The rise of deductive mathematics: Early Greek mathematics

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History of Mathematics

History and myth

Both early Greek history and history of mathematics, more generally, are full of many legendary stories that are told many times over the years, but which may have no basis in what actually happened. (Examples: The murder of Hippasus at sea for discussing incommensurability, Euclid and King Ptolemy discussing the absence of a royal road in geometry,[1] Archimedes burning the Roman ships in the harbor of Syracuse.)

These stories are part of the lore of the *discipline*. They play an important role in helping form the *identity* of the field by producing a notion of heritage, but they are not history.

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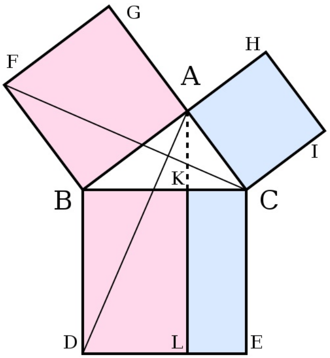
[1] μὴ εἶναι βασιλικὴν ἀσραπὸν ἐπι γεωμετρίαν.

History and heritage, 1

- We can take as an example the so-called Pythagorean theorem (**N**):
- From the perspective of heritage we can say this is the equation $a^2 + b^2 = c^2$.
- Euclid himself talks about the square on AB, the square on AC and the square on BC.
- How are these different?



History and heritage, 1

Mathematicians have a vested interest in telling the story of past mathematics in *a particular way* – one that directly sheds light on their own activity. In order to do this, they transform a past result, **N**, into one of their own conception.

Feature	**History**	**Heritage**
motivation for **N**	important issue	unimportant
types of influence	can be negative or positive; both are noted	clean them up (only direct positive influences)

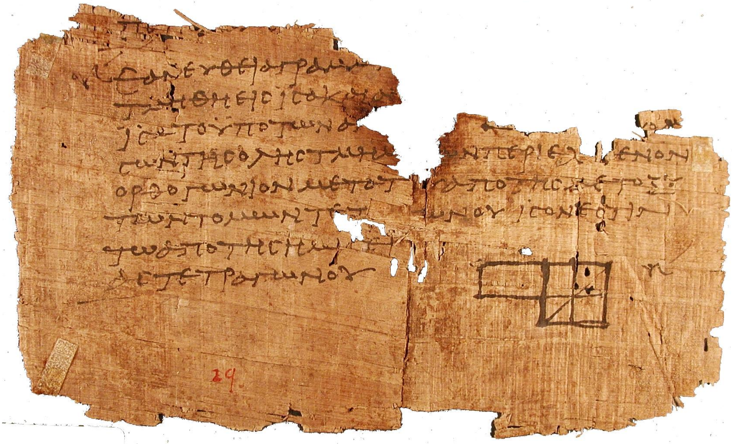
History and heritage, 2

Feature	**History**	**Heritage**
successful developments	important, but also studies failures, delays, etc.	the main concern
determined?	generally not claimed	may be determinist, “we *had* to get here”
foundations of a theory	dig through the sources down to them	lay them down and build from this solid ground
level of importance of **N**	will vary over time	not considered

The evidence for Greek mathematics

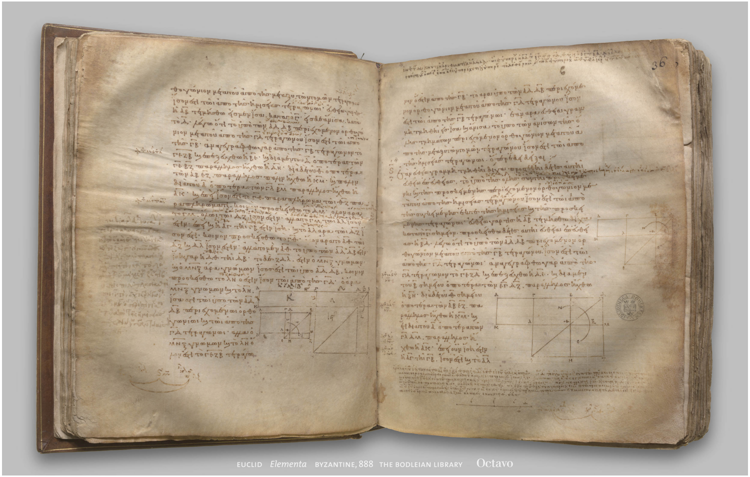
- We have some direct material evidence for *practical mathematics* – arithmetic, surveying, etc. – but very little for *theoretical mathematics*.
- Almost all of our evidence for Greek mathematics comes from the medieval manuscript tradition.
- The theoretical texts are written mostly in Ancient Greek in copies of copies of copies, many times removed. Some are in Arabic, Hebrew, Latin, Persian, etc.
- The full manuscripts that we use to make a printed edition are from as early as the 9th century CE to as late as the 16th century. (Around 1,000 to 1,500 years later than the original texts were written.)
- We have *almost* no direct sources. We have no ancient autographs. (We have a number of medieval ones.)

