MI314 – History of Mathematics:
Pascal's Treatment of the Arithmetic Triangle

The Arithmetic triangle

Definition of the triangle and its elements. *Calling each element in the triangle a* cell*, each horizontal line a* row*, and each vertical line a* column*, the cell in nth row and kth column is called $\binom{n}{k}$. After setting out the first three cells as*

$$\begin{matrix} 1 & \\ 1 & 1 \end{matrix}$$

each row n begins with a 1, and then the following cells are computed as $\binom{n-1}{k-1} + \binom{n-1}{k}$. Each row goes from $\binom{n}{0}$ to $\binom{n}{k=n} = 1$. Hence, in general,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

In this way, the first 10 rows of the table are

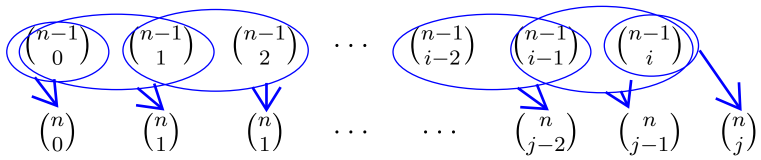
n/k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	...
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
...	...	...	...	...	...	...	...	...	...	...	...

7th Consequence. *The sum of the cells of the nth row is double that of row $n-1$:*

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2 \sum_{j=0}^{j=n-1} \binom{n-1}{j}.$$

Proof. Pascal simply gives a generalizable example to show this proposition. We can summarize this argument symbolically, although Pascal does not use such symbols.

We consider a row and and the previous row:



By definition, each cell $\binom{n}{k}$ is equal to $\binom{n-1}{k} + \binom{n-1}{k-1}$, and since $\binom{n}{0} = \binom{n-1}{0}$ and $\binom{n}{j} = \binom{n-1}{i}$, each cell of row $n-1$ is used twice when constructing the cells of row n.

□

Key. *Here we simply point out that the definition of the triangle directly implies that each cell of a row is taken twice in computing the cells of the following row.*

8th Consequence. *The sum of the cells of the nth row is 2^n:*

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2^n.$$

Proof. We will show this by perfect induction.

Step 1: Prop(0) can be shown as follows: $n := 0$, then $\binom{0}{0} = 1$, and $2^0 = 1$, so the theorem holds. Or, Prop(1), may be shown as $n := 1$, then $\binom{1}{0} + \binom{1}{1} = 2$, and $2^1 = 2$, so the theorem holds for $n := 0$ and $n := 1$.

Step 2: Prop$(n) \implies$ Prop$(n+1)$ is shown by proving that

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2^n \implies \sum_{j=0}^{j=n+1} \binom{n+1}{j} = 2^{n+1}.$$

Here, the *inductive hypothesis* is

$$\sum_{k=0}^{k=n} \binom{n}{k} = 2^n \tag{8.1}$$

But, by the 7th Cons., the sum of the cells of the next row is twice that of n, that is,

$$\sum_{j=0}^{j=n+1} \binom{n+1}{j} = 2 \sum_{k=0}^{k=n} \binom{n}{k}. \tag{8.2}$$

So, substituting (8.1), the *inductive hypothesis*, into (8.2), the 7th Cons., we have

$$\sum_{j=0}^{j=n+1} \binom{n+1}{j} = 2 \times 2^n = 2^{n+1}.$$

Hence, **Step 2** has been shown, so the theorem holds for all n. □

Key. *We show that the theorem holds in the first row, and then show that if we* assume *it holds for any row it must necessarily hold for the following row.*

The Problem of Points

The problem of points involves dividing up the stakes in a gambling game of *pure chance*, in which two or more players plan to play a number of matches and the player with the most wins takes the whole pot. We then consider the possibility that the game is interrupted before a winner has been determined, and seek to find a fair division of the pot.

Pascal gives a "generalizable example" as a proof for this proposition. We can show it, however, with perfect induction.

Definitions for problem of points. *Two or more players (**Player 1, Player 2,** etc.) are playing a* game *of pure chance, in which each player has put up an even* stake *($\frac{1}{2}$, $\frac{1}{3}$, etc.) in the total* pot. *That player who wins the most* matches *takes the whole* pot. *If the game is interrupted before a winner has been decided, we want to know what would be fair* take *for each player to receive from the pot.*

Problem of Points. *Where **Player 1** lacks $r \geqslant 1$ matches and **Player 2** lacks $s \geqslant 1$ to take the whole pot, such that there are at most $n = r + s - 1$ matches left to decide the game, then the pot should be divided such that*

$$\textbf{Player 1}\text{'s take : whole pot} = \sum_{k=0}^{s-1} \binom{n}{k} : 2^n.$$

Explanation. This means that the more matches n there are left to play the greater the row of the Arithmetic Triangle, so the more possibilities there are for dividing up the pot – or the more finely the division can be made. If we consider any particular row, **Player 1**'s take, $\sum_{k=0}^{s-1}\binom{n}{k}$, is on the righthand side and **Player 2**'s take, $2^n - \sum_{k=0}^{s-1}\binom{n}{k}$, is on the lefthand side of position where we divide the pot, which is determined by the number of matches that **Player 2** has yet to win. Moreover, the greater the number of matches **Player 2** has left to win, s, the further along the row we move to the right, so that **Player 2**'s take decreases and **Player 1**'s take increases.

