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Viète, Descartes, and the Emergence of Modern Mathematics

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François Viète (1540-1603) is often regarded as the first modern mathematician on the grounds that he was the first to develop the literal notation, that is, the use of two sorts of letters, one for the unknown and the other for the known parameters of a problem. The fact that he achieved neither a modern conception of quantity nor a modern understanding of curves, both of which are explicit in Descartes' *Geometry* (1637), is to be explained on this view "by an incomplete symbolization rather than by any obstacle intrinsic in the system."[1] Descartes' *Geometry* provides only a "clearer expression" of themes already sounded in Viète's work,[2] one that perfects Viète's literal calculus and gives it "its modern form";[3] it merely continues the "'new' and 'pure' algebra *which Viète first established as the 'general analytic art'*."[4] It can seem, furthermore, that this must be right, that had there been some obstacle intrinsic to Viète's system that barred the way to a modern conception of quantity and a modern understanding of curves, then Descartes' *Geometry* would have had to have taken a very different form than it did. As it was, Descartes had only to improve Viète's symbolism, free himself of the last vestiges of the ancient view of geometrical and arithmetical objects, and apply the new symbolism to the study of curves in order to achieve what Viète did not *but could have*. But what, really, is the status of this "could have"; what would it actually have taken for Viète to achieve Descartes' results in the *Geometry*? The interest of the question lies in its potential to better our understanding of the nature of (early) modern mathematics.

1. Viète's Analytic Art

Viète's Analytic Art comprises three stages. At the first stage, *zetetics*, a problem, whether of arithmetic or geometry, is translated into Viète's newly created symbol system or *logistice speciosa* in the form of an equation. At the second stage, *poristics*, equations are transformed according to rules into canonical forms; and finally at the third, *exegetics*, a solution to the problem is found on the basis of the derived equation. As Viète himself emphasizes, at this third stage the analyst turns either geometer, "by executing a true construction," or arithmetician, "solving numerically whatever powers, whether pure or affected, are exhibited."[5] Viète teaches the Art in eight essays first published individually between 1591 and 1631, then brought together in a single volume, the *Opera Mathematica*, in 1646.[6]

Both in the opening paragraph of the Introduction to *The Analytic Art* and in the Dedication that precedes it, Viète emphasizes the close connections between his art and the work of the Greeks. Three themes from the then newly rediscovered Greek mathematical tradition are especially relevant: Pappus' conception of the analytic method in geometry as outlined in the seventh book of his *Mathematical Collection*,[7] Diophantus' treatment of arithmetical problems using letters for the unknown and for powers of the unknown in his *Arithmetica*,[8] and Eudoxus' general theory of propositions as set out in the fifth book of Euclid's *Elements*.[9] In the spirit of Eudoxus' general theory, Viète aimed to provide a general method for the solution both of the sorts of arithmetical problems Diophantus had considered and of the sorts of geometrical problems Pappus discusses. The method itself was that of analysis, a method Pappus describes as making "the passage from the thing sought, as if it were admitted, through the things which follow in order [from it], to something admitted as the result of synthesis."[10] Diophantus' treatment of arithmetical problems provides an instructive illustration.

Diophantus' *Arithmetica* is a collection of arithmetical problems involving determinate and indeterminate equations together with their (reasoned) solutions. The text is remarkable along a number of dimensions. First, unlike earlier Babylonian and Egyptian texts dealing with similar sorts of problems, Diophantus' *Arithmetica* refers not to numbers of cattle, or sheep, or bushels of grain, but to numbers of pure monads (or unknown numbers of monads, or powers of unknown numbers of monads); and it aims to provide not merely rules for the solution of problems but a demonstration, of sorts, to show why the rule is a good one. The *Arithmetica*, in other words, is a scientific or theoretical work at least as much as it is a practical manual in the art of solving problems. It is also remarkable in employing abbreviations—for the unknown and for powers of the unknown (up to the sixth), for the monad, and so on—all of which are explicitly introduced at the beginning of the work, and in providing explicit rules for the transformation of equations (by adding equal terms to both sides and reducing like terms).[11] Diophantus uses the letter 'ς' (from ἀριθμός, number) to signify the unknown, 'Δ^{υ}' (from δύναμις, power or square) for the square of the unknown, and 'K^{υ}' (from κύβος) for the cube of the unknown. The fourth power is a square-square, the fifth a square-cube, and the sixth a cube-cube. The monad is abbreviated '$\overset{\circ}{M}$'. Negative numbers are conceived in terms of missing or lacking[12] and are signaled by a special sign (an inverted 'ψ'). Diophantus also uses Greek alphabetic numerals, and he indicates addition by concatenation.

A simple problem illustrating his analytic method is to divide a given number into two numbers with a given difference. Diophantus turns immediately to a particular instance. The given number is assumed to be, say, one hundred, and the difference forty, units:

> Let the less be taken as ςα [one unknown]. Then the greater will be ςα$\overset{\circ}{M}$μ [one unknown and forty units]. Then both together become ςβ$\overset{\circ}{M}$μ [two unknowns and forty units]. But they have been given as $\overset{\circ}{M}$ρ [one hundred units]. $\overset{\circ}{M}$ρ [one hundred units], then, are equal to ςβ$\overset{\circ}{M}$μ [two unknowns and forty units]. And, taking like things from like: I take $\overset{\circ}{M}$μ [forty units] from the ρ [one hundred] and likewise μ [forty] from the β [two] numbers and μ [forty] units. The ςβ [two unknowns] are left equal to $\overset{\circ}{M}$ξ [sixty units]. Then, each ς [unknown] becomes $\overset{\circ}{M}$λ [thirty units]. As to the actual numbers required: the less will be $\overset{\circ}{M}$λ [thirty units] and the greater $\overset{\circ}{M}$ο [seventy units], and the proof is clear.[13]

Rather than merely telling us what to do to find the desired answer as his predecessors had done (say: take forty from one hundred to give sixty, then divide by two to give thirty; the two numbers, then, are thirty and thirty plus forty, or seventy), the results of which can then be checked against the original parameters of the problem, Diophantus works the problem out arithmetically. Because he has a sign for the unknown, Diophantus can treat it as if it were known and proceed, through a series of familiar operations, to the answer that is sought. This analytical treatment of arithmetical problems through the introduction of signs for the unknown and its powers, together with Eudoxus' treatment of proportions—where a proportion, according to Viète, is "that from which an equation is composed"[14]—provides Viète with the crucial clues to his Analytic Art.

