MI314 – History of Mathematics: Introduction to proofs

A proof by definition

Definitions. *An **even number** can be written as*

$$n_1 = 2k,$$

*and **odd number** can be written as*

$$n_2 = 2k + 1,$$

where $k = \{0, 1, -1, 2, -2, ...\}$.

Theorem. *If n is even, then n^2 is even.*

Proof. Since n is even,

$$n = 2k,$$

so

$$\begin{aligned} n^2 &= (2k)(2k) \\ &= 2(2k^2). \end{aligned} \tag{1}$$

But we can set $2k^2 = m$, where m is some integer $= 0, 1, -1, 2, -2, ...$, so, substituting into (1), we have

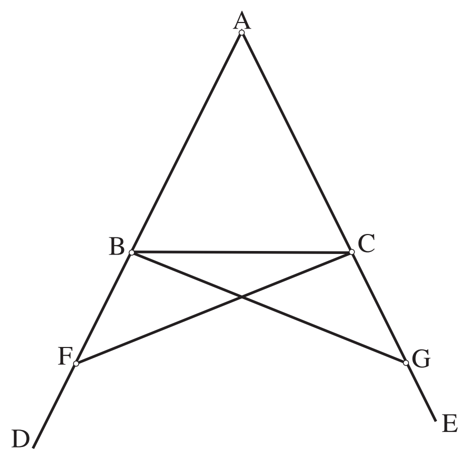
$$n^2 = 2m.$$

Therefore, n^2 is even. □

Key. *We* define *even numbers in such a way that we can show that if n satisfies this definition, than so does n^2.*

A proof by construction, *Elements* I 5

Preliminary. *If two triangles have two sides and the angle between them equal, then the triangles are congruent (≅). (SAS congruency.)*



Theorem. *In a triangle, if two sides are equal, the angles that subtend them are also equal.*

Proof. Let $\triangle ABC$ have $AB = AC$.

Take point F at random and cut off $AG = AF$. Join FC and GB.

Then, since $AB = AC$ and $\angle BAC$ is common, by the preliminary,

$$\triangle ACF \cong \triangle ABG.$$

So,

$$AF - AB = AG - AC, \text{that is, } BF = CG.$$

And since $\triangle ACF \cong \triangle ABG$,

$$FC = BG \text{ and } \angle AFC = \angle AGB.$$

Therefore,

$$\triangle BCG \cong \triangle CBF.$$

Then, since $\angle FBC = \angle GCB$,

$$\begin{aligned} 180^\circ - \angle FBC &= 180^\circ - \angle GCB \\ \angle ABC &= \angle ACB. \end{aligned}$$

□

Key. *We* construct *two congruent triangles whose properties can then be used to demonstrate the equality we are interested in.*

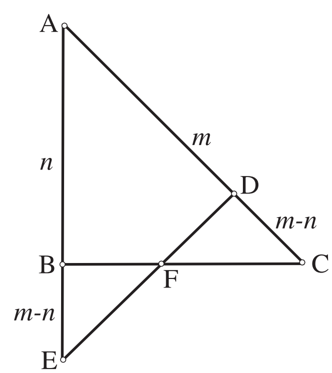
A proof by contradiction

Preliminary 1. *The previous theorem — In a triangle, if two sides are equal, the angles that subtend them are also equal.*

Preliminary 2. *In similar triangles, corresponding sides are proportional.*

Preliminary 3. *In a right isosceles triangle, the hypothenuse*

Definition. *A rational number can be expressed as* $\frac{q}{r}$*, where* $q = \{0, 1, -1, 2, -2, ...\}$*, and* $r = \{1, -1, 2, -2, ...\}$.



Theorem. *The square root of two,* $\sqrt{2}$*, is irrational. (A geometric proof.)*

Proof. Let $\triangle ABC$ be an isosceles right triangle with sides $AB = BC = n$ and hypothenuse $AC = m$, so that $m = n\sqrt{2}$.

Assume *for the sake of contradiction* that $\frac{m}{n} = \sqrt{2}$ is a rational number in least terms. (If it is not in least terms, divide through by any factors until it is expressed in least terms.)