Papyrus fragment, Euclid's *Elements* II 5 (study notes)[2]



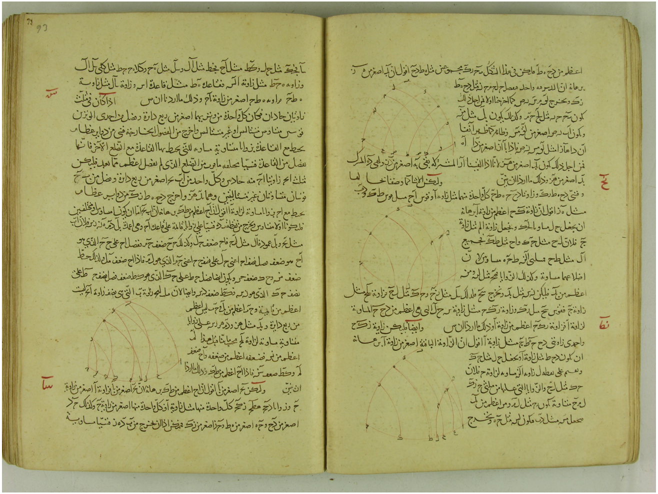
[2] From the Egyptian city of Oxyrhynchus, dated 1st or 2nd century CE.

An old manuscript of Euclid's *Elements* II 5[3]



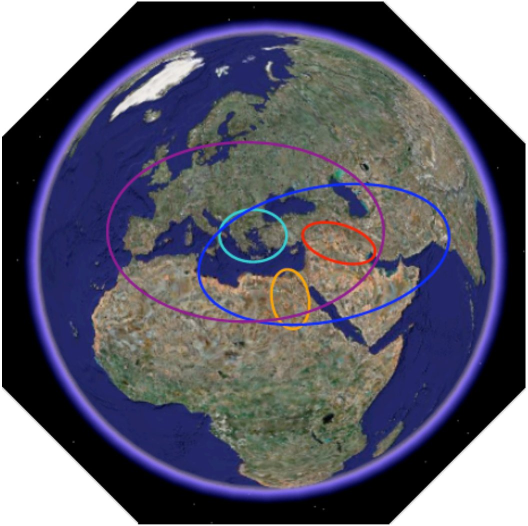
[3] Written in 888 CE. From the Bodleian Library, D'Orville 301.

Al-Harawī's version of Menelaus' *Spherics*[4]



[4] Ahmet III, 3464 (1227 CE). Propositions I 57–61.

Ancient cultures around the Mediterranean



Mesopotamia
(3000 B.C. - A.D. 100)

Egypt
(3000 B.C. - A.D. 300)

Greek states
(1000 B.C. - 330 B.C.)

Hellenistic kingdoms
(330 B.C. - 30 B.C.)

Roman Empire
(30 B.C. - A.D. 400)