Although Diophantus' method for solving a problem is clearly meant to be a general one for problems of the relevant type, his solutions are always of particular numerical problems. He has a general method but no means of expressing it in its full generality. Viète resolves the difficulty by appeal to the distinction between vowels and consonants: unknown magnitudes are to be designated by uppercase vowels and given terms by uppercase consonants, all of which are to be operated on as Diophantus operates on his signs for the unknown and its powers.[15]

Viète also greatly simplifies matters by designating the unknown and its powers not by using a variety of different signs (as Diophantus, and the cossists, had) but by using one sign, a vowel, for the unknown, followed by a word (‘*quadratum*’, ‘*cubum*’, and so on) to indicate the power of the unknown. More significantly, he also generalizes Diophantus’ method to apply not only to the sorts of arithmetical problems Diophantus considers but also to well-known geometrical problems. It is this dimension of generalization, we will see, that provides the key to an adequate understanding of Viète’s *logistice speciosa*, his symbolic language or algebra.

Perhaps the first to recognize the fundamental connection between Euclidean geometry and the new algebra, or art of the coss, was Petrus Ramus (1515-1572), the influential French pedagogue and author of textbooks of mathematics. It was Ramus who first gave the sort of algebraic reading of the *Elements*, in particular, of Books II and VI, that would become standard with Zeuthen and Tannery.[16] But it was Viète who would realize Ramus’ ambitions, both mathematical and pedagogical, by showing that algebra, or as he preferred to call it, analysis, provides a general method for the solution of problems whether geometrical or arithmetical. The aim of Viète’s Analytic Art, following Ramus, is to teach this method in a pedagogically effective fashion, that is, in a way that will enable students systematically to solve mathematical problems.[17]

As already noted, the *logistice speciosa* that Viète introduces in the Analytic Art uses two different sorts of uppercase letters, vowels and consonants, for unknown and known parameters of a problem. The various species (or powers) of unknowns are designated by a vowel followed by a word marking the power to which it is raised, for example, ‘A *cubum*’ (sometimes ‘A *cub.*’) or ‘E *quadratum*’ (‘E *quad.*’). It is clear that these expressions are comparable to our ‘x^3’ and ‘y^2’: in Viète’s system, if one multiplies, say, ‘A *quadratum*’ by ‘A’, the result is ‘A *cubum*’. ‘A’ in such expressions designates not the value but only the root. Signs for Viète’s known parameters, although they too take the form of a letter followed by a word indicating the species (e.g., ‘B *plano*’ or ‘Z *solido*’), do not function in the same way. In a sign such as ‘B *plano*’ of the *logistice speciosa*, it is the sign ‘B’ alone that designates the known parameter; ‘*plano*’ merely annotates the letter.[18] As required by the law of homogeneity according to which “homogeneous terms must be compared with homogeneous terms,” it serves to remind the analyst that if, at the last stage in solving a problem, he turns geometer (rather than arithmetician), he must put for ‘B’ something of the appropriate “scale.”[19] If the problem is arithmetical, any number can be put for ‘B’ (because among numbers there is no difference in scale, all being measured by the unit). But if the problem is geometrical, then only a plane figure (for instance, a square or a rectangle) can meaningfully be assigned to 'B'. Where Viète wishes to indicate the known parameter raised to a power, say the second, he writes 'B *quad*.'; if he wishes to indicate that a root raised to a power (say, the second) is planar, he writes 'E *plani-quad*.'.[20]

Viète's two different sorts of letters, uppercase vowels for unknowns and uppercase consonants for known parameters of a problem, function in his symbolic language in two essentially different ways. Vowels signify roots the powers of which are then indicated by the word that follows the vowel. Consonants signify the known parameter itself. The word that follows the consonant (e.g., '*plano*' or '*solido*') serves only to indicate the sort of figure that can be put for the letter at the last stage of the art in the case in which the problem is geometrical. The *logistice speciosa* serves in this way as a symbolic language that can be applied to both arithmetical and geometrical problems. It is, in this regard, quite like Eudoxus' general theory of proportions as developed in Book V of the *Elements*—though, we will see, with one essential difference.

Eudoxus' theory is general in the sense of applying generally to numbers and geometrical figures. It concerns itself not specifically with ratios of numbers or ratios of geometrical figures but more generally with ratios of any sorts of entities that can stand in the relevant relationships; it concerns numbers but not *qua* numbers because it applies equally to figures and motions, and it concerns figures and motions but not *qua* figures or motions because it applies equally to numbers. The theory is concerned with such objects insofar as they fall under a "higher universal," one that "has no name."[21] That is, it applies to such objects insofar as they belong to some genus, which has no name, of which *number* and *geometrical figure* are species much as a theory of mammals applies to cats and cows (among other things) insofar as they belong to a genus—one that does have a name, *mammal*—of which *cat* and *cow* are species. Viète's *logistice speciosa* is not general in this way. Though it does in a way apply generally to both numbers and geometrical figures, it also "generalizes" over two very different sorts of operations. Whereas in Eudoxus' theory, the notions of ratio and proportion are univocal—precisely the same thing is meant whether it is a ratio or proportion of numbers or of geometrical figures that is being considered—in the Analytic Art, the notions of addition, multiplication, and so on, are not univocal: the arithmetical operations that are applied to numbers in the Analytic Art are essentially different from those applied to geometrical figures. Whereas in arithmetic one calculates with numbers, each calculation taking numbers to yield numbers, in geometry one constructs using figures, and in the cases of multiplication and division, and in the geometrical analogue of the taking of roots, the result of a construction is a different sort of figure from that with which one began (or even a different sort of entity altogether, namely, a ratio). Furthermore, in arithmetic the result of an operation can be merely determinable, as it is in the case of the root extraction of, say, two; in geometry, all results are fully determinate. There is in the Analytic Art no genus to which numbers and figures belong such that they can be, for instance, added, multiplied, or squared. How, then, are we to read an expression of Viète's *logistice speciosa* such as 'A *quadratum*+B *plano*' given that there is no genus relative to which the mathematical operations (here, addition) can coherently be applied?

