MI314 – History of Mathematics: Episodes in Non-Euclidean Geometry, II

Circular Inversion

D.1: Definition, Circular inversion. *Point A' is the* circular inversion *of point, A, when they satisfy the relation*

$$AO \cdot A'O = r^2,$$

where O is the center of the base circle and r is its radius.

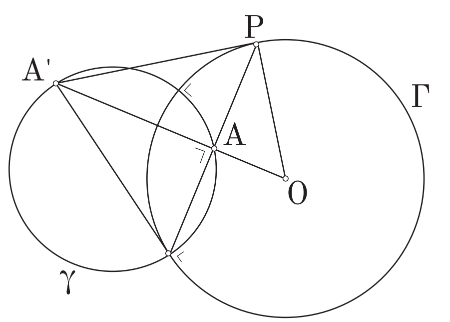

Figure 1: CL.1, circular inverse of a point

CI.1: Circular inverse of a point. *To find the circular inverse of a point.*

With base circle Γ about center O, if A is a point inside the circle, or A' is a point outside the circle, to find the circular inverse of A or A'.

1) We join OA and erect $PA \perp AO$. Join PO, then erect $PA' \perp PO$ and extend it to meet OA extended at A'.

Or, 2), we join OA' and draw a tangent from A' to Γ meeting at P. Then we drop $PA \perp OA'$ and join PO.

Proof. In either construction $\triangle OPA$ and $\triangle OPA'$ are both right and share $\angle O$. Hence they are similar, $\triangle OPA \sim \triangle OPA'$, so that

$$\frac{OA}{OP} = \frac{OP}{OA'}.$$

Which we can rewrite as

$$AO \cdot A'O = OP^2 = r^2$$

so that by definition D.1, A and A' are circular inverses of one another. □

CI.2: Circles through points in circular inversion. *Any circle passing through two points in circular inversion is perpendicular to the base circle.*

[Preliminaries: *Elem.* III.18: a tangent to a circle is perpendicular to the radius through the point of tangency. *Elem.* III.36, III.37: When from any point, O, outside a circle γ, if a tangent, OP and a secant line OAA' are drawn, then $OA \cdot OA' = OP^2$.]

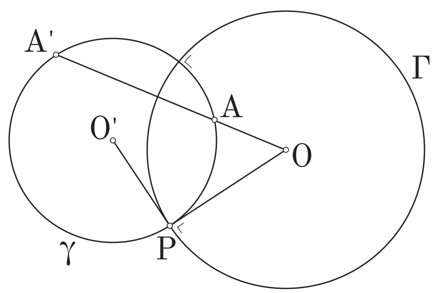

Figure 2: CL.2, circles through points in circular inversion

Proof. Let A and A' be two points in circular inversion about circle Γ. Then pass *any* circle γ through both A and A' meeting Γ at a point P. Then by definition D.1 $OA \cdot OA' = OP^2$. So, by *Elem.* III.37, OP is a tangent to γ. Therefore, if we join P to the center of γ, say O', then, by *Elem.* III.18, $O'P \perp OP$. Therefore, $\gamma \perp \Gamma$. □

D.2: Definition, Cross-ratio. *For* any *four points, A, B, P and Q, their* cross-ratio *is defined as*

$$\mathbf{CR}(AB, PQ) = \frac{AB}{AQ} \div \frac{BP}{BQ}.$$

Example. Cross-ratios are preserved in linear projection. That is, the cross-ratio of any four points, A, B, P, Q, of a line ℓ_1 will be projected from any point O onto the points A', B', P', Q' of any other line ℓ_2 such that

$$\frac{AB}{AQ} \div \frac{BP}{BQ} = \frac{A'B'}{A'Q'} \div \frac{B'P'}{B'Q'}.$$

CI.3: The cross-ratio of four points in circular inversion. *Let there be any four points A, B, P, and Q, then their circular inversions with respect to a base circle Γ have the same cross-ratio.*