The Hellenic city states, around 550 BCE

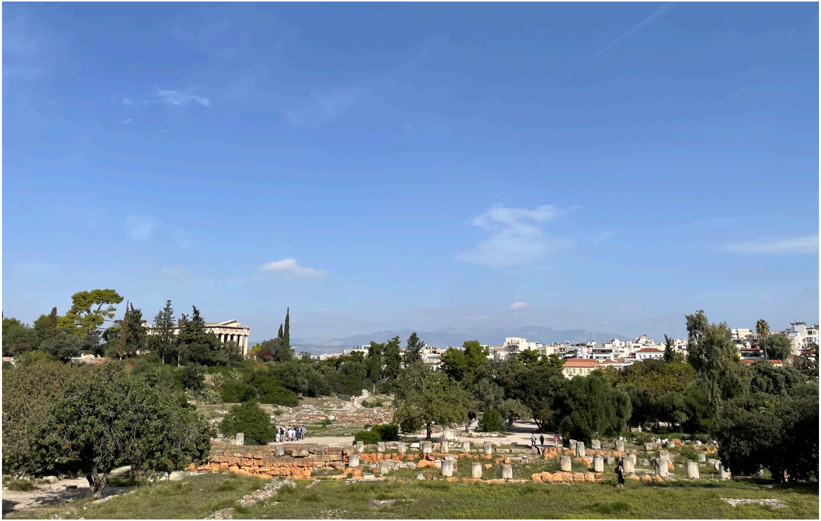
Ancient Greek political and intellectual culture

- The political organization was generally around small, independent city-states.
- The form of government differed from state to state, but many of them were democratic,[5] or participatory.
- As a result of this, Greek intellectual culture centered around open, public debates.
- They developed techniques of *rhetoric* and *dialectic* – the practice of using logical or emotional verbal constructions to convince others of one's position.

[5] In Greece, "democracy" was a limited notion since only native males possessing a certain amount of wealth, or land, were defined as "citizens" and could participate.

The public forum

The Ancient Agora, Athens, 2023



The theater

The remains of the ancient theater, Epidaurus

Remains of the theater at Delphi, 2023

The sophists[6]

The sophists were a group of traveling intellectuals who claimed that they could give the answer to any question. (Protagoras, Gorgias, Antiphon, etc.)

- ◁ They collected money for giving classes, in which they showed how to use rhetoric and dialectic to convince others that the speaker held the correct view on any given topic.
- ◁ They were especially common in Athens, which was a highly litigious city-state.

Although they were criticized by philosophers such as Socrates, Plato, Aristotle, etc., the philosophers adopted many of their methods.

Deductive mathematics was born in the context of these argumentative, public spaces.

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[6] From "sophia" (σοφία), wisdom, skill, intelligence.

The place of mathematicians in Ancient Greek culture

- Greco-Roman culture was *not* a scribal culture.
- Most citizens (and some slaves, freemen and women) could read and write. (Nevertheless, this was a *small percentage* of the total population.)
- Literacy was a form of participation in higher political and intellectual culture.
- Although there was a group of people carrying out the accounting-style mathematics we saw in the Egyptian and Mesopotamian cultures, they were usually socially and culturally distinct from the theoretical mathematicians.
- By and large, *theoretical mathematics* was carried out as a form of high culture – it was not meant to be useful, or immediately beneficial. It was something like music, theater or poetry.

An example of mathematics in public: Plato's *Meno*

- Plato's *Meno* is a dialog in which Socrates tries to answer the question, "What is (human) excellence (ἀρετή)?"
- The main characters are **Socrates** and **Meno**. (Meno is young, good-looking and well-born – perhaps *a sophist*. Socrates is old, ugly, and common – *a philosopher*.)
- Along the way, they are trying to figure out *whether or not excellence can be taught*, and, more generally, whether *anything* can be taught.
- Socrates claims that all *true knowledge* is innate – that it is already there inside us, we just need to properly remember what it is.
- In order to make his point, he asks Meno to lend him one of Meno's slaves. He then attempts to show that the **Slave Boy** is capable of "remembering" mathematics, without being told any answers.

Reading the Meno

- Three students will read out the parts of Socrates, the Slave Boy (or Girl) and Meno.
- One student will advance the slides, while I will draw diagrams on the board. (Note that the diagrams referred to in the text have no letter-names.)

- The presence of letter-names, in later texts, is an indication of the transformation from an oral to a textual tradition.

A passage from Plato's *Meno*

MENO: What do you mean when you say that we don't learn anything, but that what we call learning is recollection? Can you teach me that it is so?

SOCRATES: I have just said that you're a rascal, and now you ask me if I can teach you, when I say there is no such thing as teaching, only recollection. Evidently you want to catch me contradicting myself straight away.

MENO: No, honestly, Socrates, I wasn't thinking of that. It was just habit. If you can in any way make clear to me that what you say is true, please do.

