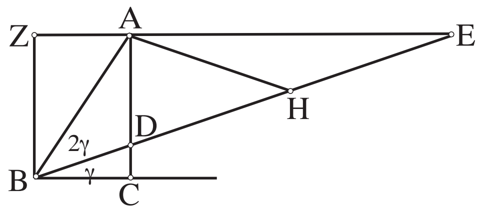
MI314 – History of Mathematics: Descartes' treatment of the Pappus locus theorem

An example of ancient analysis

Definitions. *In ancient analysis, the geometer assumes what is to be done, and then uses the mathematical consequences of this assumption to reveal what can be done to solve the problem.*

As well as the Euclidean constructions, ancient geometers allowed a type of construction called neusis. *A neuses line is a line such that a segment of a given length is cut off between two given object and its extension passes through a given point.*



Theorem. *Pappus* Collection *IV 39. To trisect a given angle.*

Proof. Let $\angle ABC$ be given. The problem is to trisect this angle.

Assume that it is *done*, and let the trisection be done by line BDE, so that $\angle ABD = \gamma$ is twice $\angle DBC = 2\gamma$.

Take a random point on BA, say A, and drop $AC \perp BC$. Complete $\square BZAC$. Extend ZA and BD to meet at point E. Take the midpoint of DE as point H, and join AH.

Since $\angle ADE$ is a right angle, $\triangle AED$ stands in a circle, so $DH = AH = HE$ (radii of a circle), and

$$\angle AHD = 2\angle AED.$$

But since $AE \parallel BC$, $\angle AEH = \angle DBC$, so that

$$\angle AHD = 2\angle DBC,$$

and therefore, by assumption,

$$\angle AHD = \angle ABD.$$

Therefore, $\triangle ABH$ is isosceles and $AH = AB$.

But $AH = DH = HE$, therefore

$$DE = 2AB.$$

Hence, if a neusis line is drawn passing through the given point B such that between given lines ZE and AC segment DE is cut off equal to $2AB$, then the problem will be solved.

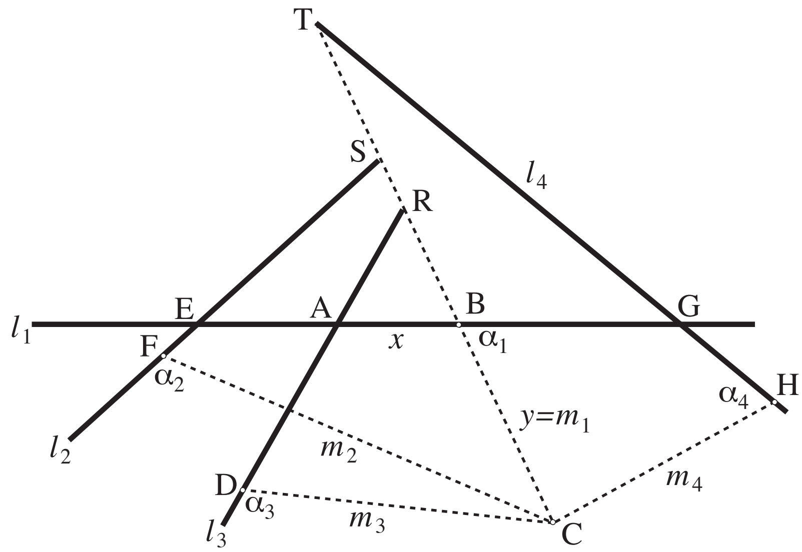
□

Key. *We begin by assuming the existence of the object we are trying to find and then explore the mathematical consequences of this assumption, until we are able to find some properties of this object that will allow us to produce it.*

The four line locus theorem

Definitions. *The general form of a conic section (parabola, ellipse, hyperbola and circle) is a second degree equation:*

$$ax^2 + bxy + cy^2 + dx + cy + e = 0.$$



Theorem. *Given four lines fixed in place, l_1, l_2, l_3 and l_4, and three variable lines, m_1, m_2, m_3 and m_4, having fixed angles with the given lines, α_1, α_2, α_3 and α_4, to show that the point, C, satisfying the condition $m_1 m_2 = m_3 m_4$ is a conic section. That is, we will show that equations of this form are second degree.*

Proof. We begin by choosing two coordinates. We will set $AB := x$ and $BC = m_1 := y$, because these lines are not given and will vary as the positions of the ms vary.

Then, since the ls are given in place and the αs are given in value, the angles of $\triangle ARB$ are given, so we can set

$$\frac{BR}{AB} := b, \text{ that is } BR = bx,$$

where a, b, c, etc. are known values. Hence

$$CR = y + bx. \tag{1}$$

Now, since the angles of $\triangle DRC$ are given, we can set

$$\frac{CD}{CR} := c, \tag{2}$$

so that, by substituting (1) into (2), we have

$$m_3 = CD = cy + cbx. \tag{1.1}$$

So, since l_1, l_2, l_3 and l_4 are fixed, we set $AE := k$ and $AG := l$, and

$$\frac{BS}{BE} := d, \tag{3}$$

$$\frac{CF}{CS} := e, \tag{4}$$

$$\frac{BT}{BG} := f, \tag{5}$$

and

$$\frac{CH}{TC} := g, \tag{6}$$

Then, looking at the figure, $BE = x + k$. So that, substituting into (3), we have $BS = dk + dx$. Then, looking at the figure,

$$CS = y + dk + dx, \tag{7}$$

and substituting (7) into (4), we have

$$m_2 = CF = ey + edk + edx. \tag{1.2}$$

And, looking at the diagram, $BG = l - x$, so substituting into (5), we have $BT = fl - fx$, and looking at the diagram,

$$TC = y + fl + fx. \tag{8}$$

Then, substituting (8) into (6), we have

$$m_4 = CH = gy + gfl - gfx. \tag{1.3}$$

Finally,

$$m_1 = CB = y. \tag{1.4}$$

Hence, in equations (1.1–4), the equations of m_1, m_2, m_3 and m_4 have been determined to be first degree equations in x and y.

Hence, any pairwise product of m_1, m_2, m_3 and m_4 will be a quadratic equation in x and y.

Therefore point C will lie on some conic section.

□

Key. *We select two coordinates, one of which is one of our variable lines,* m_1 *and the other is a length of one of the fixed lines,* l_1*, cut off between the intersection of two fixed lines,* l_1 *and* l_3*, and the intersection of the variable line,* m_1*. We then express all other lengths in terms of these two variables and show that the equation of the locus is quadratic.*