/\operatorname{sgn}(\sin\overline{CPB})}{\operatorname{sgn}(\sin\overline{APD})/\operatorname{sgn}(\sin\overline{DPB})} \\
&= \frac{\operatorname{sgn}(\overline{AC}/\operatorname{sgn}(\overline{CB})}{\operatorname{sgn}(\overline{AD})/\operatorname{sgn}(\overline{DB})} \\
&= \operatorname{sgn}(AB,CD).
\end{aligned}$$

Proof of (2).

The proof uses the fact that it is possible to compute the area of a triangle in two different ways:



(a) (b)

In the figure above,

(a) $\text{area}(\triangle XPY) = XY \cdot \frac{h}{2}$, and

(b) $\text{area}(\triangle XPY) = \frac{1}{2}XP \cdot PY \cdot \sin(\angle XPY)$,

where (b) follows from the fact that $QY = PY\sin(\angle XPY)$.

Expanding $|(AB,CD)|$, we get

$$\begin{aligned}
|(AB,CD)| &= \frac{|\overline{AC}|\,/\,|\overline{CB}|}{|\overline{AD}|\,/\,|\overline{DB}|} \\
&= \frac{(|\overline{AC}|\cdot\frac{h}{2})\ /\ (|\overline{CB}|\cdot\frac{h}{2})}{(|\overline{AD}|\cdot\frac{h}{2})\ /\ (|\overline{DB}|\cdot\frac{h}{2})} \\
&= \frac{\text{area}(\triangle APC)\,/\,\text{area}(\triangle CPB)}{\text{area}(\triangle APD)\,/\,\text{area}(\triangle DPB)}\,.
\end{aligned}$$

Calculating the areas using (b) we get

$$\text{area}(\triangle APC) = \frac{1}{2}|AP| \cdot |PC| \cdot |\sin \overline{APC}|,$$

$$\text{area}(\triangle APD) = \frac{1}{2}|AP| \cdot |PD| \cdot |\sin \overline{APD}|,$$

$$\text{area}(\triangle CPB) = \frac{1}{2}|CP| \cdot |PB| \cdot |\sin \overline{CPB}|,$$

$$\text{area}(\triangle DPB) = \frac{1}{2}|DP| \cdot |PB| \cdot |\sin \overline{DPB}|.$$

Substituting these values into the expression for $|(AB,CD)|$ above, we get

$$|(AB,CD)| = \frac{|\sin \overline{APC}| \,/\, |\sin \overline{CPB}|}{|\sin \overline{APD}| \,/\, |\sin \overline{DPB}|} = |P(AB,CD)|\,.$$

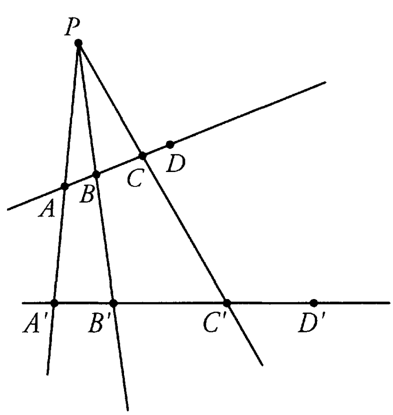
This completes the proof of the theorem. □

15.2 Applications of Cross Ratios

Four Useful Lemmas

The following lemmas connect cross ratios with concurrency and collinearity.

Lemma 15.2.1. *In the figure below, P is an ordinary point and P, A, and A' are collinear; P, B, and B' are collinear; and P, C, and C' are collinear. Transversals cut the lines PA, PB, and PC at A, B, and C, respectively. The point D is on AC, and the point D' is on $A'C'$.*



With this configuration, if $(AB,CD) = (A'B',C'D')$, then P is on DD'.

Proof. Suppose the line PD intersects $A'D'$ at some point E' (E' is not shown in the diagram). Then from Theorem 15.1.6, we have

$$P(AB,CD) = (A'B',C'E').$$

However,

$$P(AB,CD) = (AB,CD) = (A'B',C'D'),$$

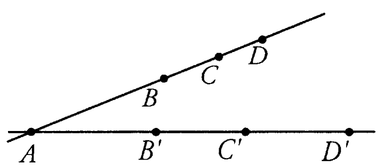
so that

$$(A'B',C'E') = (A'B',C'D'),$$

which implies that $E' = D'$, and P is on DD'.