SOCRATES: It isn't an easy thing, but still I should like to do what I can since you ask me. I see you have a large number of retainers here. Call one of them, anyone you like, and I will use him to demonstrate it to you.

MENO: (To a slave boy.) Come here.

SOCRATES: He is a Greek and speaks our language?

MENO: Indeed, yes – born and bred in the house.

SOCRATES: Listen carefully then, and see whether it seems to you that he is learning from me or simply being reminded.

MENO: I will.

SOCRATES: Now boy, you know that a square is a figure like this?
(Socrates begins to draw figures in the sand at his feet.)

BOY: Yes.

SOCRATES: It has all these four sides equal?

BOY: Yes.

SOCRATES: And these lines which go through the middle of it are also equal?

BOY: Yes.

SOCRATES: Such a figure could be either larger or smaller, could it not?

BOY: Yes.

SOCRATES: Now if this side is two feet long, and this side the same, how many feet will the whole be? Put it this way. If it were two feet in this direction and only one in that, must not the area be two feet taken once?

BOY: Yes.

SOCRATES: But since it is two feet this way also, does it not become twice two feet?

BOY: Yes.

SOCRATES: And how many feet is twice two? Work it out and tell me.

BOY: Four, Socrates.

SOCRATES: Now could one draw another figure double the size of this, but similar, that is, with all its sides equal like this one?

BOY: Yes.

SOCRATES: How many feet will its area be?

BOY: Eight.

SOCRATES: Now then, try to tell me how long each of its sides will be. The present figure has a side of two feet. What will be the side of the double-sized one?

BOY: It will be double, Socrates, obviously.

SOCRATES: You see, Meno, that I am not teaching him anything, only asking. Now he thinks he knows the length of the side of the eight-feet square.

MENO: Yes.

SOCRATES: But does he?

MENO: Certainly not.

SOCRATES: He thinks it is twice the length of the other.

MENO: Yes.

SOCRATES: Now watch how he recollects things in order—the proper way to recollect. (To the Boy:) You say that the side of double length produces the double-sized figure? Like this I mean, not long this way and short that. It must be equal on all sides like the first figure, only twice its size, that is eight feet. Think a moment whether you still expect to get it from doubling the side. BOY: Yes, I do.

SOCRATES: Well now, shall we have a line double the length of this, if we add another the same length at this end?

BOY: Yes.

SOCRATES: It is on this line then, according to you, that we shall make the eight-feet square, by taking four of the same length?

BOY: Yes.

SOCRATES: Let us draw in four equal lines, using the first as a base. Does this not give us what you call the eight-feet figure?

BOY: Certainly.

SOCRATES: But does it contain these four squares, each equal to the original four-feet one?

BOY: Yes

SOCRATES: How big is it then? Won't it be four times as big?

BOY: Of course.

SOCRATES: And is four times the same as twice?

BOY: Of course not.

SOCRATES: So doubling the side has given us not a double but a fourfold figure?

BOY: True.

SOCRATES: And four times four are sixteen, are they not?

BOY: Yes.

SOCRATES: Then how big is the side of the eight-feet figure? This one has given us four times the original area, hasn't it?

BOY: Yes.

SOCRATES: And a side half the length gave us a square of four feet?

BOY: Yes.

SOCRATES: Good. And isn't a square of eight feet double this one and half that?

BOY: Yes.

SOCRATES: Will it not have a side greater than this one but less than that?

BOY: I think it will.

SOCRATES: Right. Always answer what you think. Now tell me: was not this side two feet long, and this one four?

BOY: Yes.

SOCRATES: Then the side of the eight-feet figure must be longer than two feet but shorter than four?

BOY: It must.

SOCRATES: Try to say how long you think it is.

BOY: Three feet.

SOCRATES: If so, shall we add half of this bit and make it three feet? Here are two, and this is one, and on this side similarly we have two plus one; and here is the figure you want.

BOY: Yes.

SOCRATES: If it is three feet this way and three that, will the whole area be three times three feet?

BOY: It looks like it.

SOCRATES: And that is how many?

BOY: Nine.

SOCRATES: Whereas the square double our first square had to be how many?

BOY: Eight.

SOCRATES: But we haven't yet got the square of eight feet even from a three-feet side?

