Introduction to thinking about mathematics

Waseda University, SILS,
History of Mathematics

What is mathematics?

- Carrying out calculations
- Constructing mathematical objects
- Solving problems
- Writing and reading proofs
- Developing mathematical theories

What are mathematical objects?

- Things like numbers
- Things like geometric objects
- Things like functions

What is a mathematical proof?

Some example proofs

Carrying out calculations

- Arithmetic operations
- Taking roots
- Using more advanced functions
- Executing algorithms
- And so on…

Constructing mathematical objects

- Drawing geometric objects
- Calculating numbers
- Building functions
- Designing algorithms
- And so on…

Solving problems

- Finding a number that satisfies certain conditions
  - Finding the number(s) that solve(s) an equation
- Producing a particular geometric object
- Transforming one type of mathematical statement into another
- Finding an equation that solves a more general equation
- And so on…

Writing and reading proofs

- A mathematician might use a proof to convince someone else that a particular statement *must* be true.
- Most of the time, when we read a mathematical proof, however, we are not in any doubt about the claim in question. We read a proof to gain insight as well as conviction.

Writing and reading proofs

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- Most of the time, when we read a mathematical proof, however, we are not in any doubt about the claim in question. We read a proof to gain insight as well as conviction.
- A proof shows:
  - that something is true…
  - why something is true…
  - how to do something…
  - that something does or does not exist…
  - and so on…

Developing mathematical theories

- Organizing arguments, problems and theorems so that they can be derived from a small set of "intuitive" first principles
- Developing a body of mathematical knowledge in order to articulate the fundamental nature of a certain set of mathematical objects
- Organizing theorems and problem-solving techniques to facilitate mathematical work in a particular field
- And so on…

Things like numbers

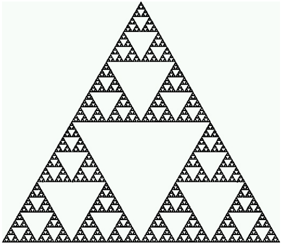
- Natural numbers or counting numbers, 1, 2, 3, … ($\mathbb{N}$)
- Whole numbers or integers, like 0, 3, 8, -3, … ($\mathbb{W}$, $\mathbb{Z}$)
- Rational numbers, $\frac{a}{b}$, where a and b are integers, … ($\mathbb{Q}$)
- Irrational numbers, like $\sqrt{2}$ … ($\mathbb{I}$)
- Transcendental numbers, like π, e
- Real numbers, like 2, $\frac{1}{2}$, $\sqrt{2}$, π, e, … ($\mathbb{R}$)
- Computable numbers
- Imaginary, and complex numbers, like $i = \sqrt{-1}$ or $x = a + bi$
- Surreal numbers, real numbers with infinite and infinitesimal numbers
- And so on…

Things like geometric objects

- A circle.
- An ellipse.

Things like geometric objects

- A circle.
- An ellipse.
- A Sierpinski Triangle?



- And so on…

Things like functions

- $y = f(x)$. We start with some number (or set of numbers), x, *do something to it (or them)* and get some other unique number, y.
- Say, $y = x^2$, a parabola

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- Say, $y = x^2$, a parabola
- The Dirichlet Function:

$$D(x) = \begin{cases} 1 & \text{for } x \text{ is rational} \\ -1 & \text{for } x \text{ is irrational} \end{cases}$$

The origin of proofs

Strangely, the concept of deductive proof as an explicitly articulated practice seems to have originated at a certain time and place in history – in Ancient Greece in the late 5th or early 4th century BCE.

The use of proofs as explicitly articulated, structured mathematical arguments largely spread through direct or indirect contact with Greek works.

The structure of a Greek proof

A proof begins with some set of starting points – axioms, definition, previously demonstrated theorems, generally accepted facts – and proceeds by logical argument to some conclusion.

The basic structure of a classical proof is:

1. proposition (a statement of the claim)
2. suppositions (some hypotheses, definitions or previous theorems)
3. constructions (we may need to introduce some new objects)
4. argument (the logical steps following from these)
5. conclusion (a statement of what has been shown)

The goals of a proof

- The most important aim of a proof is to convince the audience that the claim is true. Or rather, that if the audience accepts the truth of the starting points, they must *necessarily* accept the truth of the conclusion.
- But some people may already accept that the claim is true. Why would a proof be interesting to them?

The goals of a proof

- For these people, a good proof can help explain why the claim is true. Hence, proof can have some *explanatory* role, and understanding a proof can give insight into the relationships between the different objects and ideas.

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- For these people, a good proof can help explain why the claim is true. Hence, proof can have some *explanatory* role, and understanding a proof can give insight into the relationships between the different objects and ideas.
- Finally, the best proofs should both provide insight into the objects in question and they should be useful for doing further mathematical work – like solving problems and producing further proofs.

If n is even, then n^2 is even

- A proof by definition
- Definitions: Where k is an integer, {0, 1, -1, 2, -2,...}, an even number can be written as

$$n = 2k,$$

an odd number as

$$n = 2k + 1.$$

- The proof is an argument based directly on the definition...

The base angles of an isosceles[1] triangle are equal

- A proof by construction
- Preliminary: if two triangles have two sides and the contained angle equal, they are equal (SAS congruence)
- We begin by assuming $\triangle ABC$, such that $AB = AC$
- We make some new constructions…

[1] From a Greek root meaning "equal sides," an isosceles triangle has two equal sides. An equilateral triangle has three equal sides.

Geometric proof that $\sqrt{2}$ is irrational

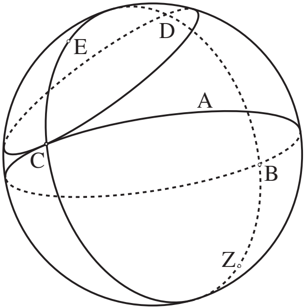
- A proof by contradiction
- Preliminaries: (1) SAS congruence, (2) in similar triangles the sides are proportional, (3) in a right isosceles triangle the hypotenuse is $\sqrt{2}$ times the side.
- Definition: a rational number can be written as $q = \frac{s}{t}$, where s and t are integers and $t \neq 0$
- We begin with an isosceles triangle, ABC, with hypothenuse m and equal sides n, so that $\frac{m}{n} = \sqrt{2}$
- We assume that m and n are integers and that $\frac{m}{n}$ is in least terms. (There are two assumptions, one is "innocent.")
- We carry out some constructions are seek a contradiction…

If a great circle in a sphere is tangent to a lesser circle, it is also tangent to another, equal and parallel to the first

- An existence proof by construction
- Definitions: A great circle is a circle whose center coincides with the center of the sphere. A lesser circle is any other circle. Lesser circles form systems of parallels about a pair of poles.
- Preliminary 1: Two circles in the sphere are tangent at a point if and only if their poles lie on a single great circle and they cut that great circle at the same point.
- Preliminary 2: Circles drawn with the same pole-distance are equal…

If a great circle in a sphere is tangent to a lesser circle, it is also tangent to another, equal and parallel to the first

- We begin with great circle *ABC* touching lesser circle *CD* at point *C*.
  - We take the pole of *CD* as point *E*.
  - We draw great circle *CEDB*, and from *B* – the intersection of the two great circles – let *BZ* be cut off equal to *CE*.



If a great circle in a sphere is tangent to a lesser circle, it is also tangent to another, equal and parallel to the first

- We take A as a pole, and ZB as pole-distance, we draw circle BH.
  - We then use the properties of the constructed objects to show that BH is equal and parallel to CD…