Cantor, the foundations of analysis, set theory, non-denumberable and transfinite numbers

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History of Mathematics

The 19^{th} century in mathematics

The 19^{th} century saw the rise of professionalization and increased specialization in all areas of the sciences, and certainly in mathematics as well.

During the 19^{th} century, the university system began to revive and many of the great mathematicians of the century worked as university professors.

Mathematics began to separate itself from the physical sciences and became much more abstract. Many new fields of mathematics were developed such as *non-Euclidean geometry*, *abstract algebra*, *vector theory*, *set theory* and *mathematical logic*. Mathematicians became increasingly occupied with the rigorous foundations of the theories and methods they used.

Set Theory

Set theory is now a theoretical foundation for many branches of mathematics. Although set theory was developed by Cantor and his contemporaries, they developed what is now called "naive" set theory, relying only on our intuitive concepts of sets and their relations.

In the early 20th century, among growing concerns about various foundational issues, it was noticed that there are a number of paradoxes that can be produced by infinite sets, sets of sets, and so on.

Throughout the 20th century, mathematicians developed various axiomatic systems to make set theory more rigorous and to free it from these paradoxes. These are the non-naive theories that are used now.

Basic Ideas of Set Theory, 1

- A set is an undefined, primitive concept, but we can understand it as *some collection of things*.
- If the thing x belongs to the set A, we write $x \in A$ and say x is a member of A, or x an element of A. We write the members of a set in curly braces, as $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- We can define sets with rules. For example, the prime numbers are

$$\{x \in \mathbb{N} | x > 1, \text{ and } y \in \mathbb{N}, y > 1, y \neq x, y \nmid x\}.$$

- $A = B$ if and only if every element of A belongs to B and every element of B belongs to A – *the members are the same*.
- A is a proper subset of B, written $A \subset B$, when $x \in A \implies x \in B$ but $A \neq B$.

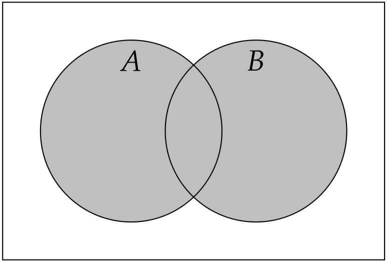
Basic Ideas of Set Theory, 2

- The set that has no members is called the empty set and is written $\{\ \}$, or $\emptyset$.
- The power set of a given set A, written $\mathcal{P}(A)$, is the set formed from all the subsets of A, including A itself and the empty set. For example, if $A = \{a, b, c\}$, then

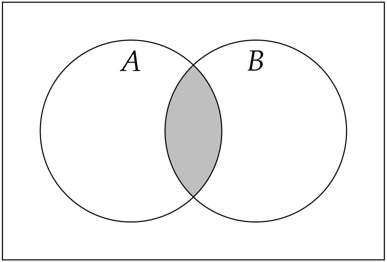
$$\mathcal{P}(A) = \{\{\ \}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

Basic Operations of Set Theory

- $\cup$ The union of set A and set B, $A \cup B$, is the set of all the members of set A, all the members of B, and any members that are in both sets. (Logical "or.")
- $\cap$ The intersection of set A and set B, $A \cap B$, is the set of all elements that are members of *both* A and B. (Logical "and.")



$A \cup B$



$A \cap B$

The foundations of analysis

Since the calculus had been developed, mathematicians often used the ideas of *limits*, *continuity*, and *real numbers*, but they became increasingly concerned that they had no clear arithmetical definitions of these fundamental concepts.

Previously, there had been the idea that the equations of analysis described curves and surfaces, which are continuous and have limits, so that we could use some sort of *intuition* of such geometric objects to understand these ideas. To many 19^{th}-century mathematicians, this was not sufficient.

Mathematicians such as Augustin-Louis Cauchy, B. Riemann, and Karl Weierstrass came to the view that equations are arithmetical, and some kind of arithmetical definition of these ideas is essential. This led to the realization that we needed a new way of thinking about real numbers.