For example, assuming that there are at most $n := 4$ matches left to play we consider the row

1		4		6		4		1
	s=1		s=2		s=3		s=1	
	r=4		r=3		r=2		r=4	

and divide up the pot between the first "1" and "4" where **Player 2** has a two match advantage, between the first "4" and "6" where **Player 2** has a one match advantage, between the second "6" and "4", where **Player 2** has a one match disadvantage and so on. (There can be no even split of the pot, $1:2$, when n is an even number, as seen from the structure of the Arithmetic Triangle.)

Example. Suppose two players, **Player 1** and **Player 2**, each put up ¥5,000 and flip coins 7 times, so that the first player who wins 4 should take the pot. Then, if when **Player 1** lacks 2 matches, $r := 2$, and **Player 2** lacks 3 matches, $s := 3$, such that at most $n = r + s - 1 = 4$ matches remain to be played, the game is stopped then the theorem asserts that

$$\textbf{Player 1}\text{'s take : whole pot } = \sum_{k=0}^{2}\binom{4}{k} : 2^4.$$

Then, since $\sum_{k=0}^{2}\binom{4}{k} = 1 + 4 + 6$,

$$\textbf{Player 1}\text{'s take : whole pot } = 11 : 16,$$

and **Player 2**'s take is $5 : 16$. Hence **Player 1** receives ¥6,875 and **Player 2** receives ¥3,125.

Proof. We again prove Pascal's claim using perfect induction.

Step 1: We show Prop(1) by setting $m := 1 = r + s - 1$, so that $r = 1$ and $s = 1$. In this situation, each player has one match left to win, so the game up to this point is evenly split, and they should should split the pot evenly. Indeed,

$$\sum_{k=0}^{1}\binom{1}{k} : 2^1 = \binom{1}{0} : 2 = 1 : 2,$$

so **Player 1**'s take is half of the pot, and **Player 2**'s take is the other half of the pot. Hence, for $m := 1$ the theorem holds.

Step 2: In order to show Prop$(m) \implies$ Prop$(m+1)$, that is

$$\sum_{k=0}^{s-1} \binom{m}{k} : 2^m \implies \sum_{k=0}^{s-1} \binom{m+1}{k} : 2^m + 1.$$

First, we need to consider what the *inductive hypothesis*, Prop(m), implies for possibilities of the next match.

Possibility 1: **Player 1** wins and lacks $r-1$ to win. Hence, for this possibility the *inductive hypothesis* implies that

$$\textbf{Player 1}\text{'s take : whole pot } = \sum_{k=0}^{s-1} \binom{m}{k} : 2^m,$$

as before. This is because for this possibility **Player 2** still lacks the same number of matches, $s-1$.

Possibility 2: **Player 2** wins, so she lacks $s-2$ matches and the *inductive hypothesis* implies that

$$\textbf{Player 1}\text{'s take : whole pot } = \sum_{k=0}^{s-2} \binom{m}{k} : 2^m.$$

□

Now, since this is a game of chance and each of these possibilities is considered to be equally probable, taken together *inductive hypothesis* implies that the ratio of **Player 1**'s take to the whole pot is the mean of these two possibilities. That is,

$$\begin{aligned}
\textbf{Player 1}\text{'s take : whole pot } &= \left(\sum_{k=0}^{s-1} \binom{m}{k} + \sum_{k=0}^{s-2} \binom{m}{k} \right) /2 : 2^m, \\
&= \sum_{k=0}^{s-1} \binom{m}{k} + \sum_{k=0}^{s-2} \binom{m}{k} : 2 \times 2^m, \\
&= \sum_{k=0}^{s-1} \binom{m}{k} + \sum_{k=0}^{s-2} \binom{m}{k} : 2^{m+1}. \qquad \text{(PoP.1)}
\end{aligned}$$

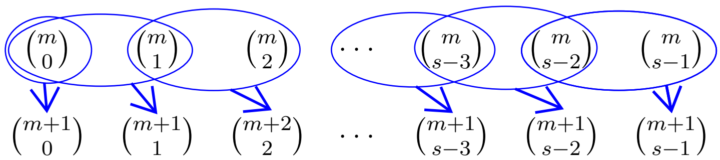
But, by the definition of the arithmetic triangle,

$$\sum_{k=0}^{s-1} \binom{m}{k} + \sum_{k=0}^{s-2} \binom{m}{k} = \sum_{k=0}^{s-1} \binom{m+1}{k}, \qquad \text{(PoP.2)}$$

or

$$\binom{m}{0} + \sum_{k=0}^{s-1} \binom{m}{k} + \sum_{k=1}^{s-2} \binom{m}{k} = \sum_{k=0}^{s-1} \binom{m+1}{k}.$$

That is, the cells of each row is computed by taking the sum of the cells of the previous row directly above and to the left,



Then, substituting (PoP.2) into (PoP.1), we have

$$\textbf{Player 1}\text{'s take : whole pot } = \sum_{k=0}^{s-1} \binom{m+1}{k} : 2^{m+1}.$$

Hence, **Step 2** has been shown, so the theorem holds for all m.

Key. *After noting that the theorem holds for the first row, we have to consider what the inductive hypothesis would imply for the next game at any row, under the assumption that the next game could go either way. Taking the mean of the possibilities for the next game, we show that this implies the theorem for the next row.*