Mathematics in Ancient Egypt and Mesopotamia

Waseda University, SILS,
History of Mathematics

Introduction

Egyptian mathematics
- Egyptian numbers
- Egyptian computation
- Some example problems

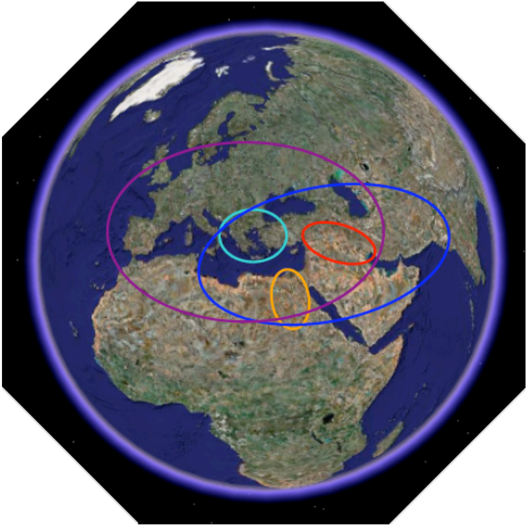
Babylonian Mathematics
- Babylonian numbers
- Babylonian computation
- Some example problems

How do historians divide up history?

The large scale periodization used for (Western) history is the following:

- Ancient: the distant past to, say, 5th or 6th century CE
- Medieval: 6th to, say, 15th or 16th century
- Modern: 16th century to the present

Ancient cultures around the Mediterranean



Mesopotamia
(3000 B.C. - A.D. 100)

Egypt
(3000 B.C. - A.D. 300)

Greek states
(1000 B.C. - 330 B.C.)

Hellenistic kingdoms
(330 B.C. - 30 B.C.)

Roman Empire
(30 B.C. - A.D. 400)

How do we study ancient history?

- What are our ancient sources?
  - material objects
  - images
  - texts
    a. found as ancient material objects
    b. transmitted by tradition
- What is the condition of the sources?
- Wherever possible, we focus on reading and understanding texts.
  - When we study objects, without any textual support or evidence, it is very easy to be mislead, or to have very open-ended and unverifiable interpretations.

A Greek multiplication table

Α	**Β**	**Γ**	**Δ**	**Ε**	**Ϛ**	**Ζ**	**Η**	**Θ**	**Ι**
Β	Δ	Ϛ	Η	Ι	ΙΒ	ΙΔ	ΙϚ	ΙΗ	Κ
Γ	Ϛ	Θ	ΙΒ	ΙΕ	ΙΗ	ΚΑ	ΚΔ	ΚΖ	Λ
Δ	Η	ΙΒ	ΙϚ	Κ	ΚΔ	ΚΗ	ΛΒ	ΛϚ	Μ
Ε	Ι	ΙΕ	Κ	ΚΕ	Λ	ΛΕ	Μ	ΜΕ	Ν
Ϛ	ΙΒ	ΙΗ	ΚΔ	Λ	ΛϚ	ΜΒ	ΜΗ	ΝΔ	Ξ
Ζ	ΙΔ	ΚΑ	ΚΗ	ΛΕ	ΜΒ	ΜΘ	ΝϚ	ΞΓ	Ο
Η	ΙϚ	ΚΔ	ΛΒ	Μ	ΜΗ	ΝϚ	ΞΔ	ΟΒ	Π
Θ	ΙΗ	ΚΖ	ΛϚ	ΜΕ	ΝΔ	ΞΓ	ΟΒ	ΠΑ	Ϙ
Ι	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ϙ	Ρ

In a stele of the 3rd–2nd century BCE, we see a “geometer” named Ptolemy (*Ptolemaiou geōmetrou*) drilling a small child on a multiplication table. Hence, *geometer* was a general expression for a teacher of mathematics.