Weierstrass's definition of *limit*

Where $f(x)$ is a function on an open interval, (x_a, x_b), and L is a real number, then L is a limit of $f(x)$,

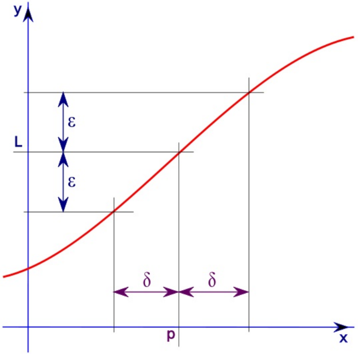
$$\lim_{x \to p} f(x) = L$$

means that, *for any real number* $\epsilon > 0$, *there exists a real number* $\delta > 0$, *such that*

$$0 < |x - p| < \delta \implies |f(x) - L| < \epsilon.$$

A function, $f(x)$, is continuous at p when $f(p)$ is defined and

$$\lim_{x \to p} f(x) = f(p).$$



The Dedekind cut, a new definition

Richard Dedekind (1831–1916) wanted to define the *real numbers* by constructing them out of sets of *rational numbers*.

Dedekind: “If all points of a straight line fall into two classes in such a way that each point of the first class lies to the left of each point of the second class, then there exists one and only one point which brings about this separation, this cutting of the line into two parts.”

He defined a *cut* as an ordered pair of sets of *rational numbers*, (A, B), with the following properties:

- A and B are all the rational numbers ($A \cup B = \mathbb{Q}$.).
- A and B have no common elements ($A \cap B = \{ \}$).
- $a \in A$ and $b \in B \implies a < b$. So, they are *ordered*.
- A has no greatest element. (Or, B has no least element.)

The Dedekind cut, *real numbers*

He then said that each cut *corresponds to*, or defines a real number.

"In this property, that not all cuts are produced by rational numbers consists the incompleteness or discontinuity of the domain $\mathbb{Q}$ of all rational numbers. Whenever, then, we have to do with a cut (A_1, B_1), produced by no rational number, we create a new, irrational number a, which we regard as defined by this cut. We say that a corresponds to this cut, or that it produces this cut."

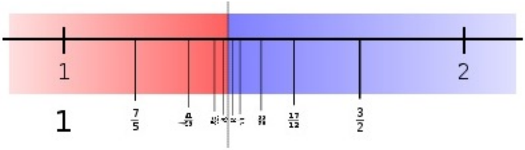
The Dedekind cut, examples

For example, the cut, (A_1, B_1), which gives the real number 2 is determined by:

$$A_1 = \{a \in \mathbb{Q} \mid a < 2\}$$
$$B_1 = \{b \in \mathbb{Q} \mid b \geqslant 2\}$$

And the real number $\sqrt{2}$ is given by (A_2, B_2), where:

$$A_2 = \{a \in \mathbb{Q} \mid a^2 < 2 \text{ or } a \leqslant 0\}$$
$$B_2 = \{b \in \mathbb{Q} \mid b^2 \geqslant 2 \text{ and } b > 0\}$$



Georg Ferdinand Lugwig Phillip Cantor (1854–1918)

- Born in St. Petersburg, Russia, and raised in Frankfurt, Germany.
- Took a Ph.D. in mathematics from the University of Berlin.
- He worked his whole career at the University of Halle.
- He founded *set theory* and the study of denumerability and transfinite sets.
- He suffered from bouts of depression and was hospitalized twice.
- He believed his mathematical results had deep philosophical and spiritual implications.