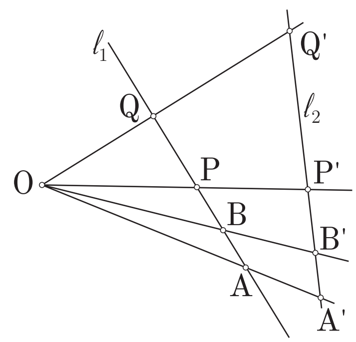

Figure 3: Example of cross-ratio in linear projection

Proof. Take any two of them, say A and P and find their circular inversions A' and P'. Hence, by definition D.1, $OA \cdot OA' = r^2 = OP \cdot OP'$, so that

$$\frac{OA}{OP} = \frac{OP'}{OA'},$$

and since the angle at O is common, $\triangle OAP \sim \triangle OP'A'$, so that

$$\frac{AP}{A'P'} = \frac{OA}{OP'}. \tag{1}$$

Take one of the other points, say Q, find its circular inverse, Q', and we can follow through the exact same steps with A and A' to show that

$$\frac{AQ}{A'Q'} = \frac{OA}{OQ'}. \tag{2}$$

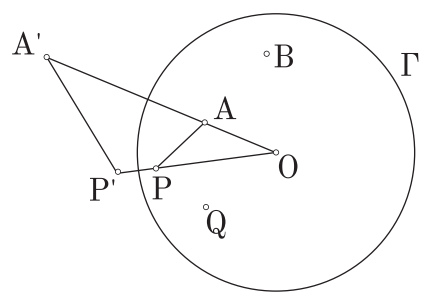

Figure 4: CI.4, cross-ratio of four points in circular inversion

Then combining (1) and (2), we have

$$\frac{AP}{A'P'} \div \frac{AQ}{A'Q'} = \frac{OA}{OP'} \div \frac{OA}{OQ'} = \frac{OA}{OP'} \cdot \frac{OQ'}{OA} = \frac{OQ'}{OP'}. \tag{3}$$

We can then do these three steps over again with a new point B to show that

$$\frac{BP}{B'P'} \div \frac{BQ}{B'Q'} = \frac{OQ'}{OP'}. \tag{4}$$

Combining (3) and (4) we have

$$\frac{AP}{A'P'} \div \frac{AQ}{A'Q'} = \frac{BP}{B'P'} \div \frac{BQ}{B'Q'},$$

or

$$\frac{AP}{A'P'} \cdot \frac{A'Q'}{AQ} = \frac{BP}{B'P'} \cdot \frac{B'Q'}{BQ}.$$

Multiplying this by $A'P'$ and BQ and dividing by $A'Q'$ and BP and rearranging, we have

$$\frac{AP}{AQ} \cdot \frac{BQ}{BP} = \frac{A'P'}{A'Q'} \cdot \frac{B'Q'}{B'P'},$$

or

$$\frac{AP}{AQ} \div \frac{BP}{BQ} = \frac{A'P'}{A'Q'} \div \frac{B'P'}{B'Q'}.$$

Hence, by definition D.4, the cross-ratio of A, B, P, and Q is equal to that of A', B', P', and Q'. □

Hilbert's Axioms

In his *Grundlagen der Geometrie* (1899), David Hilbert (1862–1943) sought to set out a complete set of the axioms of geometry. Among these we find the following:

- **Incidence.1 (I.1):** For every two points A, B there exists a line a that contains each of the points A, B.
- **Incidence.2 (I.2):** For every two points A, B there exists no more than one line that contains each of the points A, B.
- ...
- **Congruence.2 (C.2)**: If, of segments, $A'B' \cong A'''B'''$ and $A''B'' \cong A'''B'''$, then segments $A'B' \cong A'''B'''$. (*Transitivity*.)
- **Congruence.3 (C.3)**: With sets of points, A, B, C on one line and A', B', C' on another, if $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$. (*Additive property*.)
- ...
- **Axiom of Parallels (AP)**: Let a be any line and A a point not on it. Then there is at most one line in the plane that contains a and A that passes through A and does not intersect a.

In the following, we will show that it is possible to develop a model of the hyperbolic plane within the Euclidean plane, and to show that Hilbert's axioms of geometry, with the exception of the Axiom of Parallels, holds for this model – we will only show this for I.1, I.2, C.3 and C.4, above. Furthermore, we will see that while Hilbert's Axiom of Parallels does not hold, there are lines have the properties of Bolai's and Lobochevskii's parallels.

The Poincaré Disk

D.3: Definitions, Poincaré Disk, P-point, P-line. Poincaré's disk *is a model of the hyperbolic plane,* $\mathscr{H}^2$*, defined as the inside of a circle* Γ*, excluding the circumference itself. A point in the hyperbolic plane, a* P-point*, is any point inside the circle. A line in the hyperbolic plane, a* P-line*, is either a line through the center of* Γ *or a circle perpendicular to* Γ.