BOY: No.

SOCRATES: Then what length will give it? Try to tell us exactly. If you don't want to count it up, just Show us on the diagram.

BOY: It's no use, Socrates, I just don't know.

SOCRATES: Observe, Meno, the stage he has reached on the path of recollection. At the beginning he did not know the side of the square of eight feet. Nor indeed does he know it now, but then he thought he knew it and answered boldly, as was appropriate – he felt no perplexity. Now however he does feel perplexed. Not only does he not know the answer; he doesn't even think he knows.

MENO: Quite true.

SOCRATES: Isn't he in a better position now in relation to what he didn't know?

MENO: I admit that too.

SOCRATES: So in perplexing him and numbing him like the sting-ray, have we done him any harm?

MENO: I think not.

SOCRATES: We have certainly, as would seem, assisted him in some degree to the discovery of the truth; and now he will wish to remedy his ignorance, but then he would have been ready to tell all the world again and again that the double area should have a double side.

MENO: True.

SOCRATES: In fact we have helped him to some extent towards finding out the right answer, for now not only is he ignorant of it but he will be quite glad to look for it. Up to now, he thought he could speak well and fluently, on many occasions and before large audiences, on the subject of a square double the size of a given square, maintaining that it must have a side of double the length.

MENO: No doubt.

SOCRATES: Do you suppose then that he would have attempted to look for, or learn, what he thought he knew – though he did not – before he was thrown into perplexity, became aware of his ignorance, and felt a desire to know?

MENO: No.

SOCRATES: Then the numbing process was good for him?

MENO: I agree.

SOCRATES: Now notice what, starting from this state of perplexity, he will discover by seeking the truth in company with me, though I simply ask him questions without teaching him. Be ready to catch me if I give him any instruction or explanation instead of simply interrogating him on his own opinions.
(Socrates here rubs out the previous figures and starts again.)
(To the slave boy:) Tell me, boy, is not this our square of four feet? You understand?

BOY: Yes.

SOCRATES: Now we can add another equal to it like this?

BOY: Yes.

SOCRATES: And a third here, equal to each of the others?

BOY: Yes.

SOCRATES: And then we can fill in this one in the corner?

BOY: Yes.

SOCRATES: Here, then, there are four equal spaces?

BOY: Yes.

SOCRATES: And how many times the size of the first square is the whole?

BOY: Four times.

SOCRATES: And we want one double the size. You remember?

BOY: Yes.

SOCRATES: Now does this line going from corner to comer cut each of these squares in half?

BOY: Yes.

SOCRATES: And these are four equal lines enclosing this area?

BOY: They are.

SOCRATES: Now think. How big is this area?

BOY: I don't understand.

SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them?

BOY: Yes.

SOCRATES: And how many such halves are there in this figure?

BOY: Four.

SOCRATES: And how many in this one?

BOY: Two.

SOCRATES: And what is the relation of four to two?

BOY: Double.

SOCRATES: How big is this figure then?

BOY: Eight feet.

SOCRATES: On what base?

BOY: This one.

SOCRATES: The line which goes from comer to comer of the square of four feet?

BOY: Yes.

SOCRATES: The technical name for it is ‘diagonal’; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area.

BOY: That is so, Socrates.

SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own?

MENO: No, they were all his.

SOCRATES: Yet he did not know, as we agreed a few minutes ago.

MENO: True.

SOCRATES: But these opinions were somewhere in him, were they not?

MENO: Yes.

SOCRATES: So a man who does not know has in himself true opinions on a subject without having knowledge.

MENO: It would appear so.

Early history of science

- It was characteristic of Greek historians to attribute everything to the work of some *individual* inventor or originator.
- In fact, it is likely that many things whose origins were unknown were attributed to famous people of the past, merely because they were famous.
- Early Greek histories are a sort of blend of facts and fiction, woven together in what the author considers will make a good story.

- In the case of mathematics, we have reports of, and quotations from, books called the *History of Geometry* and the *History of Arithmetic*, by Eudemus of Rhodes (late 4th century BCE), which are themselves lost.

The doxographical tradition

- Later, during the Hellenistic period, the history of intellectual traditions were collected in a type of work that was called *doxography* – reporting on and writing about the thoughts of others.