One-to-one Correspondence

In order to compare different sets of numbers, Cantor introduced the idea of a one-to-one correspondence, a *bijection*. This is some function, or mapping, that can be used to set every member of the range (codomain) in relation to *one and only one* member of the domain. For example, we could make a *mapping* between the first 26 numbers and the English alphabet.

$$
\begin{array}{cccccccccccc}
X & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots, & 26 & \} \\
 & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\
Y & = & \{ & a, & b, & c, & d, & e, & f, & g, & h, & \ldots, & z & \}
\end{array}
$$

Cantor: “Two sets *M* and *N* are *equivalent* … if it is possible to put them, by some law, in such a relation to one another that to every element of each of them corresponds to one and only one element of the other.”

Comparing the natural numbers with other sets

He then used this concept to compare, for example, the natural numbers with the even numbers:

$$\begin{array}{ccccccccccc} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & \ldots, & n, & \ldots & \} \\ & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\ \mathbb{E} & = & \{ & 2, & 4, & 6, & 8, & 10, & \ldots, & 2n, & \ldots & \} \end{array}$$

Or the natural numbers with the integers:

$$\begin{array}{ccccccccccc} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & \ldots, & n, & \ldots & \} \\ & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\ \mathbb{Z} & = & \{ & 0, & 1, & -1, & 2, & -2, & \ldots, & \frac{1+(-1)^n(2n-1)}{4}, & \ldots & \} \end{array}$$

In this way, he argued that there were the *same kind of infinite* number of members in all of these sets. We express this by saying they have the same cardinality, $\bar{\mathbb{N}} = \bar{\mathbb{E}} = \bar{\mathbb{Z}} = \aleph_0$.

Comparing natural numbers with rational numbers

But what about the rational numbers? There are an infinite number of rational numbers between any two natural numbers – that is, they are *dense* – so it seems like the cardinality of $\mathbb{Q}$ must be greater. But consider the following table:

1	-1	2	-2	3	...
1/2	-1/2	[2/2]	[-2/2]	3/2	...
1/3	-1/3	2/3	-2/3	[3/3]	...
1/4	-1/4	[2/4]	[-2/4]	3/4	...
1/5	-1/5	2/5	-2/5	3/5	...

$\mathbb{N}$	=	{	1,	2,	3,	4,	5,	6,	7,	…	}
			↕	↕	↕	↕	↕	↕	↕		
$\mathbb{Q}$	=	{	0,	1,	1/2,	-1,	2,	-1/2,	1/3,	…	}

In this way, we can make a one-to-one mapping from each member of $\mathbb{N}$ to each member of $\mathbb{Q}$. Therefore, $\bar{\bar{\mathbb{N}}} = \bar{\bar{\mathbb{Q}}} = \aleph_0$.

The non-denumberbility of the continuum, 1

We define the continuum, c, as a set of real numbers $x \in \mathbb{R}$, in the open interval (a, b) such that $a < x < b$.

Theorem: The real numbers in the interval $(0, 1)$ are not denumerable.

An indirect proof. We begin my assuming that this interval *can* be set into one-to-one correspondence with $\mathbb{N}$.

Since the numbers can be expressed as non-terminating decimal numbers, we write out a list of all the real numbers that exist in our interval...

The non-denumberbility of the continuum, 2

$$\begin{array}{ccc} 1 & \leftrightarrow & 0.d_{1,1}d_{1,2}d_{1,3}d_{1,4}d_{1,5}d_{1,6}d_{1,7}\ \dots \\ 2 & \leftrightarrow & 0.d_{2,1}d_{2,2}d_{2,3}d_{2,4}d_{2,5}d_{2,6}d_{2,7}\ \dots \\ \dots & & \dots \\ \mathrm{n} & \leftrightarrow & 0.d_{n,1}d_{n,2}d_{n,3}d_{n,4}d_{n,5}d_{n,6}d_{n,7}\ \dots \\ \dots & & \dots \end{array}$$

Then we consider the number

$$r = 0.[d_{1,1}+1][d_{2,2}+1][d_{3,3}+1][d_{4,4}+1][d_{5,5}+1]\dots,$$

But, if $d_{n,m}+1 = 0$ or 9, then we use $[d_{n,m}+2]$ or $[d_{n,m}-1]$. Clearly, r must be in the interval $(1,0)$, but it is different from the first number at the first place, the second at the second, and so on. Therefore, it is not found in our list; hence, it is a counterexample to the assumption. Therefore, $\bar{c} \neq \bar{\bar{\mathbb{N}}} = \aleph_0$.