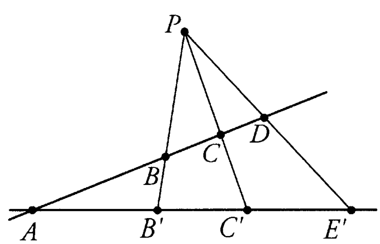
□

Lemma 15.2.2. *In the figure below, two lines intersect at the point A. The points B, C, and D are on one of the lines, while the points B', C', and D' are on the other line.*



With this configuration, if $(AB,CD) = (AB',C'D')$, then BB', CC', and DD' are concurrent.

Proof. Let $P = BB' \cap CC'$ and let $E' = PD \cap AC'$, as shown below.



It suffices to show that $E' = D'$.

We have

$$P(AB,CD) = P(AB',C'E'),$$

which implies that

$$(AB,CD) = (AB',C'E'),$$

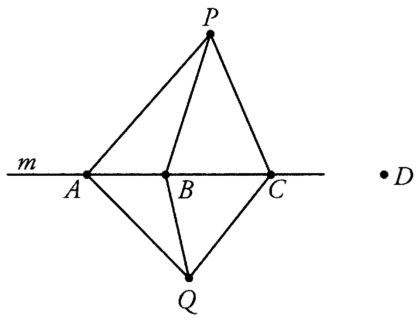
and since $(AB, CD) = (AB', C'D')$, it follows that

$$(AB', C'D') = (AB', C'E').$$

Therefore, $E' = D'$.

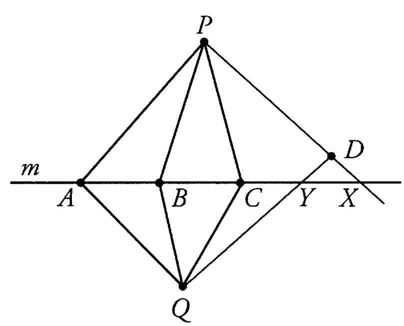
$\square$

Lemma 15.2.3. *In the figure below, A, B, and C are points on a line m. P and Q are points not on m. D is a point other than A, B, C, P, or Q.*



With this configuration, if $P(AB, CD) = Q(AB, CD)$, then D is on m.

Proof. If D is not on m, let PD intersect m at X and let QD intersect m at Y, as shown below.



We have

$$P(AB, CD) = P(AB, CX) = (AB, CX)$$

and

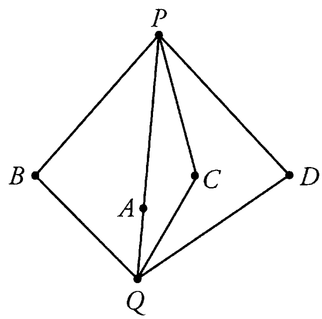
$$Q(AB, CD) = Q(AB, CY) = (AB, CY),$$

and since $P(AB, CD) = Q(AB, CD)$, we must have $(AB, CX) = (AB, CY)$.

However, this can only happen if $PD \cap QD = X = Y = D$; that is, D is on m.

$\square$

Lemma 15.2.4. *In the figure below, A, B, C, and D are points other than P and Q, and the point A is on PQ.*



With this configuration, if

$$P(AB,CD) = Q(AB,CD),$$

then B, C, and D are collinear.

Proof. Let A' be the point $PQ \cap BC$ and let m be the line BC. Then

$$P(AB,CD) = P(A'B,CD),$$
$$Q(AB,CD) = Q(A'B,CD),$$

which implies that

$$P(A'B,CD) = Q(A'B,CD),$$

which implies that D is on m, by the previous lemma.