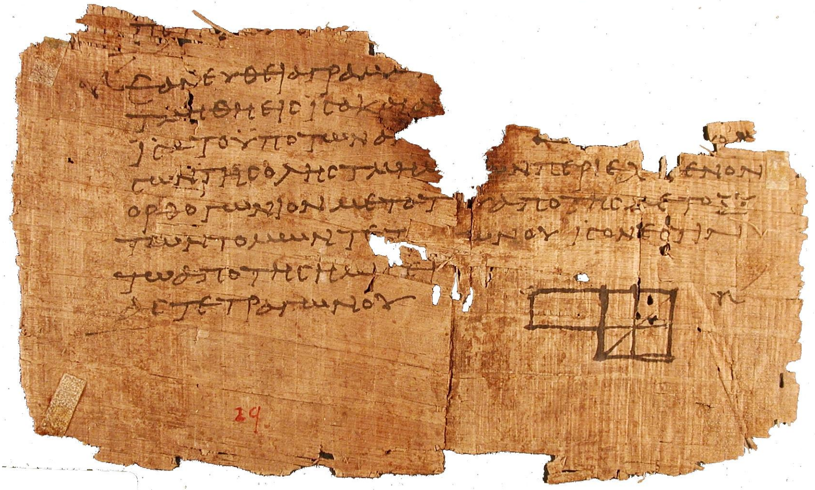
How can we interpret these objects without texts?[1]

[1] The pyramids of Giza.

Or how about these?[2]

[2] Stonehenge in Wiltshire, England.

But … how do we interpret these texts?



Things we might want to know about a text

- What does it say, what are its contents?
- Who wrote it?
- What was its purpose?
- What can we learn from the text itself about its author?
- How much do we need to know about the author in order to understand the text?
- Can we situate it in a broader context?
  - The "horizontal" context of the society in which it was composed?
  - The "vertical" context of earlier and later developments?

Evidence for Egyptian and Babylonian mathematics

- Egypt: A handful of Old Egyptian Hieratic papyri, wooden tables and leather rolls, a few handfuls of Middle Egyptian Demotic papyri, less than a hundred Greek papyri, written in Egypt and forming a continuous tradition with the older material.
- Mesopotamia: Many thousands of clay tablets containing Sumerian and Assyrian, written in cuneiform.
- All of this material is scattered around in a number of different library collections, poses many difficulties to scholars, and involves many problems of interpretation.
- These texts are written in dead languages.

The place of mathematics in Egyptian culture

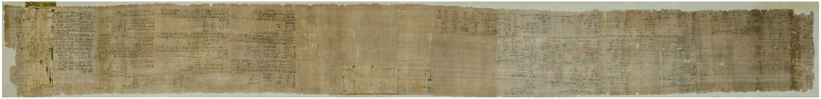
- Ancient Egypt was a fairly autocratic society ruled by lines of Pharaohs (kings), who were thought to be divine.
- Egyptian, a Semitic language, was written in two forms, Hieroglyphic and Hieratic.
- A very small group of professional scribes could read and write.
- The only evidence we have for mathematics in ancient Egypt comes from the scribal tradition.
- The study of mathematics was a key component of a scribe's education.

Our evidence for Egyptian mathematics

- Ancient Egyptian mathematics is preserved in Hieratic and Demotic on a small number of papyri, wooden tablets and a leather roll.
- Middle and late Egyptian mathematics is preserved on a few Demotic and Coptic papyri and many more Greek papyri, pot sherds and tablets.
- Ancient Egyptian is a Semitic language.

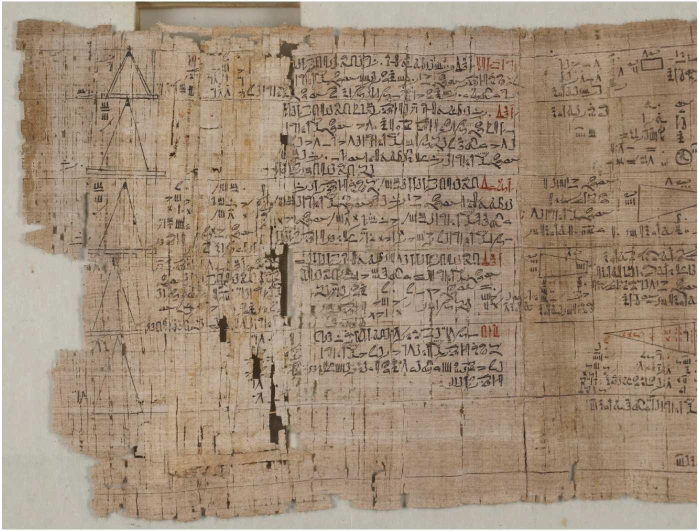
- This must be only a small fraction of what was once produced, so it is possible that our knowledge of Egyptian mathematics is skewed by the lack of evidence.