- Based on the work of Eudemus, and other sources, in later periods there developed a body of work expounding the history of early Greek mathematics.
- Our actual sources for this material often come from authors as late as Simplicius and Eutocius (6th century CE).

- Hence, our idea of the history of early Greek mathematics is based on a great deal of (re)construction – both ancient and modern.

Thales: The angle in a semicircle is a right angle

Here we look at a reconstruction by Dunham. Note that Thales himself left *no writings*. What might his proof have been like? (Euclid proves the same thing in *Elements* III 31.)

- A direct proof by (simple) construction.
- Preliminaries: (1) Radii of a circle are equal (by definition). (2) The bases of an isosceles triangle are equal (*Elem*. I.5). (3) The sum of the angles of a triangle are two right angles ($= 180° = \pi_{\text{rad}}$, *Elem*. I.32).

- We start with a semicircle on diameter BC. We take center O, and a *random point*, A, on the arc. We join line AO…

Pythagoreans: Incommensurability in the square[7]

We want to show that there is no line, no matter how small, that will *measure* both the side and the diameter of the square – so the lines are incommensurable. This is mathematically related to our notion of irrational.

- ◁ An indirect proof by construction.
- ◁ Preliminaries: (1) The square on the diagonal is twice the square on the side, *Meno*, (2) any ratio can be expressed in least terms (no common factors), (3) if n^2 is even, then n is even.
- ◁ We start with square $ABCD$ and assume that side AB and diameter AC can be expressed by some least ratio in two natural numbers, ef and g. We will then argue that, if this is the case then g is both even and odd – which is absurd…

[7] This proof comes from Euclid's *Elements* X 117. Even in antiquity, it was attributed to the Pythagoreans.

Quadrature of the lunes

- Our first evidence for logical arguments come from a report written in the 6th c. CE by Simplicius about work done by Hippocrates *some eight centuries earlier*.
- Much of what we can say about this must be reconstructed from the text.

- Hippocrates's argument relies on a number of preliminaries: (1) The so-called Phythagorean theorem, (2) angles in a semicircle are right (attributed to Thales), (3) the possibility of squaring any rectilinear figure – that is, producing the *quadrature*, (4) the areas of similar segments of circles are as the squares on their chords,

$$A_1 : A_2 = c_1^2 : c_2^2$$

Quadrature of a simple lune (Dunham)

- A direct proof by construction.
- We begin with a lune, $AECF$, formed on the side of a square drawn in circle ABC.
- We draw in the internal triangles, $\triangle AOC$ and $\triangle COB$, and argue that the areas can be related based on the geometry of the figure…

Interpreting an ancient text

- We have seen Dunham's reading of the semicircular lune. He presents a similar argument for the hexagonal lune. Both of these reconstructions are drawn from Alexander of Aphrodisias (end of 2nd centure CE).
- Simplicius argues that Hippocrates thought he could square every lune, because he squares one on the semicircle, one on a segment greater than a semicircle and one on a segment lesser than a semicircle.
- But in fact, the lunes that Simplicius discusses are "special cases," and Hippocrates must have known this.

Simplicius' account

Simplicius, *Commentary on Aristotle's Physics*

"He [Hippocrates] did this by circumscribing about a right angled isosceles triangle a semicircle and about the base of the segment of a circle similar to those cut off by the sides. Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both, the lune will be equal to the triangle.

He then goes on to describe a similar situation with a trapezoid (trapezium), such that the square on the base is three times the squares on the other three equal sides.

Two quadratures attributed to Hippocrates by Simplicius

- In fact, Hippocrates' arguments are simple and elegant – based directly on the claims that circular segments are as the squares of their bases ($A_1 : A_2 = c_1^2 : c_2^2$).
- He begins by considering the lune with an outer circumference that is divided into two equal parts and stands on a base whose square is equal to the sum of the squares on the sides.
- Then he examines a lune with an outer circumference that is divided into three equal parts.
- (The final lune in the reading is based on the same general principle but is somewhat more complicated.)

The logical structure of early proofs

- ◁ What was the idea of the completeness of the logical argument?
- ◁ What kinds of things could serve as starting points (assumptions)?
- ◁ What was the overall logical organization of early proofs?
- ◁ Were there theories, or just loose collections of theorems?