□

Theorems of Desargues, Pascal, and Pappus

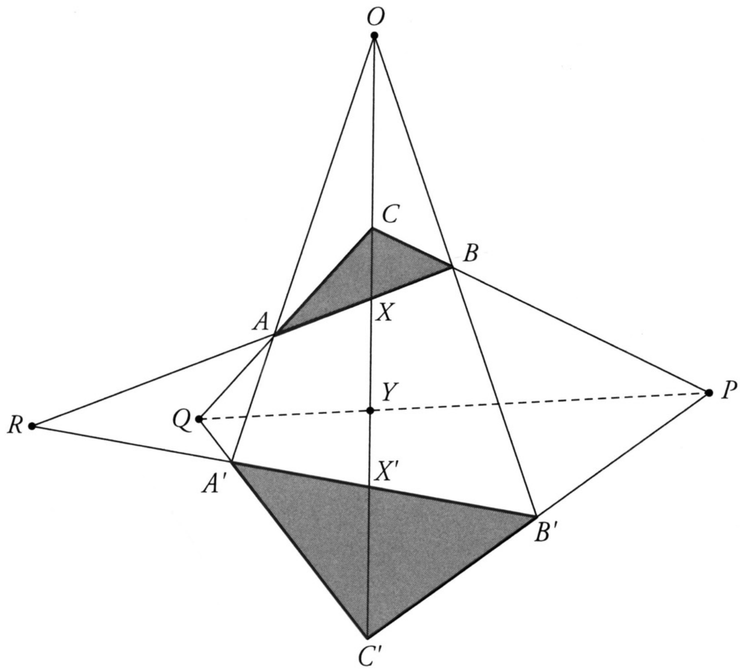
Desargues' Two Triangle Theorem (Theorem 4.4.3) was proven in Part I using Ceva's Theorem and Menelaus' Theorem. We restate it here and prove it using cross ratios.

Theorem 15.2.5. *Copolar triangles are coaxial and conversely.*

Proof. Let the copolar triangles be ABC and $A'B'C'$. Then, as in the figure on the following page,

$$AA' \cap BB' \cap CC' = O.$$

Triangles are coplanar when lines drawn through the vertices meet at a single point, called the pole of the triangles. Triangles are coaxial when the sides of the triangles extended intersect at points that are colinear - that is, points that lie on a single line. The line on which the extended sides intersect is called the axis of the triangles.



Let P, Q, and R be the points of intersection of the corresponding sides of the triangle. In order to show that P, Q, and R are collinear, we will show that R is on the line PQ.

Let X, X', and Y be as shown; that is, let $X = OC' \cap AB$, $X' = OC' \cap A'B'$, and $Y = OC' \cap PQ$.

There are three pencils of lines: one concurrent at C, one concurrent at O, and one concurrent at C'.

Since BA is a transversal for the pencil at C, we have

$$C(PY, QR) = (BX, AR),$$

and since $B'A'$ is a transversal for the pencil at O, we have

$$(BX, AR) = O(BX, AR) = (B'X', A'R) = C'(B'X', A'R),$$

and replacing B' by P, X' by Y, and A' by Q in the last expression, we have

$$C'(B'X', A'R) = C'(PY, QR).$$

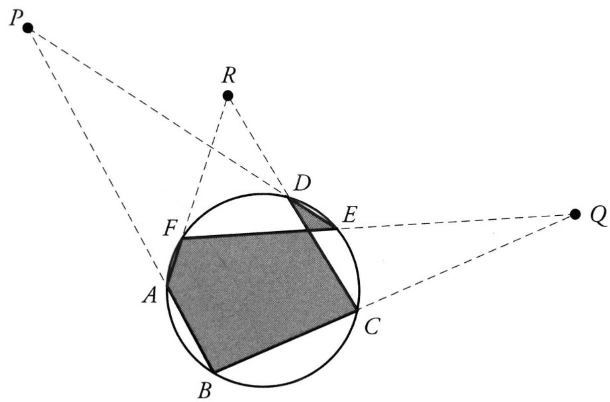
Therefore, $C(PY, QR) = C'(PY, QR)$, and by Lemma 15.2.4, P, Q, and R are collinear.

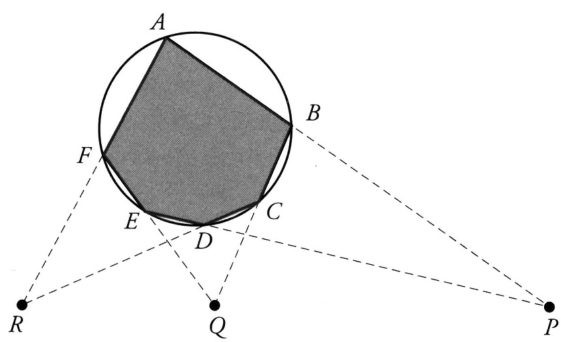
□

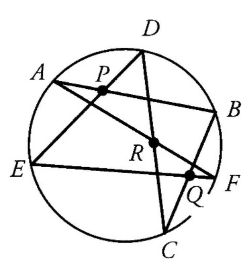
Pascal's Mystic Hexagon Theorem from Part I (Theorem 4.4.4) says the following:

Theorem 15.2.6. *If a hexagon is inscribed in a circle, the points of intersection of the opposite sides are collinear.*

There are many possible configurations, three of which are shown in the following figure.