An example: The Rhind Papyrus, complete

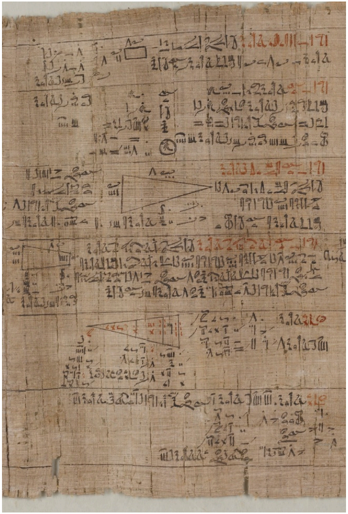


P. British Museum 10057. Copied in the 17th–16th c. BCE by a scribe called Ahmose (Ahmes), based on a lost original though to originate in the reign of pharaoh Amenemhat III, 18th c. BCE.

An example: The Rhind Papyrus, end



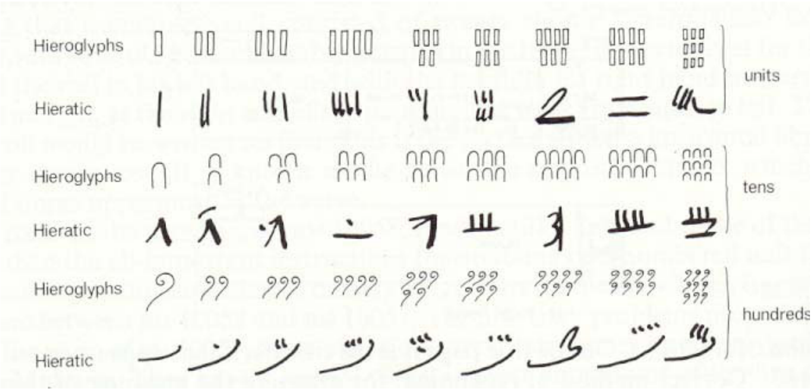
An example: The Rhind Papyrus, detail



Egyptian numeral system

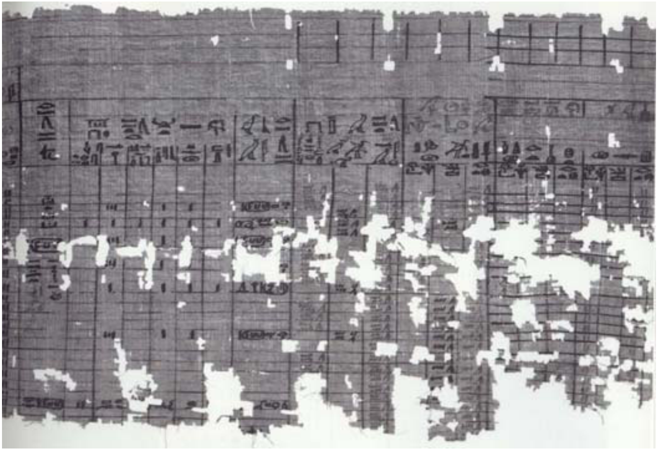
- A decimal (base-10) system; cumulative and additive.
- Not a place-value system. Every mark had an absolute value.
- Unordered

Egyptian numeral system



Some examples: An inscribed inventory

Some examples: Accounting[3]

