Non-Euclidean Geometry:
A new theory of parallel lines

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History of Mathematics

What is geometry?

Until the 19^{th} century, geometry was the study of the mathematical properties of *figures*. All figures were regarded as embedded in euclidean space. (Ex. Spherical geometry.)

During the 19^{th} century, geometry became the study of physical and mathematical *space itself*. It was an investigation of the properties of the types of spaces *in which* figures could be found. Hyperbolic space was put forward as a viable alternative to euclidean space. Geometers realized that elliptical (ex. *spherical*) spaces, and many more, were also possible.

By the end of the century, geometry had come to be seen as the abstract investigation of the consequences of some set of axioms. Space was understood as the set of points determined by these axioms.

Was there a revolution in the study of geometry?

Since antiquity, mathematicians had sought to demonstrate the parallel postulate, but by the beginning of the 19^{th} century, there was a growing anxiety that it could not be done. (Crisis.)

At the beginning of the 19^{th} century, three men – Gauss, Bolyai, Lobachevskii – independently, arrived at a geometry of hyperbolic space. In the middle of the century, Riemann showed that there are an infinite number of non-euclidean geometries, and pointed out that elliptical spaces are possible.

By the end of the century, mathematicians were no longer asking, “what is the correct geometry?” but, rather, trying to determine what sets of axioms would produce what kinds of spaces…

Topics in the non-euclidean revolution

We will look at:

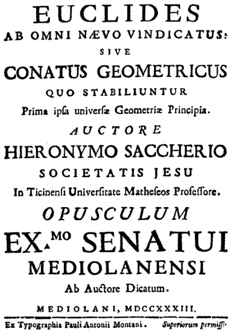
- An extended attempt to prove that euclidean space is the only “correct” space. (We can now see this as logically flawed.)
- An abstract definition of *parallel lines*, which leads to a new kind of space – hyperbolic space.
- A model for showing that a hyperbolic plane is logically consistent with, or mathematically mappable to, a euclidean plane.

Giovanni Girolamo Saccheri (1667–1733)

Saccheri was an Italian Jesuit, who worked as professor of philosophy, theology and mathematics at Turin and Pavia.

In 1733, he published *Euclides ab omni naevo vindicatus* (Euclid freed from all flaws), the first part of which was an attempt to demonstrate Euclid's 5$^{\text{th}}$ postulate. (The other "flaws" concern fourth proportionals and compound ratios.)

- Many of Saccheri ideas were predated, and perhaps influenced, by the work of Omar Khayyam and Ibn al-Haytham.



EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
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EX.$^{\text{MO}}$ SENATUI
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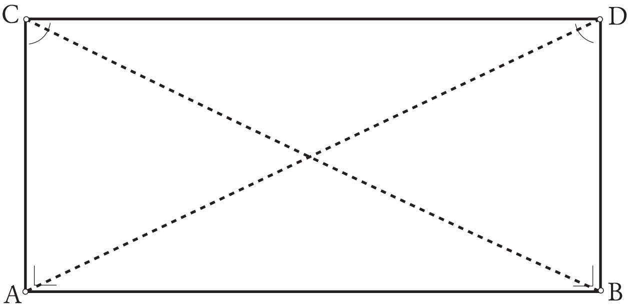
The Saccheri quadralateral



$$\text{Hypothesis of the Right Angle (HRA)} : \angle C = \angle D \quad = \quad 90^\circ$$
$$\text{Hypothesis of the Obtuse Angle (HOA)} : \angle C = \angle D \quad > \quad 90^\circ$$
$$\text{Hypothesis of the Acute Angle (HAA)} : \angle C = \angle D \quad < \quad 90^\circ$$

The structure of the "proof"

Prop. 3:

$$
\begin{array}{llllll}
\text{HRA} & \Longrightarrow & AB & = & CD & \Longrightarrow & \angle s \text{ of } \triangle = 180^\circ \\
\text{HOA} & \Longrightarrow & AB & > & CD & \Longrightarrow & \angle s \text{ of } \triangle > 180^\circ \\
\text{HAA} & \Longrightarrow & AB & < & CD & \Longrightarrow & \angle s \text{ of } \triangle < 180^\circ
\end{array}
$$

The entire first part of *Euclides vindicatus* is structured like an extended indirect proof:

- A): He uses **HRA** to demonstrate the parallel postulate.
- B): He uses **HOA** to show that the geometry so created is inconsistent with *Elem*. I 16 & 17.[1]
- C): He uses **HAA** to argue that it leads to consequences that are "repugnant to the nature of the straight line."

[1] *Elem*. I 16: an exterior $\angle$ of a $\triangle$ is greater than the sum of the 2 interior $\angle$s. *Elem*. I 17: the sum of any 2$\angle$s of a $\triangle$ are less than $2R$.

The hypothesis of the obtuse angle

- ***EV* Props. 11 & 12 (HRA, HOA):** If a line falls on two given lines such that it is perpendicular to one and makes an acute angle with the other, then the two given lines will meet in the direction of the said acute angle. (If $\angle LPA$ is right and $\angle PAD$ is acute, then AD will intersect PL.)