In the internal part of the disk each object can be seen in two ways, either as a normal Euclidean object – a point, line, circle, and so on – or as an object in the hyperbolic plain – a P-point, P-line, a P-circle, and so on.

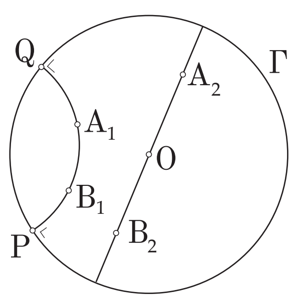

Figure 5: D.3, P-points, P-lines

PD.1: Draw a P-line through two P-points. *Given any two P-points A and B, draw the P-line joining them.*

Proof. Given two points, A and B, we find the circular inverse of one, say A', using CI.1, and then join them with a either a line, as A_1, B_1, and A'_1, or a circle, as A_2, B_2, and A'_2.

If A_1, B_1, and A'_1 are joined by a line, then this line will pass through 0, and if A_2, B_2, and A'_2 are joined by a circle, this circle will be perpendicular to Γ, by CI.1. Hence, by D.3, they are P-lines through A and B. □

PD.2: Draw a P-circle given two P-points. *Given any two P-points A and B, draw the P-circle around A and passing through B.*

Proof. Any P-circle about A will be perpendicular to every P-line passing through A. Hence, if we join the P-lines AB and AO, using A' by CI.1, then the P-circle about A will be perpendicular to each of them. Hence, the center, say C, of the P-circle must be

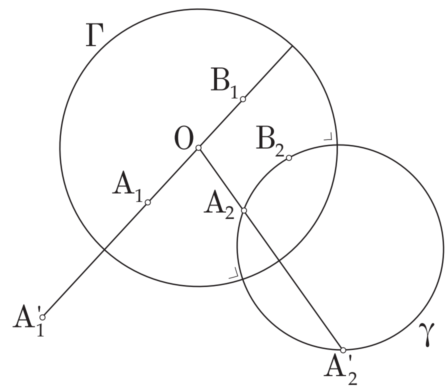

Figure 6: PD.1, A P-line through two P-points

on AO, both the line and the P-line. Then, we take the center of the P-line through AB, say C_1, by a standard Euclidean construction, say *Elem.* IV.5, and join C_1B. Next, take the perpendicular to this, such that it meets AO at C. Then, a circle drawn around C and passing through B will be perpendicular to circle AB. Hence, it will be a P-circle perpendicular to P-line AB.

Furthermore, this P-circle will be perpendicular to every other P-line through A. Take any point D on the P-circle BD and join the P-line AD using A' by CI.1. Then, taking the center of circle AD, say C_2, join C_2D, which will be tangent to circle BD. But since line $BC = CD$, $CD^2 = CA \cdot CA'1$, by *Elem.* III.36 and III.37,l so circles $AA'D$ and BD are perpendicular. Thus, P-line AD is perpendicular to P-circle BD. □

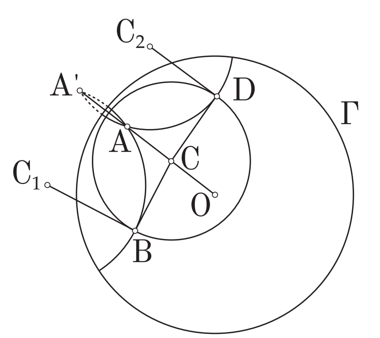

Figure 7: PD.2, A P-circle about one P-point and through another

With these two constructions we can then build up the early constructions corresponding to the early problems of *Elements* I, such as cutting off equal segments, bisecting lines and angles, producing perpendiculars two and from given points, constructing equal angles, and so on.

PD.3: Verification of axioms of Incidence I.1 and I.2. *Any two P-points are contained by one and only only P-line.*

Proof. Let there be two P-points, A and B, inside circle Γ. Then A and B (1) either lie on a diameter of Γ, or (2) not.

(1): If A_1 and B_1 lie on a diameter, then A_1B_1 goes though O and is the only line passing through A_1 and B_1, since one and only one line passes through two points. Hence, by the definition, there is a unique P-line passing through A_1 and B_1.

