Isaac Newton: Development of the Calculus and a Recalculation of π

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History of Mathematics

Introduction
Early modern Britain
Newton's life and work

Newton's mathematical development

A new method for calculating the value of π

The early modern period in Britain

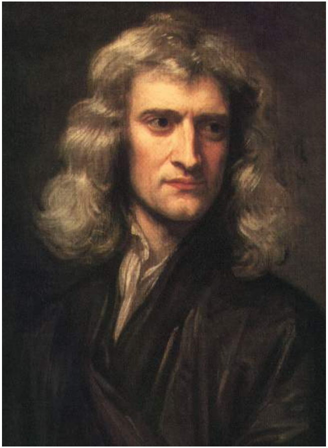
- The early modern period in Britain saw the county's role in the world vastly expanded through exploration, navel power and colonialism.
- There were a number of revolutions, which expanded the access to power of the upper and middle-classes.
- There was a rise in the standard of living, access to education, and an increase in the scope of the social role of the middle and lower classes.
- During this period, England became one of the major centers of scientific activity. A number of new institutions were founded to promote scientific activities.

The intellectual context of Newton's work

- During Newton's lifetime, England was an important center of the scientific revolution that was taking place all across Europe.
- The most recent ideas were the *mechanical* and the *experimental philosophies* and the most recent mathematical developments were the symbolic and analytical geometry of Descartes and Fermat, symbolic number theory, probability theory, and the techniques of measuring areas and finding tangents being developed by the colleagues of Mersenne and others.
- But, *Newton was not able to study any of this at university*.
- Because the universities of the time did not serve the needs of people who were interested in the sciences, a number of new institutions were created: The Royal Society of London, The Royal Observatory, etc.

Isaac Newton (1643–1727)

- Abandoned by his widowed mother.
- Alone his whole life; no family, few close friends. Deeply obsessive personality.
- 1664–1666: *Anni mirabiles*.
- Nervous breakdown $\implies$ Began a public life. Director of the Mint. President of the Royal Society.
- Made a peer of the realm. Buried in state at Westminister Abbey.



Newton's work

- Around 1665, he developed the *calculus* (also independently developed by G.W. Leibniz) and did much original work in mathematics. Wrote many papers, the majority unpublished.
- Worked continuously on alchemy and theology. Many volumes of notes, never published. (The majority of Newton's writings are of these kinds.)
- He founded a new form of mathematical dynamics, which was published in the *Philosophiæ naturalis principia mathematica* (1687). Often called the *Principa*.
- He developed a new science of optics based on the refractive properties of light, which was published early in some papers in the *Transactions of the Royal Society*, and later in *Optics* (1704).

The evidence for studying Newton's work

- Newton meticulously kept everything he wrote, so that we now have hundreds of boxes of his notes and autograph manuscripts in a number of different libraries. The vast majority of what Newton wrote was not published.
- On his death he left his papers to Trinity College, Cambridge, but they were claimed by one of his debtors and went into private hands.
- For a number of reasons, the various papers (scientific, mathematical, theological, alchemical) were separated into different collections.
- Most of the scientific and mathematical texts were given to Cambridge University in 1872. The extent of Newton's interest in astrology and theology only became clear to scholars in the second half of the 20th century.

Learning mathematics

- When Newton was an undergraduate at Cambridge, Isaac Barrow (1630–1677) was the first Lucasian Professor of Mathematics.
- Although Barrow discovered a geometric version of the *fundamental theorem of calculus*, it is likely that his university lessons focused only on Greek mathematics and that Newton did not attend them.
- Newton learned mathematics by borrowing the books of Descartes and others from the library and reading them on his own. We still posses many of the notebooks he kept during this process.
- He says that Descartes' *Geometry* was so difficult that it took him many tries to get through it. (His study notes give evidence of his various attempts.)

