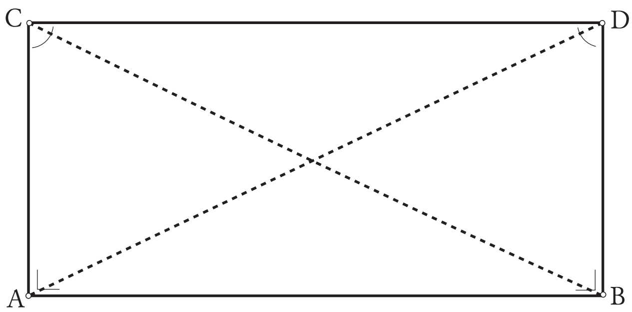
MI314 – History of Mathematics:
Episodes in Non-Euclidean Geometry, I

Giovanni Saccheri, *Euclides ab omni naevo vindicatus*

In 1733, Saccheri published *Euclides ab omni naevo vindicatus* (Euclid vindicated from all faults), in which he attempted to prove Euclid's 5^{th} postulate.

He began by introducing the geometric object now known as a *Saccheri quadrilateral*.



A Saccheri quadrilateral has two right angles at the base, $\angle A = \angle B = \mathbf{R} = 90^\circ$.

EV Prop. 1: Saccheri then showed that

$$\angle C = \angle D \iff AC = BD.$$

He then proceeds by setting out three possible *hypotheses*.

$$\begin{aligned}
\text{Hypothesis of the Right Angle (HRA): } \angle C = \angle D &= \mathbf{R} = 90^\circ, \\
\text{Hypothesis of the Obtuse Angle (HOA): } \angle C = \angle D &> \mathbf{R} = 90^\circ, \\
\text{Hypothesis of the Acute Angle (HAA): } \angle C = \angle D &< \mathbf{R} = 90^\circ.
\end{aligned}$$

EV Prop. 3: He then shows that

$$\begin{aligned} \text{HRA} &\Longrightarrow AB = CD, \\ \text{HOA} &\Longrightarrow AB > CD, \\ \text{HAA} &\Longrightarrow AB < CD, \end{aligned}$$

which can be used to show that

$$\begin{aligned} \text{HRA} &\Longrightarrow \text{sum of angles of a } \triangle = 2\mathbf{R} = 180^\circ, \\ \text{HOA} &\Longrightarrow \text{sum of angles of a } \triangle > 2\mathbf{R} = 180^\circ, \\ \text{HAA} &\Longrightarrow \text{sum of angles of a } \triangle < 2\mathbf{R} = 180^\circ. \end{aligned}$$

The rest of this part of the text is structured as an extended *reductio ad absurdum*.

Part 1 He uses HRA to prove Euclid's 5^{th} postulate.

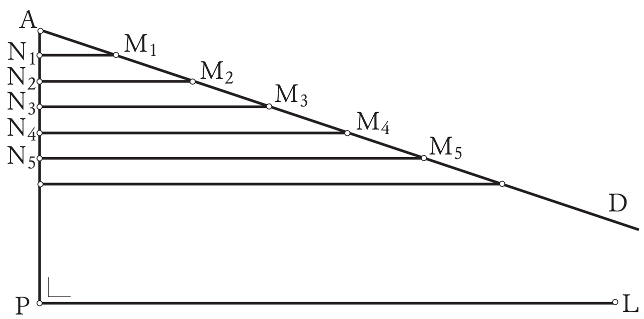
Part 2 He uses HOA to show that the geometry so created is inconsistent with *Elem.* I 16 & 17. [And, hence, Saccheri believes HOA is false.]

Part 3 He uses HAA to argue that it leads to consequences that are "repugnant to the nature of the straight line." [And, hence, Saccheri believes HAA is false.]

If his arguments were sound, this would lead to a proof of the 5^{th} postulate, however, there are problems with the conclusions of Parts 2 and 3.

We will examine some of the arguments from these two parts to see if we can uncover where Saccheri has gone wrong.

EV Props. 11 & 12



Props. 11 & 12 (HRA, HOA). *If a line falls on two given lines such that it is perpendicular to one and makes an acute angle with the other, then the two given lines will meet in the direction of the said acute angle.*

Proof. Let line AP fall on AD and PL, such that $AP \perp PL$ and $\angle PAD$ is an acute angle. To show that AD will intersect PL.

