Many Geometries: Intrinsic concepts of curvature, and mappings between different geometries

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Many geometries

Throughout the 19th century there were a number of different approaches to geometry. Gauss, Riemann and others developed intrinsic geometries of surfaces using the concepts of curvature and a metric (a measurement between any two points). Analytical and differential geometry continued to be developed by many mathematicians. There were various projects to extend and develop projective geometry. And there was increasing emphasis placed on the foundations of geometry – that is, the treatment of what assumptions to start with, and in what order to develop the theorems of geometry.

Moreover, it came to be recognized that there were as many possible geometries as we please, but that we can use the concept of curvature to delineate them: irregular and constant curvature, positive or negative curvature, or no curvature at all.

Topics geometry in the 19th and early 20th century

We will look at:

- Bernard Riemann's general conception of geometry,
- inversions on a circle (a topic in projective geometry),
- David Hilbert's axioms of geometry, and
- Henry Poincaré's disk model of the hyperbolic plane, showing that it is logically consistent with, or mathematically mappable to, a euclidean plane.

(Strictly speaking, we take the last two out of chronological order, but it is important to note that many of the key ideas of Hilbert's foundations of geometry were already circulating in various forms throughout the 19th century.)

Riemann's reorientation of geometry

In 1854, G.F. Bernhard Riemann (1826–1866) gave a lecture for his Habilitation called "On the hypotheses which lie at the foundations of geometry" – a topic chosen by Gauss, who sat in the audience. (The lecture was later published and translated.)

Reimann set the ideas of an infinitesimal metric, angle, and Gauss's concept of curvature at the center of his approach, allowing him to treat curvature intrinsically – that is, from within the surface whose curvature should be measured analytically. He argued that length is measured only at the limit, which allows him to define the distance between two points, the metric, through integration. He distinguished between unbounded and infinite spaces, and divided geometries into those with positive, zero, and negative curvature. He argued that there are infinitely many geometries.

Foundations of geometry in the 19th century

Throughout the 19th century, many geometers – especially in France, Germany, and Italy – worked extensively on projective geometry, and began to become convinced that it was the most fundamental geometry. In 1873, Luigi Cremona (1830–1903) published his *Elementi di Geometria Projettiva*, in which he showed that all proofs of projective geometry could be done using purely projective methods.

Geometers reread Euclid's *Elements* and found that it was full of logical gaps and illicit appeals to intuition. There were a number of efforts to state all of the assumptions explicitly and deduce a geometry from these – such as Moritz Pasch's *Vorlesungen über neuere Geometrie*, 1882, and Guiseppe Peano's *I principii di geometria logicamente esposti*, 1889.

Some axioms (Hilbert, *Grundlagen der Geometrie*, 1899)

- **Incidence.1:** For every two points A, B there exists a line a that contains each of the points A, B.
- **Incidence.2:** For every two points A, B there exists no more than one line that contains each of the points A, B.
- **Congruence.2**: If, of segments, $A'B' \cong A'''B'''$ and $A''B'' \cong A'''B'''$, then segments $A'B' \cong A'''B'''$. (*Transitivity*.)
- **Congruence.3**: With sets of points, A, B, C on one line and A', B', C' on another, if $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$. (*Additive property*.)
- Congruence.5: If for two triangles ABC and $A'B'C'$ the congruences $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ are valid, then the congruence $\angle ABC \cong \angle A'B'C'$ is also satisfied. (Used to show SAS congruence, *Elements* I.4.)

Pointwise circular inversion

- (D1) *Definition*: For a circle Γ with center O and radius r, if $OA \cdot OA' = r^2$ then point A is the *circular inverse* of point A'.
- (P1) Then, given either A, or A', we can use the tangent and the similarity of right triangles to find the other; so that

$$A \overset{\Gamma}{\mapsto} A' \text{ and } A' \overset{\Gamma}{\mapsto} A.$$

  Hence, every point inside Γ is mapped to a point outside, except O, which maps to an *undefined* point at infinity. The points of Γ map to themselves.
- (P2) Any circle γ that passes through both A and A' is perpendicular to Γ, and maps to itself ($\gamma \overset{\Gamma}{\mapsto} \gamma$). [Preliminaries: *Elem.* III.18, tangent $\perp$ radius; *Elem.* III.36, III.37, $OA{\cdot}OA' = OP^2 \iff OP$ is tangent.]