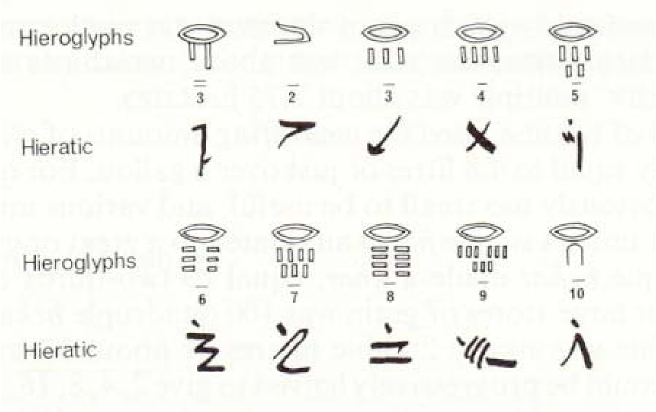


[3] This text uses both Hieroglyphic and Hieratic forms.

Egyptian fraction system

- The Egyptians only used unit fractions (we would write $\frac{1}{n}$).
- They wrote a number with a mark above it, and had special symbols for $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{2}{3}$.
- This makes working with their fractions different from what we learned in school.
- Doubling an "even-numbered" fraction is simple (ex. $\frac{1}{10} + \frac{1}{10} = \frac{1}{5}$), however doubling an "odd-numbered" fraction is not straightforward (ex. $\frac{1}{5} + \frac{1}{5} = \frac{1}{3} + \frac{1}{15}$) …
- And, collections of unit fractions are often not unique (ex. $\frac{1}{5} + \frac{1}{5} = \frac{1}{3} + \frac{1}{15} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$).
- We often write Egyptian fractions as $\bar{\bar{3}} = \frac{2}{3}$, $\bar{2} = \frac{1}{2}$, $\bar{3} = \frac{1}{3}$, $\bar{4} = \frac{1}{4}$, etc., so that $n \times \bar{n} = 1$. Why not just use $\frac{1}{n}$?

Egyptian fraction system



Multiplication

The Egyptians carried out multiplication by a series of doublings and then additions, as well as inversions (flipping reciprocals, $2\ \ 24 \rightarrow \overline{24}\ \ \overline{2}$). For example, to multiply 12 by 12, we find, in *P. Rhind* #32:

.	12	
2	24	
\ 4	48	
\ 8	96	
		sum 144

- The scribe has written out a list of the successive doubles of 12 then put a check by the ordering numbers (1, 2, 4, 8, ...) that total to 12.
  - The doubles that correspond to these ordering numbers are then summed ($48 + 96 = 144$).

Division

- Division is an analogous process, using halving and unit fractions. For example, to divide 19 by 8 we find, in *P. Rhind* #24:

$$\begin{array}{rrl} . & 8 & \\ \backslash\ 2 & 16 & \\ \bar{2} & 4 & \\ \backslash\ \bar{4} & 2 & \\ \backslash\ \bar{8} & 1 & \\ & & \text{sum } 19 \end{array}$$

- In this case, the numbers on the right side sum to 19 $(16 + 2 + 1 = 19)$, so the ordering numbers must be added together to give the answer $(2 + \bar{4} + \bar{8} = 2.375)$. The answer is not stated explicitly.

The method of false position, 1

- A class of problems, known as *ʿḥʿ* problems (*ʿḥʿ* means "heap" or "quantity," sometimes read *aha*), reveals a method for solving problems of the form $x + ax = b$. For example, in *P. Rhind* #26, we have:

A quantity, its $\bar{4}$ [is added] to it so that 15 results. What is the quantity? [That is, $x + \frac{x}{4} = 15$]
Calculate with 4. [The assumed, "false" value.]
You shall calculate its $\bar{4}$ as 1. Total 5.
Divide 15 by 5.

\ .	5
\ 2	10

3 shall result …

The method of false position, 2

… Multiply 3 times 4

	.	3
	2	6
\	4	12

12 shall result

\	.	12
\	$\bar{4}$	3

Total 15

- ◁ We start with an assumed value, 4, and find out what part of the final result, 15, it produces. Then we correct by this part.
- ◁ Why would the text treat such trivial calculations in this kind of detail?