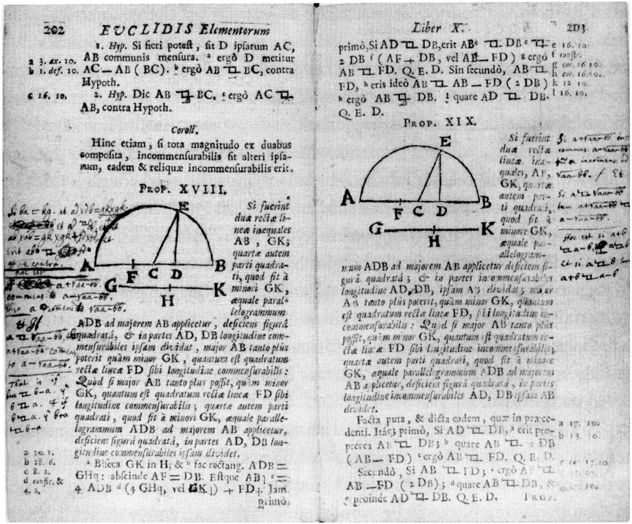
Developing the calculus

- When he was an undergradate, during the plague years, he developed a general, symbolic treatment of the differential and integral calculus, known as the method of *fluxions*.
- Although he was doing mathematical work that he must have known was more advanced than anything currently available, he did not publish it.
- The example of his calculation of the value of π is taken from this early period, although published much later.

Reading the classics and writing the *Principia*

- When he was working as the Lucasian Professor of Mathematics, following Isaac Barrow (1630–1677), as he became more interested in alchemy and theology, he also began to read classical Greek mathematics: Euclid, Archimedes and Apollonius.
- Somehow, he became convinced that this ancient geometrical approach was more appropriate for describing the *physical world*.
- When he composed the *Principia*, it was in the classical style, with little indication of the more modern and symbolic approach that had lead him to his new ideas. (He also included a short section showing that some of the problems that Descartes was most proud of solving could also be solved using ancient methods: the locus problems.)

A page from Newton's copy of the *Elements*, Book X


202 EUCLIDIS Elementorum

1. Hyp. Si fieri potest, sit D ipsarum AC, AB communis mensura. a ergò D metitur AC — AB (BC). b ergò AB ⊓ BC, contra Hypoth.

2. Hyp. Dic AB ⊓̸ BC. c ergò AC ⊓̸ AB, contra Hypoth.

a 3. ax. 10. b 1. def. 10. c 16. 10.

Coroll.

Hinc etiam, si tota magnitudo ex duabus composita, incommensurabilis sit alteri ipsarum, eadem & reliquæ incommensurabilis erit.

Prop. XVIII.

Si fuerint duæ rectæ lineæ inæquales AB, GK; quartæ autem parti quadrati, quod fit à minori GK, æquale parallelogrammum ADB ad majorem AB applicetur, deficiens figurâ quadratâ, & in partes AD, DB longitudine commensurabiles ipsam dividat, major AB tanto plus poterit quàm minor GK, quantum est quadratum rectæ lineæ FD sibi longitudine commensurabilis: Quòd si major AB tanto plus possit, quàm minor GK, quantum est quadratum rectæ lineæ FD sibi longitudine commensurabilis; quartæ autem parti quadrati, quod fit à minori GK, æquale parallelogrammum ADB ad majorem AB applicetur, deficiens figurâ quadratâ, in partes AD, DB longitudine commensurabiles ipsam dividet.

a Biseca GK in H; & b fac rectang. ADB = GHq: abscinde AF = DB. Estque ABq c = 4 ADB d (4 GHq, vel GKq) + FDq. Jam primò,

a 10. 1. b 28. 6. c 8. 2. d constr. & 4. 2.

Liber X. 203

primò, Si AD ⊓ DB, erit AB e ⊓ DB e ⊓ 2 DB f (AF + DB, vel AB — FD) g ergò AB ⊓ FD. Q. E. D. Sin secundò, AB ⊓ FD, h erit ideò AB ⊓ AB — FD (2 DB) k ergò AB ⊓ DB. l quare AD ⊓ DB. Q. E. D.

e 16. 10. f constr. g cor. 16. 10. h cor. 16. 10. k 12. 10. l 16. 10.

Prop. XIX.