Cross-ratio, its circular inversion

- (D2) *Definition*: For any four points in a plane, A, B, P, Q, the cross-ratio is defined as follows:

  $$(AB, PQ) = AP/AQ \div BP/BQ.$$

  Example: If four points on a line, $ABPQ$, are projected through a point O onto four points of another line, $A'B'P'Q'$, then their cross-ratios are equal.

- (P3) For *any four points in a plane*, the circular inversion about Γ preserves the cross-ratio. Assume three points, A, P, and Q, and use basic geometry to set up certain proportions, (1) and (2), and then take the compound ratio that relates these, (3). Do the whole thing again with B, (4). Combine (3) and (4) to show the equality of the cross-ratios.

Jules Henri Poincaré (1854–1912)

- Born to an influential, bourgeois family.[1]
- Graduated from the École Polytechnique and the École des Mines.
- He eventually became a professor at the Sorbonne.
- He was an influential figure in French mathematical sciences and also wrote many popular works.

[1] His sister married the philosopher Emile Broutroux and his cousin Raymond Poincaré became president of France.

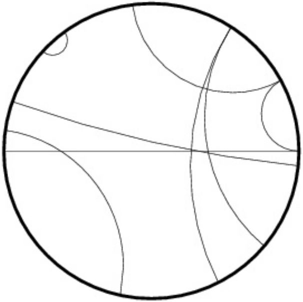
The Poincaré disk

The hyperbolic plane, $\mathcal{H}^2$, is *defined* as the set of points of the disk inside circle Γ, but not including the circumference itself.

The definition of a *P-point* is any point inside the circle.

The definition of a *P-line* is any circle or line γ that is *perpendicular* to Γ.

[The euclidean points, lines and circle are simply referred to as such.]



Consistency with the axioms incidence

- We can use the model to demonstrate all of Hilbert's axioms of euclidian geometry. For example, we show that two P-points always determine a unique P-line (Incidence.1 & Incidence.2).
- Suppose we have two P-points, A and B.
  - a) If line AB goes through center O, then ABO is the only line through A and B, and it is also a P-line.
  - b) If not, we find the polar inverse of A as A' and draw circle γ through A', A and B. Then γ is the only circle through A', A and B and it is orthogonal to Γ.
- Therefore, there is always a unique P-line through any two P-points.
- We can used the construction set out in (P2), above, to construct the P-line between two points.

Consistency with the axioms congruence

- Two P-segments, AB and $A'B'$, on P-lines PQ and $P'Q'$ are *defined* as congruent when their cross-ratios are the same, that is

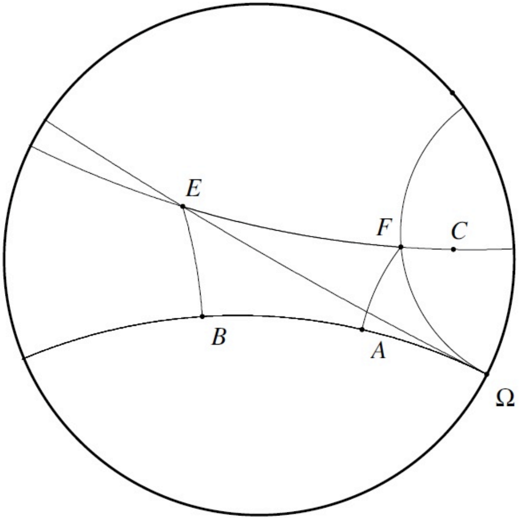
$$(AB, PQ) = (A'B', P'Q'),$$
$$AP/AQ \div BP/BQ = A'P'/A'Q' \div B'P'/B'Q'.$$

- Then, we can show that the *transitivity* of congruent P-segments, Congruence.2, follows by definition.
- We can show that when congruent P-segments are added to congruent P-segments the wholes are congruent P-segments, Congruence.3, by assuming that P-segments are added by multiplying their cross ratios.
- We can do similar proofs for all of Hilbert's axioms, except the parallel axiom.