For Viète, as for the ancients, relations depend essentially on the objects that are their relata; there are no relations independent of the objects they relate. It follows that Viète can have no generic notion of an arithmetical operation, say, addition, that serves as the genus, as it were, of which arithmetical and geometrical addition are species. A sign such as '+' in Viète's *logistice speciosa* cannot signify either arithmetical addition or geometrical addition to the exclusion of the other; it cannot be merely equivocal or ambiguous; and there is no genus that might be signified instead. The only plausible reading of Viète's *logistice speciosa* is a reading of it as a formal theory or uninterpreted calculus. The first stage of the Analytic Art, *zetetics*, takes one out of a particular domain of inquiry, either arithmetical or geometrical, into a purely formal system of uninterpreted signs that are to be manipulated according to rules laid out in advance, and only at the last stage, *exegetics*, are the signs again provided an interpretation, either arithmetical or geometrical. As Mahoney explains,

> the elevation of algebra from a subdiscipline of arithmetic to the art of analysis deprives it of its content at the same time that it extends its applicability. Viète's specious logistic, the system of symbolic expression set forth in the *Introduction* is, to use modern terms, a language of uninterpreted symbols.[22]

Bos makes essentially the same point in *Redefining Geometrical Exactness*:

> [w]hile considering abstract magnitudes Viète could obviously not specify how a multiplication (or any other operation) was actually performed but only how it was symbolically represented. Thereby the "specious" part of the new algebra was indeed a fully abstract formal system implicitly defined by basic assumptions about magnitudes, dimensions, and scales . . . and by axioms concerning the operations . . . [of] addition, subtraction, multiplication, division, root extraction, and the formation of ratios.[23]

Viète's *logistice speciosa* is not, then, properly speaking a language at all. It is an uninterpreted calculus, a tool that is useful for finding solutions to problems but within which (that is, independent of any interpretation that might be given to it) neither problems nor their solutions can be stated. Indeed, its usefulness is a direct function of its being an uninterpreted calculus. Because the *logistice speciosa* has no meaning or content of its own, the results that are derivable in it may be interpreted either arithmetically or geometrically. It is in just this way that, as Viète proudly announces, "the Analytic Art claims for itself the greatest problem of all, which is

To solve every problem."[24]

2. Descartes' Geometry[25]

Descartes' *Geometry*, which first appeared as one of three appendices to his *Discourse on Method*, aims to illustrate Descartes' new method for the discovery of truths. And although as its title announces the *Geometry* is concerned only with geometry, his method in that work can seem to be essentially that of Viète. A geometrical problem is to be reduced to an algebraic equation which is then manipulated according to rules. The required roots of the equation are then constructed geometrically. Despite the differences in aim and orientation between Descartes and Viète, it can seem, in other words, that these differences do not penetrate to the use of the symbolic language itself, that Descartes' symbolic language (which we still use today: letters early in the alphabet for known parameters, letters late in the alphabet for unknowns, and numerical superscripts for powers) is essentially that of Viète. Certainly any formula of Viète's Analytic Art is easily translated into the symbolism Descartes uses. Nevertheless, we will see, Descartes' understanding of his symbolic language is very different from Viète's understanding of the *logistice speciosa* introduced in *The Analytic Art*. Whereas Viète abstracts from the particular subject matter, either that of arithmetic or that of geometry, Descartes transforms the subject matter of geometry. An expression such as 'a^2+bc' of Descartes' symbolism is not an empty formalism interpretable either arithmetically or geometrically; it is itself fully meaningful, a representation of an arbitrary line segment.

We have seen that Viète's symbolic language must be read as an uninterpreted calculus, as a purely formal language the symbols of which are to be manipulated according to stated rules and can be interpreted either geometrically or arithmetically. The first indication that Descartes' symbolic language is essentially different is the fact that Descartes despised formalism. To understand why, we need to look briefly at Sextus Empiricus, from whom Descartes borrows many of his criticisms.

Sextus argues that an inference such as this:

> Everything human is an animal.
>
> Socrates is human.
>
> Therefore, Socrates is an animal.

either is circular or has a redundant premise. Suppose, first, that the major premise, that everything human is an animal, is merely accidental, that it is true in virtue of a collection of merely contingent facts about actual humans, that each and every one is an animal. In that case, Sextus claims, the argument is circular because the conclusion, that Socrates (one of the humans) is an animal, "is actually confirmatory of the universal proposition in virtue of the inductive mode."[26] Because one cannot establish that everything human is an animal without first establishing that Socrates, one of the humans, is an animal, the major premise presupposes already the truth of the conclusion. As Descartes puts the point in the *Regulae*, dialecticians "are unable to formulate a syllogism with a true conclusion unless they are already in possession of the substance of the conclusion, i.e., unless they have previous knowledge of the very truth deduced in the syllogism" (AT X 406; CSM I 36-7).[27] It follows, as Descartes immediately points out, that "ordinary dialectic is of no use whatever to those who wish to investigate the truth of things."[28]

But as Sextus indicates, the major premise of a syllogism such as that given above need not be construed as a generalization derived from our knowledge of particular propositions. It can instead be conceived as a law to the effect that, as Sextus thinks of it, being an animal follows being human; and if it is, then the major premise is redundant because "at the same time it is said that Socrates is human, it may be concluded that he is an animal."[29] If the major premise expresses not a generalization but instead an inference license—not a claim *from which* to reason but a principle or rule *according to which* to reason, as Mill would later put it[30]—then one can conclude, on the basis of the minor premise alone, that Socrates is an animal. Because what the rule licenses just is one's concluding that Socrates, say, is an animal given that he is human, it would be inconsistent with its status as such a license to require its inclusion among the premises. In the Second Replies, Descartes suggests that the thought 'I think; therefore, I am' is valid in just this way. 'I think; therefore, I am' is not a syllogism (that is, enthymematic, in need of a major premise to the effect that anything that thinks exists) but instead some sort of immediate inference: a person "learns it from experiencing in his own case that it is impossible that he should think without existing" (AT VII 140-1; CSM II 100). The inference is valid because, whether or not one explicitly formulates it as a rule or inference license, that one (oneself) thinks entails that one (oneself) exists.[31]