"I met a man with seven wives…"[4]

P. Rhind #79

"There were 7 houses, in each house 7 cats, each cat caught 7 mice, each mice ate 7 bags of *emmer* [a type of grain], and each bag contained 7 *heqat* [a measure of grain]. How many were there altogether?"

- Answer (modern notation): $7 + 7^2 + 7^3 + 7^4 + 7^5 = 19607$. The problem is not worked out in the text.
- Is this a *real* problem? What is the point of it?

[4] An old English nursery rhyme: "As I was going to St. Ives: I met a man with seven wives: Each wife had seven sacks: Each sack had seven cats: Each cat had seven kits: Kits, cats, sacks, wives: How many were going to St. Ives?"

The place of mathematics in Mesopotamian culture

- Ancient Mesopotamian societies had two primary institutions, the King and the Temple, but wealthy merchants also played an important role in society.
- Mathematics was practiced by clans of literate scribes.
- They made their living working as priests, scribes and accountants—mathematics was a side product of their primary roles.

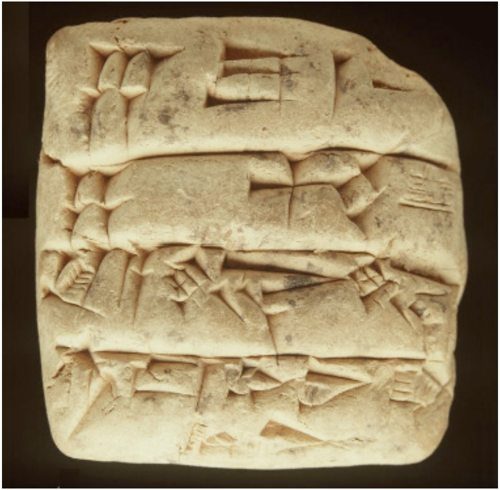
The place of mathematics in Mesopotamian culture

- Mathematics was used for both practical purposes and to create professional distinctions.
- In the century before and after the conquests of Alexander, Mesopotamian scholars, working mostly as priests, applied their skills to the production of a highly detailed mathematical astronomy.

Our evidence for Babylonian mathematics

- Ancient Mesopotamian mathematics was written with a stylus on clay tablets.
- We have hundreds of thousands of tablets, the majority of which have numbers on them and many of which have still not been read or understood.
- The tablets are written in the Cuneiform script, mostly in the Sumerian and Assyrian languages. (Latin *cuneus* means wedge.) Assyrian is a Semitic language, while Sumerian is unrelated to any known language.

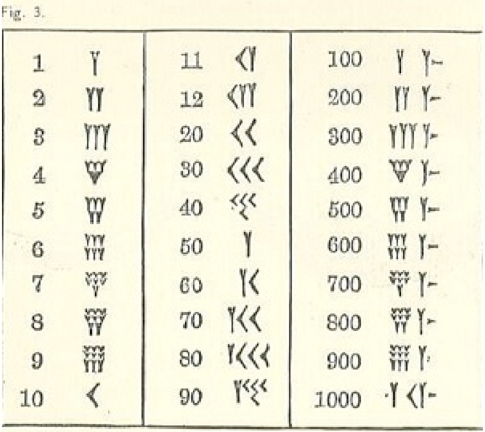
An Example: A Sumerian tablet



The Babylonian numeral system

- A sexagesimal (base-60) number system, although the numerals had decimal characteristics; cumulative positional.
- It was a place-value system, but a place holder (like our 0) was only inconsistently used in the early period; later it became more standardized.
- Also, there was no clear division between the integer and fractional parts – no *radix character*, such as "." or ";". This means that in computational practice, it was a pure floating point, or simply floating, system.
- Ordered (Left to right, top to bottom)