Si fuerint duæ rectæ lineæ inæquales, AB, GK, quartæ autem parti quadrati, quod fit à minore GK, æquale parallelogrammum ADB ad majorem AB applicetur deficiens figurâ quadratâ; & in partes incommensurabiles longitudine AD, DB, ipsam AB dividat; major AB tanto plus poterit, quàm minor GK, quantum est quadratum rectæ lineæ FD, sibi longitudine incommensurabilis: Quòd si major AB tanto plus possit, quàm minor GK, quantum est quadratum rectæ lineæ FD sibi longitudine incommensurabilis, quartæ autem parti quadrati, quod fit à minore GK, æquale parallelogrammum ADB ad majorem AB applicetur, deficiens figurâ quadratâ, in partes longitudine incommensurabiles AD, DB ipsam AB dividet.

Facta puta, & dicta eadem, quæ in præcedenti. Itáq; primò, Si AD ⊓̸ DB, a erit propterea AB ⊓̸ DB; b quare AB ⊓̸ 2 DB (AB — FD) c ergò AB ⊓̸ FD. Q. E. D.

Secundò, Si AB ⊓̸ FD; c ergò AB ⊓̸ AB — FD (2 DB); d quare AB ⊓̸ DB, & e proinde AD ⊓̸ DB. Q. E. D.

a 17. 10. b 13. 10.

Prop.

Newton's *Principia*, 1687, 1713, 1726

[12]

AXIOMATA SIVE LEGES MOTUS

Lex. I.

Corpus omne perſeverare in ſtatu ſuo quieſcendi vel movendi uniformiter in directum, niſi quatenus a viribus impreſſis cogitur ſtatum illum mutare.

Projectilia perſeverant in motibus ſuis niſi quatenus a reſiſtentia aeris retardantur & vi gravitatis impelluntur deorſum. Trochus, cujus partes cohærendo perpetuo retrahunt ſeſe a motibus rectilineis, non ceſſat rotari niſi quatenus ab aere retardatur. Majora autem Planetarum & Cometarum corpora motus ſuos & progreſſivos & circulares in ſpatiis minus reſiſtentibus factos conſervant diutius.

Lex. II.

Mutationem motus proportionalem eſſe vi motrici impreſſæ, & fieri ſecundum lineam rectam qua vis illa imprimitur.

Si vis aliqua motum quemvis generet, dupla duplum, tripla triplum generabit, ſive ſimul & ſemel, ſive gradatim & ſucceſſive impreſſa fuerit. Et hic motus quoniam in eandem ſemper plagam cum vi generatrice determinatur, ſi corpus antea movebatur, motui ejus vel conſpiranti additur, vel contrario ſubducitur, vel oblique adjicitur, & cum eo ſecundum utriuſq; determinationem componitur.

Lex. III.

[13]

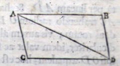
Lex. III.

Actioni contrariam ſemper & æqualem eſſe reactionem: ſive corporum duorum actiones in ſe mutuo ſemper eſſe æquales & in partes contrarias dirigi.

Quicquid premit vel trahit alterum, tantundem ab eo premitur vel trahitur. Siquis lapidem digito premit, premitur & hujus digitus a lapide. Si equus lapidem funi allegatum trahit, retrahetur etiam & equus æqualiter in lapidem: nam funis utrinq; diſtentus eodem relaxandi ſe conatu urgebit Equum verſus lapidem, ac lapidem verſus equum, tantumq; impediet progreſſum unius quantum promovet progreſſum alterius. Si corpus aliquod in corpus aliud impingens, motum ejus vi ſua quomodocunq; mutaverit, idem quoque viciſſim in motu proprio eandem mutationem in partem contrariam vi alterius (ob æqualitatem preſſionis mutuæ) ſubibit. His actionibus æquales fiunt mutationes non velocitatum ſed motuum, (ſcilicet in corporibus non aliunde impeditis:) Mutationes enim velocitatum, in contrarias itidem partes factæ, quia motus æqualiter mutantur, ſunt corporibus reciproce proportionales.

Corol. I.

Corpus viribus conjunctis diagonalem parallelogrammi eodem tempore deſcribere, quo latera ſeparatis.