The argument concerning Socrates, on Sextus' second reading of it, has a redundant premise because in that case, given the inference license in the major premise, the conclusion follows directly from the minor premise. But although the major is not required, it can nonetheless be included among the premises. To include it is to transform a materially valid inference into one that is formally valid; and this, it may seem, is an unalloyed good because in that case one can directly see, on the basis of the form of the argument alone, that it is valid without having to attend to what in particular the premises state. Significantly, Descartes rejects the move. As he writes in his remarks on Rule Ten in the *Regulae*,

> some will perhaps be surprised that in this context, where we are searching for ways of making ourselves more skilful at deducing some truths on the basis of others, we make no mention of any of the precepts with which dialecticians suppose they govern human reason. They prescribe certain forms of reasoning in which the conclusions follow with such irresistible necessity that if our reason relies on them, even though it takes, as it were, a rest from considering a particular inference clearly and attentively, it can nonetheless draw a conclusion which is certain simply in virtue of the form. But, as we have noticed, truth often slips through these fetters, while those who employ them are left entrapped in them. (AT X 405-6; CSM I 36)

Descartes' aim is to become "more skilful at deducing some truths on the basis of others"; so, it might seem, he would be helped by having ready to hand knowledge of the various valid forms of argument that are outlined by dialecticians precisely because these forms enable one's reason to take "as it were, a rest from considering a particular inference clearly and attentively." In fact, Descartes suggests, the opposite is true. To appeal to such forms is to become *less* skilled at deducing truths precisely because they promote inattention: "our principle concern here is thus to guard against our reason's taking a holiday while we are investigating the truth about some issue; so we reject the forms of reasoning just described as being inimical to our project" (ibid.).

Sextus suggests that any formally valid, non-circular inference has a valid (non-formal) counterpart in which the conclusion is inferred directly from the minor premise. To rely on formally valid patterns of inference nonetheless, Descartes thinks, is to risk error through inattention. Because one is in that case unlikely to draw any conclusions that do not follow in virtue of the (valid) form of argument, the source of the error would seem to lie instead in one's assenting to premises that are not in fact true. Descartes claims exactly that: "none of the errors to which men . . . are liable is ever due to faulty inference; they are due only to the fact that men take for granted certain poorly understood observations, or lay down rash and groundless judgments" (AT X 365; CSM I 12). We err in reasoning not because we are ignorant of the rules of formally valid reasoning but because we take to be true that which is poorly understood or draw inferences that are not in fact materially valid. It is "those who make a judgment when they are ignorant of the grounds on which it is based [who] are the ones who go astray" (AT VII 147; CSM II 105). To avoid error requires that one assent only to that which one clearly and distinctly perceives to be true and to that which clearly follows (by a materially valid rule of inference) from something clearly and distinctly perceived to be true.

Descartes' opposition to the logic of the schools is not an opposition to deductive reasoning. His opposition is to the dialectician's focus on formally valid patterns of deduction on the grounds that, by emphasizing valid forms, the dialectician promotes inattention both to the truth, or falsity, of one's premises and to the validity, or invalidity, of the material rules of inference employed in the deduction. Further evidence that this is indeed the nature of his critique of formalism is provided by Descartes' account in his Second Replies of the analytic and synthetic methods of demonstration in geometry.[32] As he explains, the order of argumentation in the two cases is the same: "the items which are put forward first must be known entirely without the aid of what comes later; and the remaining items must be arranged in such a way that their demonstration depends solely on what has gone before" (AT VII 155; CSM II 110). Whether the method of demonstration is analytic or synthetic, it is deductive ("what comes later depends solely on what has gone before") and non-circular (what is first put forward "must be known entirely without the aid of what comes later"). Nevertheless, Descartes thinks, the two methods are very different.

> Analysis shows the true way by means of which the thing in question was discovered methodically and as it were *a priori*, so that if the reader is willing to follow it and give sufficient attention to all points he will make the thing his own and understand it just as perfectly as if he had discovered it for himself. But this method contains nothing to compel belief in an argumentative or inattentive reader; for if he fails to attend even to the smallest point he will not see the necessity of the conclusion. Moreover there are many truths which—although it is vital to be aware of them—the method often scarcely mentions, since they are transparently clear to anyone who gives them his attention. (AT VII 155-6; CSM II 110)

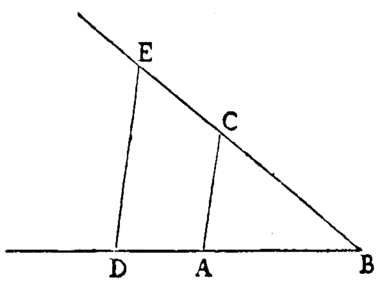
In the *Meditations*, Descartes tells us, it was the analytic method that was followed, and the method is "*a priori*" because, as Descartes indicates in his account of synthesis, it is prior to the method of synthesis.

> Synthesis, by contrast, employs a directly opposite method where the search is, as it were, *a posteriori* (though the proof itself is often more *a priori* than it is in the analytic method). It demonstrates the conclusion clearly and employs a long series of definitions, postulates, axioms, theorems, and problems, so that if anyone denies one of the conclusions it can be shown at once that it is contained in what has gone before, and hence the reader, however argumentative or stubborn he may be, is compelled to give his assent. (AT VII 156; CSM II 110-1)

In a synthetic demonstration everything on which the proof depends, save for the rules of formally valid inference employed, is explicitly stated in advance in axioms, definitions, postulates, (previously demonstrated) theorems, and (previously solved) problems. Because it is, the proof compels assent by anyone, no matter how argumentative or stubborn, who accepts those axioms, definitions, postulates, theorems, and problems. But of course one can set out everything on which a proof depends only after one has the proof itself, only after one has achieved, analytically, knowledge of the "primary notions" and has drawn (materially valid) conclusions from them. It is for just this reason that the analytic method requires, in a way the synthetic does not, the closest attention to each point and enables, in a way the synthetic does not, a reader to "make the thing his own and understand it as perfectly as if he had discovered it for himself." The analytic method employs (often unstated) rules of material inference to draw conclusions from evidently true premises. In this case, one is to *see* that the truth of the premises entails the truth of the conclusion, but in order to see that, one must actually think about what the premises mean, and thereby about what they entail. The synthetic method, because and insofar as it relies only on formally valid rules of inference, does compel assent; but because it does not require that one actually think about what is being claimed in the premises, it does not ensure understanding. It follows that the synthetic method "is not as satisfying as the method of analysis," that the analytic method is "the best and truest method of instruction" (AT VII 156; CSM II 111).