(2): If A_2 and B_2 do not lie on a diameter, we find the polar inverse of A_2 at A_2' and pass a circle γ, through the three points A_2, A_2' and B_2. Since only one circle passes through three points, circle γ is unique. But circle $\gamma \perp \Gamma$, by CI.2, so is a P-line passing through A_2 and B_2, by D.3. Hence, there is one and only one P-line passing through A_2 and B_2. □

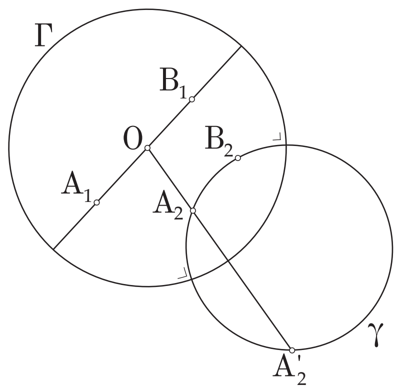

Figure 8: PD.3, Axioms of incidence

D.4: Definition, Congruent P-segments. *Two P-segments are congruent when their cross-ratios are equal. That is, for two P-segments, AB and $A'B'$,*

$$AB \cong A'B' \iff \frac{AB}{AQ} \div \frac{BP}{BQ} = \frac{A'B'}{A'Q'} \div \frac{B'P'}{B'Q'}.$$

PD.4: Transitivity of congruence for P-segments. *If two P-segments, AB and $A'B'$, are congruent to a third, $A''B''$, they are congruent to each other. That is*

$$AB \cong A'B' \text{ and } A'B' \cong A''B'' \implies AB \cong A''B''.$$

Proof. Since $AB \cong A'B'$ and $A'B' \cong A''B''$, by definition D.4,

$$\frac{AB}{AQ} \div \frac{BP}{BQ} = \frac{A'B'}{A'Q'} \div \frac{B'P'}{B'Q'} \text{ and } \frac{A'B'}{A'Q'} \div \frac{B'P}{B'Q} = \frac{A''B''}{A''Q''} \div \frac{B''P''}{B''Q''},$$

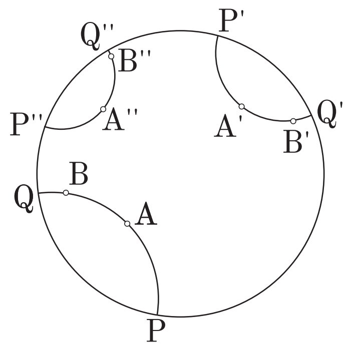

Figure 9: D.4, PD.4, Congruent segments, Transitivity of congruent P-segments

so,

$$\frac{AB}{AQ} \div \frac{BP}{BQ} = \frac{A''B''}{A''Q''} \div \frac{B''P''}{B''Q''}.$$

therefore, again by definition D.4,

$$AB \cong A''B''.$$

□

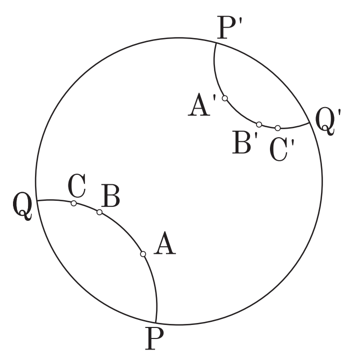

Figure 10: PD.5, Additivity of congruent P-segment

PD.5: Concatenating congruent P-segments. *When congruent P-segments are concatenated, or added, to congruent P-segments, the resulting P-segments are congruent. That is,*

$$AB \cong A'B' \textit{ and } BC \cong B'C' \implies AC \cong A'C'.$$

Proof. Since $AB \cong A'B'$ and $BC \cong B'C'$, by definition D.4,

$$\frac{AP}{AQ} \div \frac{BP}{BQ} = \frac{A'P'}{A'Q'} \div \frac{B'P'}{B'Q'} \text{ and } \frac{BP}{BQ} \div \frac{CP}{CQ} = \frac{B'P'}{B'Q'} \div \frac{C'P'}{C'Q'}.$$

To concatenate, or add, P-segments, $AB + BC$ and $A'B' + B'C'$, we need to multiple the cross-ratios, so

$$\left(\frac{AP}{AQ} \div \frac{BP}{BQ}\right)\left(\frac{BP}{BQ} \div \frac{PC}{CQ}\right) = \left(\frac{A'P'}{A'Q'} \div \frac{B'P'}{B'Q'}\right)\left(\frac{B'P'}{B'Q'} \div \frac{P'C'}{C'Q'}\right),$$

so, cancelling out the middle terms,

$$\frac{AP}{AQ} \div \frac{PC}{CQ} = \frac{A'P'}{A'Q'} \div \frac{P'C'}{C'Q'}.$$

Therefore, by definition D.4, $AC \cong A'C'$. □

Using methods such as these, one can show all of Hilbert's axioms in the Poincaré disk, except the Axiom of Parallels.