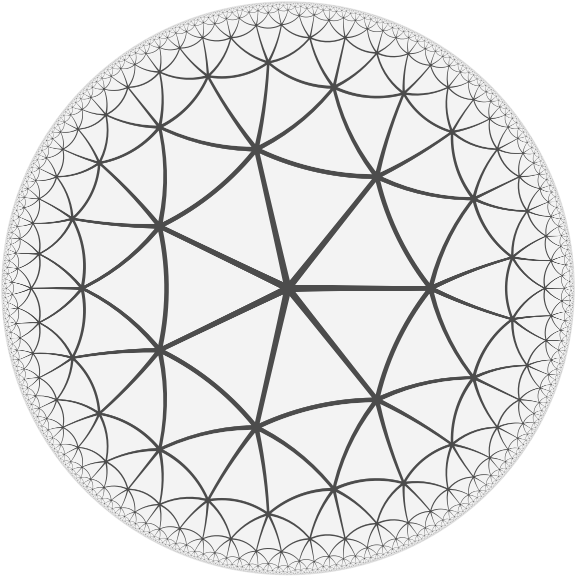
Parallels in the Poincaré disk

Hilbert's parallel axiom: "Let a be any line and A a point not on it. Then there is at most one line in the plane that contains a and A that passes through A and does not intersect a."

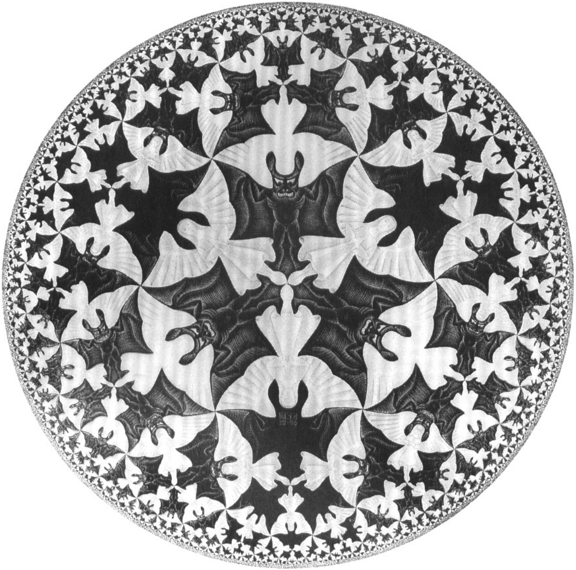
- Two P-lines that are *defined* as limiting parallel when they intersect circle Γ at the same point. (Hence, they are tangent circles.)
- There are many P-lines through a given P-point that do not intersect a given P-line. (Proof by construction.)
- There are two P-limiting parallels to any P-line PQ passing through any point A, in opposite directions. (Proof by construction.)
- The two angles of parallelism, $\Pi(p)$, associated with a given P-point and P-line are equal. (Proof by contradiction.)



A Saccheri quadrilateral in the Poincaré disk



A tessellation of *congruent* equilateral triangles on the hyperbolic plane



M.C. Escher's drawing of angles and devils on the hyperbolic plane

Overview

In the 19th century, there were a number of attempts to develop non-euclidean geometries and to show that these were valid. Mathematicians became increasingly concerned with the foundations of mathematics and stressed *validity* as opposed to *truth*. They began to work on modeling one type of geometry in another – that is, producing mappings of one geometry to another.

Around the turn of the 20th century, there was a renewed emphasis on foundational work on euclidean geometry. This lead, in turn, to new work on showing the logical consistency between various types of geometries. The mechanism for this was a map, or a model. Since *projective geometry* supplies such maps, it became clear that euclidean, hyperbolic and elliptic geometries can all be expressed in projective geometry.