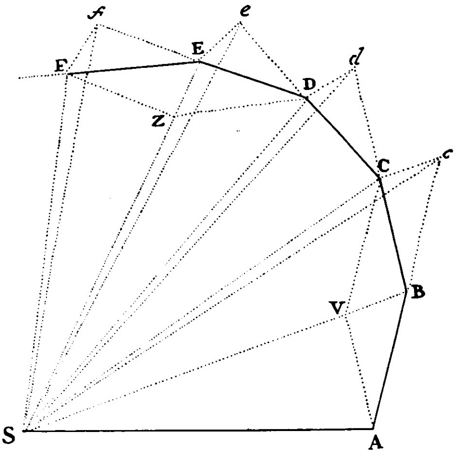


Si corpus dato tempore, vi ſola M, ferretur ab A ad B, & vi ſola N, ab A ad C, compleatur parallelogrammum $ABDC$, & vi utraq; feretur id eodem tempore ab A ad D. Nam quoniam vis N agit ſecundum lineam AC ipſi BD parallelam, hæc vis nihil mutabit velocitatem accedendi ad lineam illam BD a vi altera genitam. Accedet igitur corpus eodem tempore ad lineam BD ſive vis N imprimatur, ſive non, atq; adeo in fine illius temporis reperietur alicubi in linea

illa

Newton's *Principia*, Prop. 1

- He used the ideas of limits developed in the calculus to develop a *geometry of forces*.
- *Principia*, Prop. 1 shows that a body which is *continuously* acted upon toward a center of force will move in a *closed curve*.



Calculating π, overview of the problem

- (1) We use Descartes' techniques of analytical geometry to express the equation of a circle.
- We use Newton's *general* binomial theorem to develop this equation as an infinite series. [1st preliminary]
- We use Newton's new ideas of the calculus to calculate the value of the area of a part of the circle *to the level of precision that we desire*. [2nd preliminary]
- (2) We use basic geometry to find the value of the *same area* in terms of π.
- (3) Then we can set up an equation involving π that we can use to produce a numeric value.

The general binomial theorem, 1st preliminary, 1

Newton expressed the general binomial theorem as

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \ldots$$

which can be rewritten as

$$(1+x)^{m/n} = 1 + \frac{m}{n}x + \frac{(\frac{m}{n})(\frac{m}{n}-1)}{2}x^2 + \frac{(\frac{m}{n})(\frac{m}{n}-1)(\frac{m}{n}-2)}{3\times 2}x^3 + \ldots$$

For example,

$$(1-x)^{1/2} = 1 + \frac{1}{2}(-x) + \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2}(-x^2) + \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(-x^3) + \ldots$$

The general binomial theorem, 1st preliminary, 2

That is,

$$(1-x)^{1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \ldots$$

We can also use this theorem to get accurate calculations of roots. For example, $\sqrt{3}$. Since, $3 = 4(3/4) = 4(1-1/4)$, $\sqrt{3} = 2(1-1/4)^{1/2}$, which, using the binomial theorem, we write as

$$\sqrt{3} = 2\left(1 - \frac{1}{2}\left(\frac{1}{4}\right) - \frac{1}{8}\left(\frac{1}{4}\right)^2 - \frac{1}{16}\left(\frac{1}{4}\right)^3 - \frac{5}{128}\left(\frac{1}{4}\right)^4 - \frac{7}{256}\left(\frac{1}{4}\right)^5 - \ldots\right)$$

that is,

$$\sqrt{3} \approx 1.73206\ldots$$

Basic rules of integral calculus, 2nd preliminary

- **Rule 1**: If a curve is given by $y = ax^{m/n}$ then the area up to x is given by Area(y) $= \frac{an}{m+n}x^{(m+n)/n}$.
- For example, if $y = x^{1/2}$, then Area(y) $= \frac{2}{3}x^{3/2}$, or if $y = \frac{1}{2}x^{3/2}$, then Area(y) $= \frac{1}{2}(\frac{2}{5}x^{5/2})$.
- **Rule 2**: If a curve is a polynomial sum of terms of the form $ax^{m/n}$, then the area under the curve is made up of the sum of the individual terms.
- For example, if $y = x^2 + x^{3/2}$, then Area(y) $= \frac{1}{3}x^3 + \frac{2}{5}x^{5/2}$, etc.