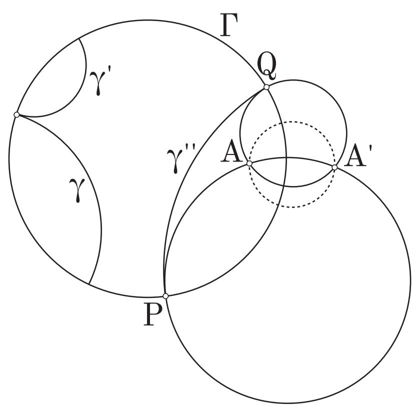

Figure 11: D.5, Definition of P-parallel; PD.6, Hilbert's axiom fails; PD.7, Two P-parallels, one in each direction

D.5: Definition, Parallel P-lines. *Two P-lines γ and γ' are P-parallel when they meet on Γ, which meeting point is not a P-point in the hyperbolic plane $\mathscr{H}^2$.*

We can then show that parallel P-lines satisfying this definition agree with Lobochevskii's definition of parallel lines, and that Hilbert's Axiom of Parallels fails on the Poincaré Disk.

PD.6: Hilbert's Axiom of Parallels fails. *Given a P-line γ'' and a P-point A not on it, any number of P-lines can be drawn through A such that they do no intersect γ''.*

Proof. Let there be a P-line γ'' intersecting Γ at P and Q, which are not P-points, and a P-point A not on γ''. Take A' as the polar inverse of A. Then, the smallest circle between A and A', shown as a dotted circle, will have its center on the midpoint of the line drawn from A to A'. Then, taking any point on Γ between the dotted circle and either P or Q, we can draw a circle through these three points, which will be a P-line. All such circles will be P-lines through A that do not meet γ''. □

PD.7: Two P-parallels, one in each direction. *Given a P-line γ'' and a P-point A not on it, there are two P-lines passing through A and P-parallel to γ'' in opposite directions.*

Proof. Let there be a P-line γ'' intersecting Γ at P and Q, which are not P-points, and a P-point A not on γ''. Take A' as the polar inverse of A. Then draw two circles one through each of A, A', P, and A, A', Q. Then these are two P-lines, by definition D.3, and since they meet γ'' at Γ, one is parallel in the direction of P and the other is parallel in the direction of Q, by definition D.5. □

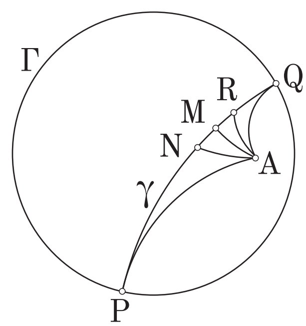

Figure 12: PD.8, angles of P-parallel are equal

PD.8: Both angles of parallelism are equal. *Given a P-line γ and a P-point A not on it, the two angles of parallelism, $\Pi(p)$, in opposite directions, are equal.*

Proof. Let there be a P-line γ intersecting Γ at P and Q, which are not P-points, and a P-point A not on γ. Using constructions involving P-lines and P-circles, PD.1 and PD.2, above, drop the P-segment $AM \perp \gamma$. Then, we need to show that $\angle MAP \cong \angle MAQ$. For the sake of contradiction, assume that it is not, say $\angle MAP > \angle MAQ$. Then, again using constructions involving P-lines and P-circles, PD.1 and PD.2, construct $\angle NAP \cong \angle MAQ$, intersecting γ at N. Then, let P-segment $MR \cong MN$ be cut off. Then, since $MR \cong MN$ and $\angle AMN \cong \angle AMR$, being right, and P-segment AM is common, by side-angle-side congruence, of P-triangles $\triangle MAN \cong \triangle MAR$. Therefore, $\angle NAM \cong \angle MAR$, but $\angle MAR \cong \angle MAQ$, so $\angle MAR \cong \angle MAQ$, the whole and the part, which is impossible. Therefore, $\angle MAP \cong \angle MAQ$. Hence, these are $\Pi(AM)$ in each direction. □