The non-denumerability of the real numbers

By providing a one-to-one mapping from the interval $(0, 1)$ to the real numbers we can show that the reals, $\mathbb{R}$, have the same cardinality as the continuum, c.

In fact, we can use the following function to map the numbers $x \in (0, 1)$ to all the real numbers:

$$y = f(x) = \frac{2x - 1}{x - x^2}$$

Therefore, $\overline{c} = \bar{\mathbb{R}}$.

The union of two denumerable sets is denumerable

Theorem: If A and B are each denumerable (countable) sets and C is the set of all elements belonging to one or the other, or both, $C = A \cup B$, then C is itself denumerable (countable).

A proof by construction. The assumed denumerability of each A and B means that they can be set into one-to-one correspondence with $\mathbb{N}$. By alternately taking one element from each we can make a one-to-one mapping between $\mathbb{N}$ and C.

$$\begin{array}{cccccccccccc} \mathbb{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots & \} \\ & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\ C & = & \{ & a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & a_4, & \ldots & \} \end{array}$$

Therefore, $\bar{\mathbb{N}} = \bar{C} = \aleph_0$.

The non-denumerability of the irrationals

We can then use this to show that the irrational numbers are not denumerable – that is, not countable.

Theorem: The irrational numbers are non-denumerable.

Proof by indirect argument. Assume that the irrationals are denumerable. Then the union of the irrationals and the rationals are denumerable. But this union is the set of the real numbers, $\mathbb{R}$, and we have shown that the reals are not denumerable. Therefore, the irrationals are not denumerable.

Comparing Cardinality

If A and B are sets, the cardinality of A is less than or equal to the cardinality of B, $\overline{A} \leqq \overline{B}$, when we can make a one-to-one mapping between *all* of the elements of A and *some subset* of the elements of B, an *injection*. (For example, we make such a correspondence between $x \in \mathbb{N}$ and $1/x \in \mathbb{Q}$.)

If A and B are sets, the cardinality of A is strictly less than the cardinality of B, $\overline{A} < \overline{B}$, when $\overline{A} \leqq \overline{B}$, *but* there is no one-to-one mapping between all the members of B and a subset of the members of A, namely $\overline{B} \nleqq \overline{A}$. (For example, we can map all $x \in \mathbb{N}$ to, say, $1/x\pi \in \mathbb{R}$, but there is *no mapping* such that $y \in \mathbb{R} \to \mathbb{N}$, because of the nondenumerability of $\mathbb{R}$.)

The Equality of Cardinality

Shröder-Bernstein theorem: If A and B are sets, the cardinality of A is equal to the cardinality of B, $\bar{A} = \bar{B}$, when $\bar{A} \leqq \bar{B}$ *and* $\bar{B} \leqq \bar{A}$. (The proof is fairly difficult for infinite sets.)

This gives a manageable way to *check* whether or not two sets have equal cardinality that relies on the basic concept of one-to-one correspondence.

Irrationals and Reals, Planes and Lines

Irrationals and Reals: We can show that the irrational numbers, $\mathbb{I}$, have the same cardinality as the real numbers, $\mathbb{R}$. Since $\mathbb{I} \subset \mathbb{R}$, $\overline{\overline{\mathbb{I}}} \leqq \overline{\overline{\mathbb{R}}}$. Then we produce a mapping of all the real numbers onto specially constructed irrational numbers ($a_0.a_1 0 a_2 11 a_3 000 a_4 1111 a_5 00000...$), so that $\overline{\overline{\mathbb{R}}} \leqq \overline{\overline{\mathbb{I}}}$.