Equation of the circle

Descartes had shown that a circle has an equation of the form $(x-a)^2+(y-b)^2-r^2=0$, where a and b are the x and y coordinates of the center point and r is the length of the radius. Newton decided to use the circle

$$(x-1/2)^2+(y-0)^2-1/2^2=0,$$
$$x^2-x+1/4+y^2-1/4=0.$$

That is,

$$\begin{aligned} y &= \sqrt{x-x^2} \\ &= \sqrt{x}\sqrt{1-x} \\ &= x^{1/2}(1-x)^{1/2} \end{aligned}$$

Calculation of the area by calculus, 1

In order to find the area under this curve, we need to expand it into a polynomial. Using the binomial theorem, as above, we have

$$y = x^{1/2}(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - ...)$$

That is

$$y = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2} - ...$$

Applying rules 1 & 2, to find the area gives

$$\frac{2}{3}x^{3/2} - \frac{1}{2}(\frac{2}{5}x^{5/2}) - \frac{1}{8}(\frac{2}{7}x^{7/2}) - \frac{1}{16}(\frac{2}{9}x^{9/2}) - \frac{5}{128}(\frac{2}{11}x^{11/2}) - ...$$

Calculation of the area by calculus, 2

If we choose some value for x, we can use this expression to find the area under the curve up to that point. Newton takes $x = 1/4$, since $1/4^{1/2} = 1/2$, giving

$$\frac{2}{3}(\frac{1}{2})^3 - \frac{1}{5}(\frac{1}{2})^5 - \frac{1}{28}(\frac{1}{2})^7 - \frac{1}{72}(\frac{1}{2})^9 - \frac{5}{704}(\frac{1}{2})^{11} - \ldots$$

In this way, we can carry out the series to *as many terms as we please*. If we take eight terms, we have

$$\begin{aligned} Area(ABD) &\approx \frac{1}{12} - \frac{1}{160} - \frac{1}{3584} - \frac{1}{36864} - \frac{3}{1441792} - \\ &\quad \frac{7}{13631488} - \frac{7}{83886080} - \frac{33}{2281701376} - \ldots \\ &\approx 0.0767731067786\ldots \end{aligned}$$

Calculation of the area by geometry, 1

Now we consider the geometry of the figure in order to relate $Area(ABD)$ to π. Where $BC = 1/4$ and $DC = 1/2$

$$BD = \sqrt{DC^2 - BC^2} = \sqrt{(1/2)^2 - (1/4)^2} = \sqrt{3/16} = \frac{\sqrt{3}}{4}$$

So that

$$\begin{aligned} Area(\triangle DBC) &= 1/2 BD \times BC, \\ &= (1/2)(\sqrt{3}/4)(1/4), \\ &= \frac{\sqrt{3}}{32}. \end{aligned}$$

Calculation of the area by geometry, 2

Since in right $\triangle DBC$, $BC = 1/2DC$ (the hypothenuse), angle DCB is 60°, so that

$$\begin{aligned} Area(sector\ DCA) &= 1/3Area(semicircle) = (1/3)(1/2)\pi r^2, \\ &= (1/3)(1/2)\pi(1/2)^2, \\ &= \frac{\pi}{24}. \end{aligned}$$

Therefore,

$$\begin{aligned} Area(ABD) &= Area(sector\ DCA) - Area(\triangle DBC), \\ &= \frac{\pi}{24} - \frac{\sqrt{3}}{32}. \end{aligned}$$

Calculation of the value of π

Now we have two expressions for $Area(ADB)$, one of which contains π, so that

$$\frac{\pi}{24} - \frac{\sqrt{3}}{32} \approx .076773107786...$$

Using the binomial theorem to approximate $\sqrt{3}$, as above, we can calculate

$$\pi \approx 3.141592668...$$