- ***EV* Props. 13 (HRA, HOA):** If a line falls on two given lines such that it makes the two interior angles on the same side less than two right angles, then the two lines will meet in the direction of the said interior angles. (If $\angle AXL + \angle XAD < 2R$, then AD will meet XL.)
  - Contradiction with *Elem.* I.17, which uses *Elem.* I.16.

- ***EV* Props. 14 (HOA):** "The hypothesis of the oblique angle is absolutely false, because it destroys itself."

The hypothesis of the acute angle

- ***EV* Prop. 32 (HAA):** If HAA is true, there there will exist a line with the following properties:
  - AX is a limit to the set of lines passing through point A that meet BX and also a limit to the set of lines passing through point A that have two distinct perpendiculars with line BX – one in each direction.
  - AX meets BX at one point, infinitely distant.
  - AX is asymptotic to BX.[2]
  - AX is a *straight line*.

- ***EV* Prop. 33 (HAA):** "The hypothesis of the acute angle is absolutely false, *because it is repugnant to the nature of a straight line*."

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[2]That is the distance between the two lines will become less than any given value.

New Geometries

In the early part of the 19th century, a number of mathematicians began to explore geometries that used different conceptions of parallel lines. Carl Friedrich Gauss (1777–1855), claimed in a number of letters to his colleagues to have developed results that we can now recognize as non-euclidean geometry, but he did not publish these. He did publish work on the intrinsic curvature of a surface.

The first intrinsic work on a non-euclidean plane and 3-space geometry was published independently by János Bolyai (1802–1860) and N.I. Lobachevskii (1792–1856), who both used a definition of parallel lines that had been introduced by Otto Száz. They (1) developed basic propositions of the plane geometry, (2) showed that a euclidean plane can be embedded in this 3-space, and (3) produced the trigonometry of the hyperbolic plane.

Nikolai Ivanovich Lobachevskii (1792–1856)

- Born to a provincial, middle class family. His father died when he was 8.
- Graduated from Kazan University with a degree in physics and mathematics.
- Worked at Kazan University his whole life.
- He developed his ideas on non-euclidean geometry from 1826 to 1855.
- Died in poverty.

An abstract theory of parallels

- Lobachevskii begins by defining a line, ℓ_p, through a given point P, as *parallel* to a given line, ℓ_0, which is joined to P by p, when ℓ_p divides all the lines that pass through point P into two mutually exclusive sets, those which are intersecting, ℓ_i, and those which are non-intersecting, ℓ_n. ($\ell_p \parallel \ell_0$.) That is, where the angle at P is written $\Pi(p)$:
  - lines, ℓ_i, where $\alpha_1 < \Pi(p)$, intersect ℓ_0, and
  - lines, ℓ_n, where $\alpha_2 > \Pi(p)$, do not intersect ℓ_0.
- Where $\Pi(p) = \pi/2 = 90°$, then there is a *unique* parallel, which is parallel in *both directions*.
- Where $\Pi(p) < \pi/2 = 90°$, then there is another line on the opposite side at the same angle that is parallel in the *opposite direction*.
- With respect to ℓ_0, all lines through P can be classified as 1) intersecting, 2) parallel, or 3) non-intersecting.

Properties of the hyperbolic plane, $\mathscr{H}^2$

- The sum of the angles of a triangle are less than $180° = \pi$.
- There is no distinction between *similarity* and *congruency*. (There are four congruence theorems SAS, SSS, AAS, and AAA.)
  - As triangles get larger and smaller, the angles also change.
- There are no straight lines everywhere equidistant from one another.
- Lines parallel to the same line need not be parallel to one another. (Parallelism is *non-transitive*, because of directionality.)
  - Two lines which *intersect* one another may both be parallel to the same line. (Again, because of directionality.)
  - Two lines may be parallel to the same line but in opposite directions, but be neither parallel, nor meet.
- and so on…

A model of the euclidean plane in hyperbolic space

Lobachevskii then constructed a three dimensional *model of the euclidean plane* within his hyperbolic space. (Just as we can construct a three dimensional model of *spherical surface* within Euclidian space.)

- He shows that the planes containing three non-coplanar parallel lines contain dihedral (solid) angles that sum to 180°. (Prism theorem.)
- He develops a *horocycle* as a curve perpendicular to a set of coplanar parallel lines, and a *horosphere* as the solid of rotation of a horocycle.
- He shows that the geometry of the horosphere is euclidean by mapping it to the euclidean plane. In this way, he shows that a euclidean plane can be embedded in hyperbolic space – just as a spherical surface can be embedded in euclidean space.

Overview

In the 18th century, it became increasingly clear that it would not be possible to prove that euclidean geometry was the only valid geometry. We looked at Saccheri's attempt to prove this. We have noted how in retrospect we can see these proofs as unsuccessful.

In the beginning of the 19th century, there were a number of attempts to develop non-euclidean geometries from a synthetic perspective and to show that these were valid. There were continuing questions about whether or not these new geometries were fully valid. Hence, mathematicians became increasingly concerned with *validity* as opposed to *truth*, and with modeling one type of geometry in another.