$AB \cap DE = P$

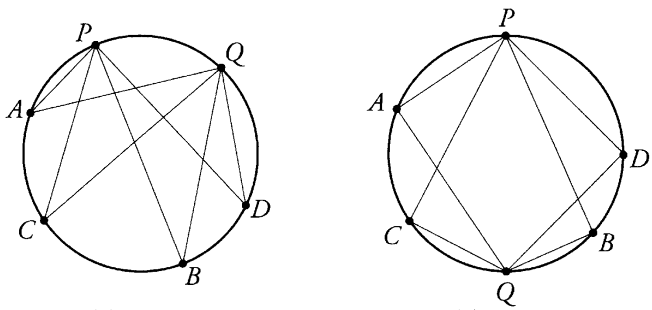
$BC \cap EF = Q$

$CD \cap FA = R$

In order to prove the theorem using cross ratios we first need the following lemma:

Lemma 15.2.7. *If A, B, C, D, and P, Q are distinct points on a circle, then $P(AB,CD) = Q(AB,CD)$.*

Proof. There are two cases to consider, as illustrated in the following figures.



(1) (2)

Case (1). P and Q are not separated by any of the points A, B, C, or D.

In this case, Thales' Theorem implies that

$$\begin{aligned}\angle\overline{APC} &= \angle\overline{AQC},\\ \angle\overline{CPB} &= \angle\overline{CQB},\\ \angle\overline{APD} &= \angle\overline{AQD},\\ \angle\overline{DPB} &= \angle\overline{DQB}.\end{aligned}$$

Thus,

$$P(AB,CD) = \frac{\sin\overline{APC}/\sin\overline{CPB}}{\sin\overline{APD}/\sin\overline{DPB}} = \frac{\sin\overline{AQC}/\sin\overline{CQB}}{\sin\overline{AQD}/\sin\overline{DQB}} = Q(AB,CD).$$

Case (2). P and Q are separated by some of the points A, B, C, or D.

The proof here is similar to that for Case (1), but now we have

$$\begin{aligned}\angle\overline{APC} &= \angle\overline{AQC},\\ \angle\overline{CPB} &= 180 + \angle\overline{CQB},\\ \angle\overline{APD} &= 180 + \angle\overline{AQD},\\ \angle\overline{DPB} &= \angle\overline{DQB}.\end{aligned}$$

The positive signs in the second and third equations arise since the signed angles are in opposite directions. From the second equation, we get

$$\sin\overline{CPB} = \sin(180 + \overline{CQB}) = -\sin\overline{CQB}.$$

Similarly, from the third equation, we get

$$\sin \overline{APD} = \sin(180 + \overline{AQD}) = -\sin \overline{AQD}.$$

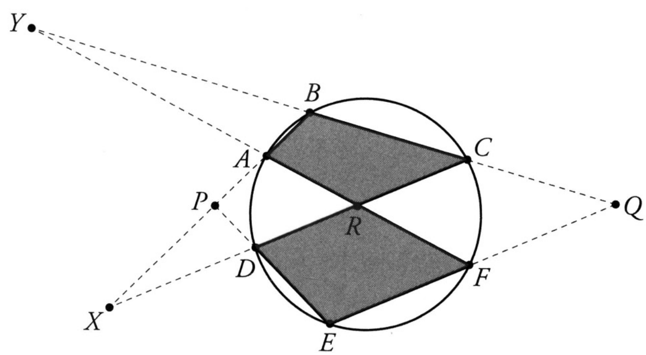
The lemma now follows from the definitions of $P(AB, CD)$ and $Q(AB, CD)$ as in Case (1).

□

For convenience, we restate Pascal's Theorem.

Theorem 15.2.8. *If $ABCDEF$ is a hexagon inscribed in a circle, then the points of intersection of the opposite sides are collinear.*

Proof. The proof works for any configuration, and for clarity we use the one in the figure below.



Let $X = AB \cap CD$ and $Y = BC \cap AF$.