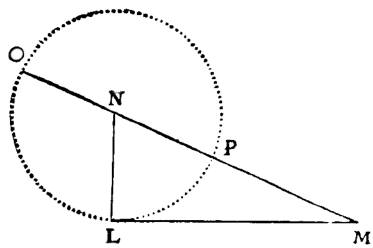
We have seen that Descartes despised formalism because it promotes thoughtlessness and error, and because it does not lead to understanding. Such charges could be leveled against Viète's *logistice speciosa* as easily as they are by Descartes against the logic of the schools. Viète's Analytic Art teaches the (essentially mechanical) manipulation of symbols according to rules and because it does, it is prone to error and does not ensure understanding. Descartes' method is to begin instead with that which is simplest and most easily known, then to proceed in order, by evidently valid (though not necessarily formally valid) steps, to that which is more complex and hence less easily known. Because, as Descartes notes in his conversation with Burman, "it is in mathematics that examples of correct reasoning, which you will find nowhere else, are to be found" (AT V 177; CSM III 352),[33] it is in mathematics that Descartes' method finds its first application.

The concern of mathematics, according to Descartes, "is with questions of order and measure and it is irrelevant whether the measure in question involves numbers, shapes, stars, sounds, or any other object whatever" (AT X 378; CSM I 9). In cases in which these proportions are to be considered separately, they are to be taken, Descartes tells us, "to hold between lines, because I did not find anything simpler, nor anything I could represent more distinctly to my imagination and senses" (AT VI 20; CSM I 121). Where such proportions are to be considered together they are to be designated "by the briefest possible symbols." "In this way," he explains, "I would take over all that is best in geometrical analysis and in algebra, using the one to correct all the defects of the other" (ibid.).[34] Suppose, for example, that one wished to represent the relationship between two numbers and their product. This can be achieved graphically, as a relation among line lengths, where AB is to be understood as the unit length and DE is drawn parallel to AC:[35]



Because BE is to BD as BC is to BA (because angle B is common to both triangles and the lengths DE and AC are parallel), it follows that BE is the product of BD and BC. (If BE:BD=BC:BA, that is, BE/BD=BC/BA, and BA is the unit length, then BE·1=BC·BD; that is, BE is the product of BC and BD.) But, as Descartes goes on, this same relationship can also be expressed symbolically: where a is the length of BD and b that of BC, the product of the two lengths can be given as ab. Descartes claims, in other words, that the geometrical relationship between the line lengths that is presented in the above diagram is not merely analogous to but an alternative expression of that which is expressed symbolically. And the same is true of the graphic and symbolic representations in Descartes' geometry of a sum, of the difference between two lengths, of the division of one length by another, and of a square root.

We are then shown how to construct, geometrically, the positive roots of various quadratic forms. Suppose, for example, that $z^2=az+b^2$. To find the root we construct a right triangle NLM with LM equal to b and LN equal to $1/2a$, and then extend MN to O so that NO is equal to NL.[36]

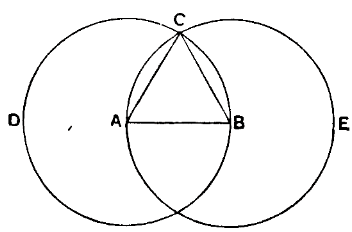


Because OM is to LM as LM is to PM, OM·PM=LM^2, that is, $z(z–a)=b^2$, or $z^2–az=b^2$. The desired root, then, is the length OM, that is, ON+NM, or $1/2a+\sqrt{}(a^2+b^2)$.

Our first indication that Descartes' symbolic language is quite different from that which Viète employs was that Descartes despised formalism in reasoning because it is mechanical and promotes thoughtlessness. According to Descartes, one needs to reflect attentively on the contents of one's expressions in order to determine whether what is revealed thereby is true. What we have just seen is that whereas Viète's *logistice speciosa* functions as an uninterpreted calculus, one that can be interpreted either geometrically or arithmetically, Descartes' symbolic language is always already interpreted. As Descartes employs them, letters and combinations of letters signify something in particular, namely, line lengths (themselves conceived as representing arbitrary quantities), either those that are given or those that are sought. If we are to understand the relationship between Viète's art and Descartes' science, then, we need better to understand the relationship between the numbers and geometrical figures that are the ultimate subject matter of Viète's Analytic Art, on the one hand, and the line lengths relations among which are the topic of Descartes' geometry, on the other.

Although it was long understood (following Ramus, Zeuthen, and Tannery) as "algebra in geometrical dress," Euclidean geometry is in fact a science of various sorts of (self-subsistent) geometrical objects.[37] A

Euclidean circle, for example, is not a one-dimensional closed (algebraic) curve all points of which are equidistant from a center, but instead a two-dimensional object, a “plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another” (Def I.15).[38] A Euclidean straight line in turn is a line, that is, a breadthless length, that “lies evenly with the points on itself” (Def I.2, Def I.4). Euclidean straight lines have length but lengths are not to be conceived in the *Elements* in measure-theoretic terms; they are only equal or unequal in length, and it is circles that provide the context for the determination of them as equal. For example, to construct an equilateral triangle on a given line length AB, one first draws a circle with center A and radius AB, then another with center B and radius AB. It follows that all straight lines from center A to the circumference of circle A and all straight lines from center B to the circumference of circle B are equal in length. In particular, straight lines from points A and B to the point C on the circumference of the two circles at which they cut one another are equal in length, both to each other and to the original line length AB. The triangle ABC, then, is equilateral.[39]



Now Euclid does also employ drawn line lengths in demonstrations about numbers in Books VII-IX, but it is evident that these drawings function in his demonstrations in a very different way from those found in his demonstrations regarding geometrical objects. In a demonstration such as that just outlined, a drawn line length signifies a geometrical object. In a demonstration regarding numbers—for instance, that showing that “any number is either a part or parts [i.e., either a submultiple or proper fraction] of any number, the less of the greater” (Prop VII.4)—the drawn line length serves instead to signify an arbitrary number, that is, a number but no number in particular. Unlike a geometrical line length, such a number cannot be conceived now as a radius of a circle, now as the side of a triangle. It is a completely different sort of object, namely, a collection of units. Just as a collection of (say) four dots:

. . . .

is an instance of four on the ancient conception of number, so a line length of four units:

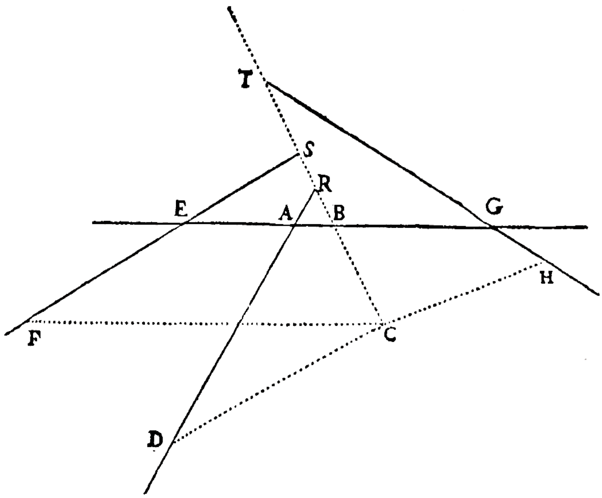
is an instance of four. But whereas a collection of dots must in the nature of things be some particular number of dots, the number of units in a line length is relative to the unit of measure. Because it is, a line length can serve in a demonstration as an arbitrary number, as an instance of number about which to reason, but as no number in particular. Anything that can be shown regarding that number must then hold of all numbers. Whereas a geometrical line length is a continuous spatial magnitude that is infinitely divisible into parts, a numerical line length is a plurality divisible into a finite number of discrete parts. The two sorts of lines can seem to be one and the same only because in a demonstration regarding numbers it is often not given just how many parts the (numerical) line length divides into.

In Euclid's demonstrations, drawn line lengths serve two essentially different functions, either as geometrical objects or as arbitrary numbers. In Viète's *logistice speciosa*, we have seen, one abstracts from these differences. An expression such as 'A *quadratum*+B *plano*' can be interpreted either geometrically (as involving figures classically conceived) or arithmetically (as involving numbers conceived as collections of units). Independent of any interpretation, the expression is a mere form. In Descartes' geometry, in virtue of the introduction of a unit length, an essentially new mathematical notion is introduced, that of a geometric quantity that is, as numbers are, dimension-free. Just as operations on numbers yield numbers in turn, so operations on line segments in Descartes' geometry yield line segments in turn. (We saw this in the example of a product BE of two line lengths BD and BC above; in that example, AB was assumed to be the unit length.) That is why there is in Descartes' symbolism nothing corresponding to Viète's annotations '*plano*', '*solido*', and so on. In Descartes' *Geometry* a letter such as 'a' or a combination of signs such as '$(a+b)^2$' is not an uninterpreted expression that can be interpreted either geometrically or arithmetically; it is a representation of (an indeterminate) line length where a line length is to be distinguished both from a Euclidean line segment and from a number classically conceived as a collection of units.

Descartes introduces a unit length into geometry and as a result can understand the product (say) of two line lengths as itself a line length rather than, as on Viète's and the ancient understanding, a plane figure. As Descartes understands them, all the basic operations on line lengths (addition, subtraction, multiplication, division, and the extraction of roots) yield only line lengths. In consequence, although he pays lip service to Viète's law of homogeneity, he has no need of it: "unity can always be understood, even where there are too many or too few dimensions; thus, if it be required to extract the cube root of a^2b^2-b, we must consider the quantity a^2b^2 divided once by unity, and the quantity b multiplied twice by unity" (AT VI 299; G 6).[40] As we can think of it, in Descartes, the interest has shifted from a concern with geometrical objects such as lines, circles, and triangles, to a concern with relations among line segments, themselves to be understood as representative of arbitrary quantities. Descartes' concern is not with geometrical figures but with the relations among line segments that can be "read off" such figures, relations that can be expressed algebraically. A different example will help to highlight the essential point.

Imagine a drawing of a triangle ABC that is right at B. The ancients would understand such a drawing as a depiction of a certain object, a particular planar area, one with a certain property, namely, that expressed in the Pythagorean theorem. Descartes' view of it seems to be different. The drawing is not conceived as a depiction of a certain planar figure but instead as presenting three line segments in a certain relation, one that can be expressed algebraically: substituting a for BA, b for BC, and c for AC, $a^2+b^2=c^2$. (This fact was appealed to above in the construction of the root of the quadratic $z^2=az+b^2$.) Whereas for the ancients, the Pythagorean theorem tells us something about a certain sort of object, namely, a right triangle, for Descartes the triangle—more exactly, the fact that the three line segments are so configured—tells us something about the (algebraically expressible) relationship among those line segments. From Descartes' perspective, the drawn triangle seems not to be an *object* at all but only one of the ways, an especially interesting and revealing way, that line segments can be related to one another. Descartes' treatment of "indeterminate" problems, that is, problems in two (or more) unknowns, problems that are systematically ignored by Viète, further manifests this new approach to problems. Pappus' locus problem provides a familiar example.

We begin with four lines given in position, and the problem is to find a point C from which straight lines can be drawn making given angles with the given lines such that the product of two of them is equal to the product of the other two.[41] Descartes immediately identifies two lines as "principal lines" to which all others are to be referred, creating thereby a frame of reference for thinking about the various relationships among the line segments relevant to the problem. (This is, of course, the key insight behind our Cartesian coordinates.) This move, together with his introduction of a unit length, enables Descartes to treat this problem in two unknowns in essentially the same way that he deals with problems in one unknown, problems such as finding the root of the quadratic $z^2=az+b^2$. Again, we begin with four line lengths given in position:[42]



The length AB, a segment (of unknown length) of one of the principal lines, and the length BC, a segment of the other principal line that also is sought, are identified as the unknowns, x and y. Now we form expressions for the lengths CD, CF, CB, and CH, which can then be combined in the equation CD·CF=CB·CH. We find the length CD, for example, as follows. The ratio AB:BR, given by the terms of the problem, is set equal to $z:b$. So, RB=bx/z and CR=CB+BR=$y+bx/z$. Taking the ratio CR:CD equal to $z:c$, it follows that CD=$cy/z+bcx/z^2$. (That is, CR:CD=$z:c$, so CD=$c/z\cdot$CR; but CR=$y+bx/z$, so CD=$c/z(y+bx/z)=cy/z+bcx/z^2$.) In essentially the same way, an expression of the form $ax+by+c$ is derived for the other three line segments CF, CB, and CH; and the four expressions are then combined according to the original terms of the problem, namely, CD·CF=CB·CH, to yield an equation in the two unknowns, x and y. Again, the crucial step here is the first, that of identifying two (extendable) lines AB and BC as the principal lines to which all others are referred. Once that is done, the problem of finding the needed equation reduces to that of transforming, or translating, various geometrical relations into (symbolically expressible) relations among line segments.