Cut off $AE = AB$ and $AE = AC$, and join ED. Then, since $\angle BAD$ is common, by the first preliminary,

$$\triangle EAD \cong \triangle CAB.$$

Since $\angle EBF = 90°$ and $\angle AED = 45°$,

$$\angle BFE = 45°.$$

Therefore, both $\triangle EBF$ is a right isosceles triangle,

$$\triangle EBF \cong \triangle EBF.$$

Hence, $BE = BF = DF = DC = m - n$. So that,

$$FC = BC - BF = n - (m - n) = 2n - m.$$

Hence, in $\triangle FDC$, which is a right isosceles triangle, similar to $\triangle ABC$, so by the second preliminary,

$$\frac{FC}{DF} = \frac{2n - m}{m - n} = \sqrt{2}.$$

But, since n and m are integers, $2n - m$ and $m - n$ must also be integers. But $2n - m < m$ and $m - n < n$. Therefore, $\frac{m}{n} = \sqrt{2}$ cannot be a rational number in least terms, contradicting the assumption.

Therefore $\sqrt{2}$ cannot be rational. □

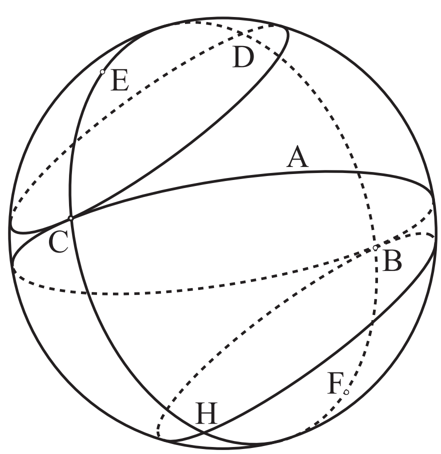
Key. *We* construct *a smaller triangle that is similar to the original triangle and show that if we assume that that the ratio of the sides of the original triangle are a rational number in least terms, then the sides of the smaller triangle must be a* smaller *rational number* in least terms. *Which is absurd.*

A proof of existence, by construction, Theodosius *Spherics* II 6

Definitions. *In sphere, a* ***great circle*** *is a circle whose plane passes through the center of the sphere, an a* ***lesser circle*** *is any other circle. Lesser circles form bundles of parallels sharing the same two poles. The* ***pole-distance*** *of a circle is the line from the circumference of a circle to its pole.*

Preliminary 1. *In a sphere, circles drawn with the same pole-distance are equal.*

Preliminary 1. *In a sphere, two circles are tangent at a point if and only if their poles lie on a single great circle and they cut that great circle at the same point.*



Theorem. *In a sphere, if a great circle is tangent to a lesser circle, then it is also tangent to another equal, and parallel, lesser circle.*

Proof. Let great circle ABC be tangent to lesser circle CD at point C.

Take the pole of circle CD at point E and let a great circle be drawn through the two points E and C. Call it great circle $CEDBFH$, where point B is the intersection of this new great circle with the original great circle ABC.

Cut off arc BF equal to arc CE.

With pole F and pole-distance FB, draw lesser circle BH. Then we much *show* that lesser circle BH is (a) equal and (b) parallel to lesser circle CD, and (c) that it is tangent to great circle ABC.

Since circles ABC and CD are tangent, great circle $CEDBFH$ will go through their poles.

(c): But, since ABC and BH meet at a point and have their poles on $CEDBFH$, they are tangent.

(b): Since $CE = BF$,

$$\begin{aligned} CE + EB &= BF + EB \qquad (2) \\ CB &= EF \end{aligned}$$

Hence, since CB is a semicircle, EF is a semicircle and E and F will be the two poles of a bundle of parallel circles, of which CD and BH are members. Therefore, CD and BH are parallel.

(a): But, the pole distances BF and CE are equal, so the circles are equal. □

Key. *We* construct *the object whose existence we are trying to demonstrate, and then show that it has the properties that we claim it will have.*