Plane and line: We can show that a plane has the same cardinality as a line. We consider a square of ordered points (x, y), where $0 < x < 1$ and $0 < y < 1$, and a line of points z in an open interval $(0, 1)$. We map the line to, say, $(z, 1/2)$. Then we map each point of the square, (x, y), where $x = 0.a_1a_2a_3...$ and $y = 0.b_1b_2b_3...$ to some point $z = 0.a_1b_1a_2b_2a_3b_3...$ of the line.

The Cardinality of Power Sets, 1

Theorem: If A is *any* set, then $\bar{A} < \overline{\mathcal{P}(A)}$. (The theorem is fairly obvious for finite sets, so we will will only look at the proof for infinite sets.)

Part 1: We first prove that $\bar{A} \leqq \overline{\mathcal{P}(A)}$. For example, let $A = \{a, b, c, d, \ldots\}$, then we could map

$$
\begin{array}{ccccccccccc}
A & = & \{ & \text{a}, & \text{b}, & \text{c}, & \text{d}, & \text{e}, & \text{f}, & \text{g}, & \ldots & \} \\
 & & & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \\
B & = & \{ & \{a\}, & \{b\}, & \{c\}, & \{d\}, & \{e\}, & \{f\}, & \{g\}, & \ldots & \}
\end{array}
$$

Since B is clearly a subset of $\mathcal{P}(A)$, then $\bar{A} \leqq \overline{\mathcal{P}(A)}$.

The Cardinality of Power Sets, 2

Part 2: We prove that there is no one-to-one mapping between all of A and all of $\mathcal{P}(A)$. Proof by contradiction. We assume there is such a mapping, say (this is an arbitrary example)

A		$\mathcal{P}(A)$
a	$\leftrightarrow$	$\{b, c\}$
b	$\leftrightarrow$	$\{d\}$
c	$\leftrightarrow$	$\{a, b, c, d\}$
d	$\leftrightarrow$	$\{\ \}$
...		...

such that *all* of the elements of A are mapped to *all* the elements of $\mathcal{P}(A)$. Then we consider set $B \in \mathcal{P}(A)$, such that:

> *B is the set of every element in A that is not a member of the subset of $\mathcal{P}(A)$ to which it is mapped.*

The Cardinality of Power Sets, 3

Therefore, $B \subset A$ and it must be that $B \in \mathcal{P}(A)$. Therefore, B is in the list, hence there is some $y \in A$ that is matched to B. Is $y \in B$ or $y \notin B$?

Case 1: Suppose $y \notin B$. Therefore by the definition of B, $y \in B$, which is a contradiction.

Case 2: Suppose $y \in B$. But B contains only elements which are not matched to it, therefore it must be that $y \notin B$, which is a contradiction.

Therefore, $\bar{A} \leqq \overline{\mathcal{P}(A)}$ but there is no one-to-one mapping between A and $\mathcal{P}(A)$, so $\bar{A} < \overline{\mathcal{P}(A)}$.

The Transfinite Realm

In this way, Cantor claimed that given any set, another set of greater cardinality could always be created by taking the power set.

So that we can create a series of infinite sets with increasingly greater cardinality,

$$\bar{\mathbb{N}} < \bar{\mathbb{R}} < \overline{\mathcal{P}(\mathbb{R})} < \mathcal{P}\overline{(\mathcal{P}(\mathbb{R})})...$$

or,

$$\aleph_0 < \aleph_1 < \aleph_2 < ...$$

Overview

- Cantor developed naive set theory, and tools to compare infinite sets.
- Cantor showed that certain classes of numbers, such as irrational and real numbers, are not countable.
- Cantor developed a style of indirect argument – known as a *diagonal argument* – that can be used to show the limits of certain sets, types, or structures in mathematics. This style of proof was later used by Kurt Gödel (1906–1978), Alan Turing (1912–1954), and others.