In the first examples we considered we saw that because Descartes assumes a unit length he can conceive the results of all operations on line segments as themselves line segments. More generally, we saw, what is of interest to Descartes is not geometrical objects themselves but the algebraically expressible relations among the line segments that are their boundaries. In the Pappus problem Descartes goes further, identifying two lines as those to which all others are to be referred, and so can understand even an indeterminate problem solely in terms of relations among line segments that are expressible in equations in his symbolic language. It is on the basis of just this understanding of its subject matter that Descartes claims, in the opening sentence of the *Geometry*, that "any problem in geometry can easily be reduced to such terms that a knowledge of the length of certain straight lines is sufficient for its construction" (AT VI 297; G 2).

3. The Emergence of Modern Mathematics

We have seen that although they are superficially similar, Viète's *logistice speciosa* and Descartes' symbolic language in fact function very differently. Whereas Viète's is a formal language or uninterpreted calculus that can be interpreted either geometrically or arithmetically (as these are classically conceived), Descartes' language is always already interpreted. It concerns relations among line segments where line segments are to be understood neither as geometrical figures classically conceived nor as numbers conceived (following the ancients) as collections of units, but as arbitrary quantities, relations among which are expressed in his symbolic language. As we have also seen, understanding line segments as Descartes does requires both the introduction of a unit length and, for the case of problems in more than one unknown, a stipulation of principal lines to which all others can be referred. But what is the significance of these differences? Was it merely an accident of history that Viète read his notation as he did, that he did not, as Descartes did, win through to a modern conception of quantity and of curves? I will suggest that it was not; more exactly, that Viète could have achieved the modern view *but only* by way of a radical reorientation in his thinking, and that it is this reorientation enabling a new way of reading the notation, rather than the literal notation itself, that marks the emergence of modern mathematics.

According to the traditional conception, a number, that is, a discrete quantity, is literally a collection of units. Because it is, there can be no zero but only an absence of any number, and there can be no negative numbers. Similarly, on the traditional conception of continuous quantity, there can be no negative quantities, or as Descartes describes them, roots that are "false or less than nothing," that are "the defect of a quantity" (AT VI 372; G 159). The modern conception of quantity, which comprehends negative, or "false," and "imaginary" quantities, as well as zero, is very different. It is achieved, Dantzig has argued, through the introduction of the literal notation:

> As long as one deals with numerical equations, such as
>
> (I) $x+4=6$ (II) $x+6=4$
>
> $2x=8$ $2x=5$
>
> $x^2=9$ $x^2=7$,
>
> one can content himself (as most medieval algebraists did) with the statement that the first group of equations is possible, while the second is impossible.
>
> But when one considers the literal equations of the same types:
>
> $$x+b=a$$
>
> $$bx=a$$
>
> $$x^n=a$$
>
> the very indeterminateness of the data compels one to give an *indicated* or *symbolic* solution to the problem:
>
> $$x=a-b$$
>
> $$x=a/b$$
>
> $$x=\sqrt[n]{a}.$$
>
> In vain, after this, will one stipulate that the expression $a-b$ has a meaning only if a is greater than b, that a/b is meaningless when a is not a multiple of b, and that $\sqrt[n]{a}$ is not a number unless a is a perfect nth power. The very act of writing down the *meaningless* has given it a meaning; and it is not easy to deny the existence of something that has received a name.[43]

Because already in Viète's Analytic Art we find the literal notation required in the formulation of this argument and thereby the resources needed (on this view) for the realization of the modern number concept, Dantzig concludes that it is Viète's Analytic Art and not Descartes' geometry that marks "the turning-point in the history of algebra."[44] Of course, Viète himself did not make this turn; according to him, only a smaller quantity can be subtracted from a larger, and only positive roots can be given as solutions to problems. Obviously, then, he can have no inkling of "imaginary" roots, no inkling of the fact that, as Descartes saw, an equation of *n*th degree can have, or at least be conceived to have, *n* roots.[45] Although Viète abstracts from the particular objects involved, whether numbers or geometrical figures, his thinking remains oriented toward such objects. As he himself understood it, his symbolism was only a useful device, a tool for solving problems. As such it lacked just the autonomy that is required by Dantzig's argument to realize the modern number concept.

Although his *logistice speciosa* abstracts from any particular mathematical objects, Viète's thinking remains oriented towards such objects as they are classically conceived. Descartes' perspective is very different. He begins not with objects but instead with relations among arbitrary quantities. As a result, instead of limiting the scope of his mathematical operations in light of antecedently available quantities (the scope of subtraction, for example, to those cases that yield a "natural" solution), he extends the domain of quantities to include any that can be the result of such operations. Descartes recognizes "false" roots as geometrically real because they are producible as line segments through the application of his constructive procedures; they make geometrical sense, at least as Descartes understands geometry. They are nonetheless "false" because they are not solutions to the problem as originally conceived but instead to topological variants of it.[46] Because the relationship between an imaginary root and a "true" or "false" root cannot be understood in the same way—"however we may increase, diminish or multiply them . . . [they] remain always imaginary" (AT VI 380; G 125)—it follows, Descartes thinks, that no definite quantity corresponds to such roots. Such roots are merely imaginary.