Consider the pencils at D and F. Since P is on DE and X is on DC, then

$$D(AE, CB) = (AP, XB),$$

and from Lemma 15.2.7, we have

$$D(AE, CB) = F(AE, CB).$$

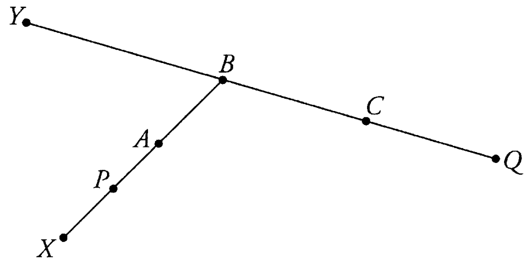
Since Y is on FA and Q is on FE, then

$$F(AE, CB) = (YQ, CB),$$

and, therefore,

$$(AP, XB) = (YQ, CB).$$

Note that we have the following configuration:



Since $(AP, XB) = (YQ, CB)$, then from Lemma 15.2.2 it follows that AY, PQ, and XC are concurrent. However, $AY \cap XC = R$, so PQ passes through R.

□

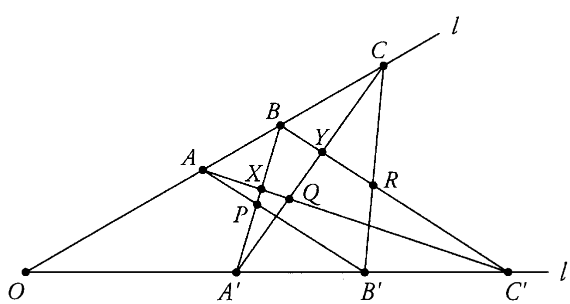
As a final example, we prove Pappus' Theorem (Theorem 4.4.5) using cross ratios:

Theorem 15.2.9. *Given points A, B, and C on a line l and points A', B', and C' on a line l', then the points of intersection*

$$P = AB' \cap A'B, \quad Q = AC' \cap A'C, \quad R = BC' \cap B'C$$

are collinear.

Proof. Introduce points $X = AC' \cap A'B$ and $Y = CA' \cap C'B$, as shown in the figure below.



Using the pencil through A, replace B by O, X by C', and P by B'. Then

$$A(BX, PA') = A(OC', B'A'),$$

and since these are transversals for the pencil through A, we have

$$(BX, PA') = (OC', B'A').$$

Using the pencil through C, replace O by B, B' by R, and A' by Y. Then

$$C(OC', B'A') = C(BC', RY),$$

and since these are transversals for the pencil through C, we have

$$(OC', B'A') = (BC', RY).$$

Thus,

$$(BX, PA') = (OC', B'A') = (BC', RY).$$

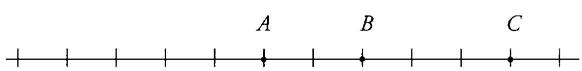
It follows from Lemma 15.2.2 that XC', PR, and $A'Y$ are concurrent, and since $XC' \cap A'Y = Q$, then Q is on PR; that is, P, Q, and R are collinear.

□

15.3 Problems

1. Given $(AB, CD) = k$, find (BC, AD) and (BD, CA).

2. Given three points A, B, and C, as shown below,



   find points D_i, $i = 1, 2, 3, 4$ such that

   (a) $(AB, CD_1) = 5/6$,

   (b) $(AB, CD_2) = -5/3$,

   (c) $(AB, CD_3) = 10/3$,

   (d) $(AB, CD_4) = 5/3$.

3. Using the definition of the cross ratio, show that $(AB, CD) = (CD, AB)$.

4. Show that for collinear points A, B, C, D, and E we have

   (a) $(AB, CE) \cdot (AB, ED) = (AB, CD)$,

   (b) $(AE, CD) \cdot (EB, CD) = (AB, CD)$.

5. Find x if $(AB, CD) = (BA, CD) = x$.

6. Let L, M, and N be the respective midpoints of the sides BC, CA, and AB of $\triangle ABC$. Prove that

$$L(MN, AB) = -1.$$

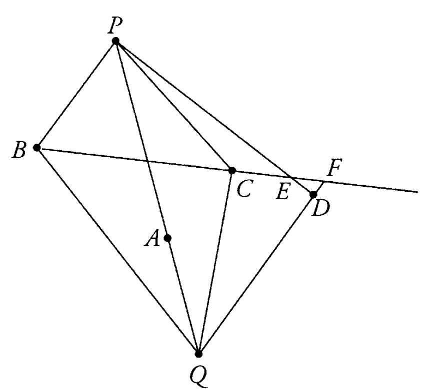
7. If P, Q, and R are the respective feet of the altitudes on the sides of BC, CA, and AB of $\triangle ABC$, show that

$$P(QR, AB) = -1.$$

8. Given $\mathcal{C}(O, r)$ and ordinary points A, B, C, and D on a ray through O, inverting into A', B', C', and D', respectively, show that $(AB, CD) = (A'B', C'D')$; that is, that the cross ratio is invariant under inversion.