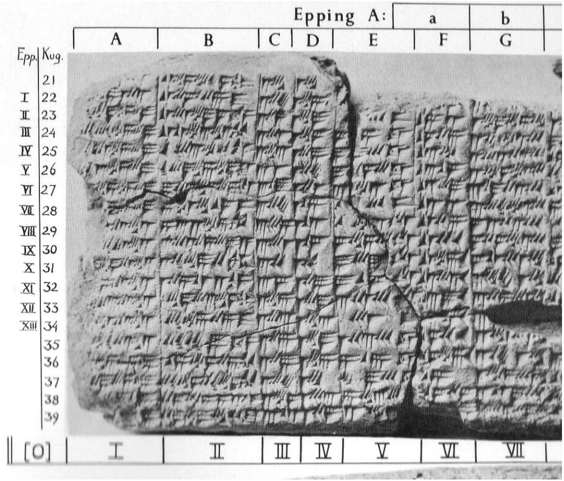
Babylonian numeral system



Some examples: Sets of “Pythagorean” triples[5]

[5] Plimpton 322 (Columbia University). “Pythagorean” triples are sets of three integers, $\{a, b, c\}$, such that $a^2 + b^2 = c^2$.

Some examples: Babylonian lunar theory[6]



[6] Neugebauer, *Astronomical Cuneiform Texts* (ACT), 122.

The Babylonian fraction system

- A base-60 fractional system.
- Modern scholars sometimes use a system of commas (,) to separate the places from each other and and colons (;) to indicate which number are integers and which are fractional parts.[7] So a number of the form $x_n, \ldots, x_2, x_1, x; x_{f1}, x_{f2}, \ldots x_{fm} =$ $x_n \times 60^n + \ldots + x_2 \times 60^2 + x_1 \times 60 + x + \frac{x_{f1}}{60} + \frac{x_{f2}}{60^2} + \ldots + \frac{x_{fm}}{60^m}$
- For example,
  $1;24,51,10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} = 1.414212963\ldots,$
  $1,12;15 = 1 \times 60 + 12 + \frac{1}{4} = 72.25$, or
  $8,31;51 = 8 \times 60 + 31 + \frac{17}{20} = 511.85.$
- We can simply use commas, dots, or colons for a pure floating system.

[7] Due to Otto Neugebauer.

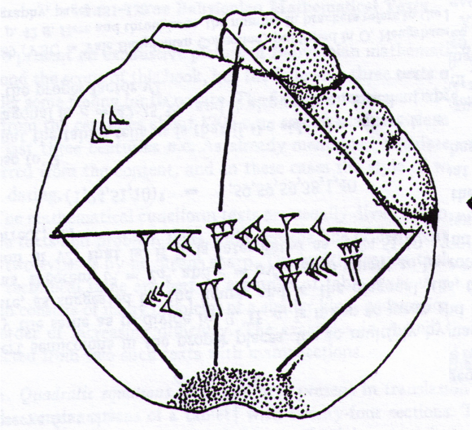
Computation in base-60

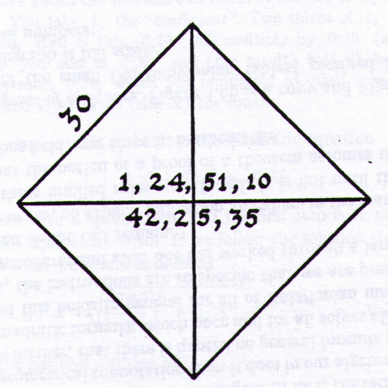
- Because it was a place-value system, the Babylonian system allowed simpler calculations, in *some* ways similar to contemporary styles.
- One difficulty was in the multiplication table, which if complete would have had 60 by 60 terms.
- Since memorizing 3,600 terms was unmanageable, instead they worked with certain “principal” tables.
- They also composed tables of pairs of reciprocals ($n \times {}^1/_n = 1$; $n \times \bar{n} = 1$), where in the place of 1 could be 60, $^1/_{60}$, 3,600, $^1/_{3,600}$, etc. This is because they regarded division as multiplication by the reciprocal.

An interesting tablet, *YBC*, 7282[8]

[8] Yale Babylonian Collection, 7282.