We cut of segments $AM_1 = M_1M_2 = M_2M_3$, and so on; and draw $M_1N_1 \parallel M_2N_2 \parallel M_3N_3$, and so on. Then

$$AN_1 \leqslant N_1N_2 \leqslant N_2N_3 \leqslant N_3N_4 \leqslant \ldots$$

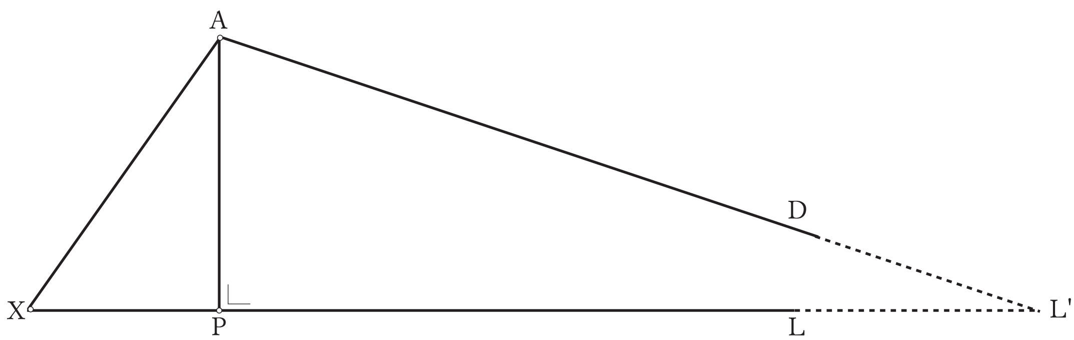
If this process is continued indefinitely, some N_n will fall beyond P. Then if we draw $N_nM_n \parallel PL$, M_n will fall beyond L and AM_n will meet PL.

□

Key. *We construct a sequence of segments from A towards P which increases at least arithmetically, and hence some segment will fall beyond P.*

Does this argument work on the sphere? Are there constructions in the argument that are not permissible on the sphere? Note that the theorem may still be true, even if the argument does not work.

EV Prop. 13 (HRA, HOA)



Prop. 13. *If a line falls on two given lines and makes the internal angles in the same direction less than two right angles, then the two given lines will meet. Moreover, this implies that in a triangle, two angles can be equal to two right angles.*

Proof. Let line AX fall on given lines AD and XL, such that $\angle AXL + \angle XAD < 2\mathbf{R}$. I say that AD will meet XL.

We drop $AP \perp XL$, so that, by *EV* Props. 11 & 12, AD will meet PL.

But in HOA, $\angle PAX + \angle AXP > \mathbf{R}$, since the angles of a $\triangle$ are greater than $2\mathbf{R}$.

Therefore, we can set

$$\angle DAP + \angle PAX + \angle AXP = 2\mathbf{R}.$$

But since AD meets XL at some point, say L', then we can say

$$\angle L'AP + \angle PAX + \angle AXP = 2\mathbf{R}.$$

But $\angle L'AP + \angle PAX = \angle XAL'$ (see the figure), so

$$\angle XAL' + \angle AXL' = 2\mathbf{R}.$$

So in $\triangle XAL'$, two angles are equal to $2\mathbf{R}$.

But in *Elements* I 17, it is shown that two angles of a triangle are less than two right angles. □

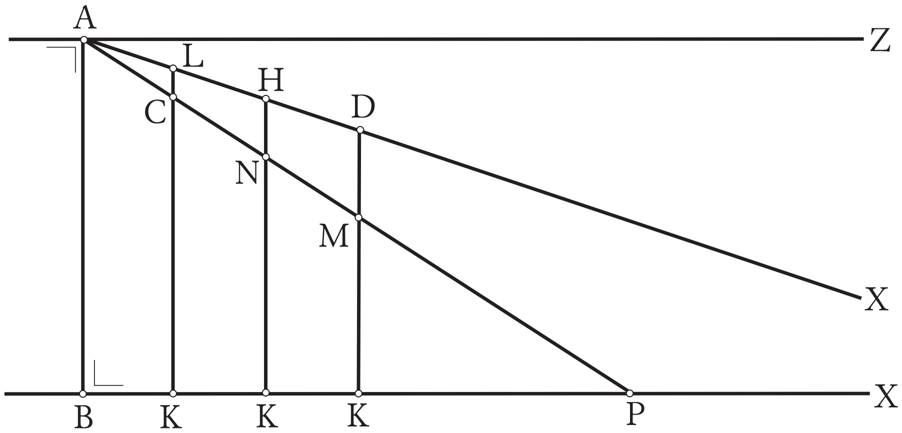
Key. *We use the properties of HOA to construct a triangle that has two angles equal to two right angles.*

EV Prop. 14

Prop. 14. *"The hypothesis of the obtuse angle is absolutely false, since it destroys itself."*

In fact, we have shown that HOA contradicts *Elements* I 17. Hence, we should look closely at *Elements* I 17 and its use of *Elements* I 16, to see what this contradiction really means.

EV Props. 32 & 33 (HAA)



Prop. 32. *We show that if HAA is true, then there will exist a line, AX, with the following properties:*

- *AX is a limit to the set of lines which meet BX and also to a set of lines which have two distinct perpendiculars, like AZ.*

- *It meets BX at* one point*, infinitely distant.*
- *It always approaches closer and closer to BX.*
- *It is a straight line.*

Prop. 33. *"The hypothesis of the acute angle is absolutely false, because it is repugnant to the nature of a straight line."*

What is the real basis of this objection? What do we know about the "nature of a straight line," and how do we know it?