   Hint: Use the distance formula $A'B' = \dfrac{r^2}{OA \cdot OB} AB$.

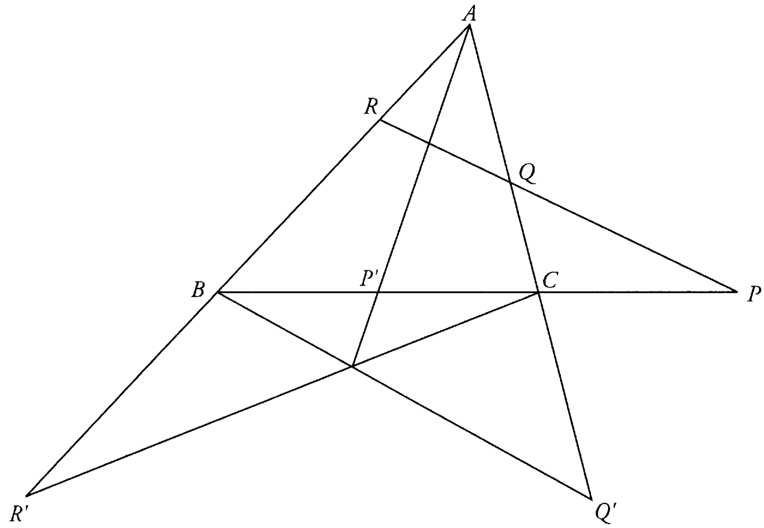
9. Prove the following: If PA, PB, PC, PD and QA, QB, QC, QD are two pencils of lines, and if $P(AB, CD) = Q(AB, CD)$ and A is on PQ, then B, C, and D are collinear.



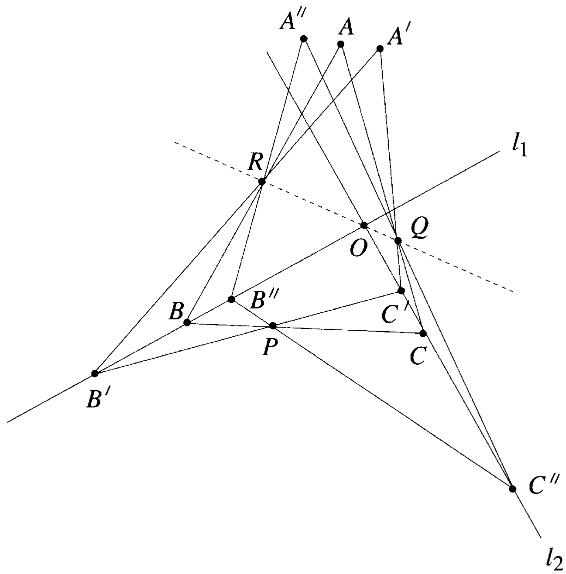
10. In $\triangle ABC$ on the following page we have

$$(BC, PP') = (CA, QQ') = (AB, RR') = -1.$$

   Show that AP', BQ', and CR' are concurrent if and only if P, Q, and R are collinear.



11. Given a variable triangle $\triangle ABC$ whose sides BC, CA, and AB pass through fixed points P, Q, and R, respectively, then if the vertices B and C move along given lines through a point O collinear with Q and R, find the locus of the vertex A.



12. If $V(AB, CD) = -1$ and if VC is perpendicular to VD, show that VC and VD are the internal bisector and external bisector of $\angle AVB$.

13. The bisector of angle A of $\triangle ABC$ intersects the opposite side at the point T. The points U and V are the feet of the perpendiculars from B and C, respectively, to the line AT. Show that U and V divide AT harmonically; that is, that $(AT, UV) = -1$.

14. A line through the midpoint A' of side BC of $\triangle ABC$ meets the side AB at the point F, side AC at the point G, and the parallel through A to side CB at the point E. Show that the points A' and E divide FG harmonically; that is, that $(FG, A'E) = -1$.

15. Prove the second part of Desargues' Theorem using cross ratios; that is, show that coaxial triangles are copolar.