An interesting tablet, *YBC*, 7282





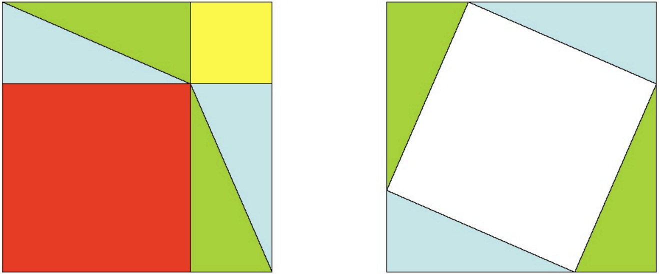
An interesting tablet

- We have three numbers

$$\begin{aligned} a &= 30, \\ b &= 1,24,51,10, \\ c &= 42,25,35. \end{aligned}$$

- If we write as 30;0 and 1;24,51,10 and 42;25,35, then $c = ab$.
- $\therefore b \approx \sqrt{2}$. Indeed, $(1,24,51,10)^2 = 1,59,59,59,38,1,40$.
- This, and many other tablets, indicate that the Babylonian mathematicians knew something like the so-called Pythagorean theorem, $a^2 + b^2 = c^2$.

How did they know this?



- We do not know, but perhaps they used a "cut-and-paste" argument. Since the two squares are "obviously" the same size, the red plus yellow squares must equal the white squares. Is this a proof?

Geometrical algebra, 1

- One of the significant mathematical accomplishments of the Mesopotamian scribes was a general method for solving certain systems of algebraic equations (although they did not think of them in this abstract way).
- It has only been fairly recently,[9] that scholars have reached the consensus that they used a kind of geometrical algebra.
- That is, they imagined the problem as represented by a rectangular figure and then "cut-and-pasted" parts of this figure in order to solve the problem.

[9] Since the 1990s.

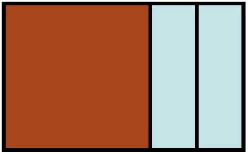
Geometrical algebra, 2

- The technique was probably originally developed for the very common types of problems where:
  - We know the *area* a plot of land and the *sum* or *difference* of the sides and we want to know the length and width, individually. (Is this a real, or practical, problem?)
  - That is, given $xy = a$ and $x \pm y = b$, to find x and y.

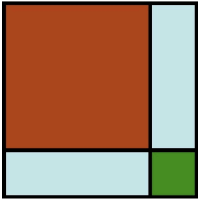
Geometrical algebra, 2

Assuming $xy = a$ and $x - y = b$, they probably reasoned as follows:

Suppose we start with the complete upper rectangle, a, we mark off a square section (brown) at one end of the plot, then we divide what is left over into two equal strips (light blue).



Then we move one of the strips and complete the square (green).



Then we can determine the area of the green square, which, along with the area of the whole rectangle, allows us to determine the *sides* of the original rectangle.

A geometrical algebra problem

These types of techniques could be used to solve a wide variety of algebraic problems. YBC, 6967 reads as follows:

A reciprocal exceeds its reciprocal by 7. What are the reciprocal and its reciprocal? You: break in two the 7 by which the reciprocal exceeds its reciprocal so that 3;30 will come up. Combine 3;30 and 3;30 so that 12;15 will come up. Add 1,00, the area, to the 12;15 which came up for you so that 1,12;15 will come up. What squares to 1,12;15? 8;30. Draw 8;30 and 8;30, its counterpart, and then take away 3;30, the holding-square, from one, and add to one. One is 12, the other is 5. The reciprocal is 12, its reciprocal is 5.

- That is, we start with a relation of the form $n - {}^{60}/_{n} = 7$ and we want to determine the value of n and ${}^{60}/_{n}$.

Advanced mathematics in Mesopotamia

- Mesopotamian scribes developed methods for solving many problems that had no immediate practical application.
- Mathematical problem-solving appears to have become a mark of distinction in the scribal profession.
- One major area of application, however, was the development of mathematical astronomy, which was used to predict significant events of the heavenly bodies and was of great value as an aid to divination.
- In these ways, the production and teaching of mathematics had a well-defined place in Mesopotamia culture.