As is manifested both by his recognition of "false" roots and by his refusal to recognize, in the same way, "imaginary" roots, Descartes' conception of quantity is, as it were, top-down, driven by an understanding of his constructive procedures, rather than bottom-up, grounded in an antecedent understanding of quantities. Geometrical "space" as Descartes understands it is not to be built up out of collections of objects; it is instead given by operations, in particular, by the constructive procedures that articulate that space. Because it is, Descartes achieves, as Viète does not, a modern concept of quantity. Descartes could, then, make something like the argument Dantzig makes, *but only* because he has already achieved the requisite conception of quantity. He can understand, in a way that Viète cannot, that a larger quantity can be subtracted from a smaller because he has already given up the classical, object-based, conception of quantity in favor of a modern, top-down, conception in terms of permissible constructive operations. It is not the notation itself that makes the essential difference, but instead the way it is read, either in accord with the ancient understanding or instead in light of the modern conception.

The case of curves is similar. On the ancient view that Viète adopts, it is not curves that are classified but instead problems. Plane problems are those the solution of which requires recourse only to lines and circles, entities that are taken to be self-subsistent geometrical objects. Next are solid problems, so-called because their solutions require appeal also to conics (i.e., hyperbolas, ellipses, and parabolas) conceived as intersections of a cone (that is, a solid, a figure having length, breadth, and depth, the limit of which is a surface) and a plane. Conics, on this view, are plane figures and so in that respect like circles, but unlike circles they are intelligible only by reference to a solid figure, a cone, and are for this reason an essentially different sort of plane figure. The last sorts of problem are the so-called line problems, the solution of which involves "lines," that is, loci of points to which no known figures, or boundaries of figures, correspond. Because such lines do not form the boundaries of any figures, they were regarded with deep suspicion by ancient geometers; they were taken to be mechanical rather than properly geometrical.

In Book III of his *Geometry*, Descartes continues this traditional line of thought; he classifies traditional determinate construction problems by the ancient criteria. In Book II, however, a radically new classification is given, a classification not of problems but of curves. According to this classificatory scheme, the simplest class of geometrical curves includes the circle, the parabola, the hyperbola, and the ellipse because all these curves are given by equations the highest term of which is either a product of the two unknowns or the square of one; they are one and all expressible in an equation of the form: $ax^2+by^2+cxy+dx+ey+f=0$.[47] Whereas on the ancient view that Viète follows, conics are essentially different from circles, from Descartes' perspective in Book II, they are all essentially alike. And they can be because a geometrical curve, rather than being conceived as the boundary of a figure, is to be conceived instead as a curve all points of which "must bear a definite relation to all points of a straight line," where this relation in turn "must be expressed by means of a single equation" (AT VI 319; G 48). As Descartes conceives it, a curve is not an edge of a thing but instead the locus of points in space conceived as an antecedently given whole. Such a curve can be traced by a continuous motion but it is not constituted by the relative positions of points on it; it is constituted by the equation that expresses it, that is, by the location of each point, independent of all the others, directly in space (relative to some arbitrarily given principal lines). From the perspective Descartes provides, a problem such as that posed by the sort of quadratic equation that was the focus of Viète's interest can be thought of as a limit case, with one unknown set equal to zero, of an indeterminate equation; for example, $x^2+ax=b^2$ as the limit case of $x^2+ax-b^2=y$, in which y is set equal to zero. Geometry, once the study of relations among certain sorts of objects, has become the study directly of relations.

As is most evident in his treatment of curves as loci of points sharing a property that is expressed in an equation, Descartes' mathematical orientation is fundamentally different from that of Viète and the ancient mathematicians he followed.[48] As the point has been put here, Descartes' is a top-down understanding grounded directly in relations among (arbitrary) line segments, as contrasted with the bottom-up, or object-oriented, approach of Viète. Both the difference between these conceptions and their essential connection can be clarified by reflection on two radically different but intimately related conceptions of space in our everyday experience of it.

It is a familiar fact of our everyday experience that any reasonably large portion of the landscape can be taken in only piecemeal in our movements from landmark to landmark. But although we experience such a portion of the landscape only serially, we learn in time to synthesize the various routes we have mastered into one unified whole; we learn to conceive the relevant layout of land as an integrated whole, each of the landmarks as having a place relative to all the others. We achieve in this way a kind of bird's eye view of the lay of the land, a sense of how things would look from a position above the land through which we move and thereby the capacity to draw a familiar sort of picture map. In a map so conceived, signs for the various landmarks are arranged so as to represent their relative positions; and space on this conception is given by the relative positions of objects (landmarks). It is a bottom-up or object-based conception, one that is achieved by synthesizing into one all the various paths from landmark to landmark that one learns by traversing the landscape.

We achieve a conception of a given space as a unified whole by integrating our knowledge of various paths through it from one landmark to another; and a map presenting the relative locations of landmarks provides a kind of picture of the relevant space so conceived, the way things would look from a bird's eye view. *Having* achieved this conception, however, one can learn to read the map differently, as a presentation of space as an *antecedently* given whole, that is, as a whole that is prior to its parts, within which landmarks are directly located, each independent of all the others. To achieve this second conception, one engages in a kind of global figure/ground switch: having in place a map of objects (landmarks) in their relative locations, one reconceives the whole by beginning instead with the (given) space itself and construing the various landmarks as having, each independently, a position (relative to some arbitrary principal lines) in that given space. Such a "view" of space is not literally a view, not a presentation of how things look, at all. It is, as we might say, "the view from nowhere," space conceived as an object of thought rather than as an object of sensory experience.

Whereas on the bird's eye view one begins with objects (landmarks) and locates them relative to one another, on the view from nowhere one begins instead with space itself abstractly conceived as a given, irreducible whole within which individual objects can be, but need not be, directly located, each independently of all the others. Although this latter conception of space is necessarily late, achieved only through a thoroughgoing transformation of the bottom-up, object-based conception of space, space on this second conception is clearly intelligible prior to, and so independent of, any reference to objects. Viète, I am suggesting, has the first, bottom-up, object-based conception, a bird's eye view; it is Descartes' achievement to have realized (on that basis) the second, top-down, holistic conception, the view from nowhere.

We have seen that in Viète's symbolic language it is demanded that all terms in an equation be homogeneous, and to that end annotations on letters for known parameters are introduced to indicate the relevant "scale" or dimension. Descartes, having assumed a unit length, can treat all operations on geometrical entities as essentially similar to operations on numbers. Geometrical quantities as Descartes understands them are not distinguished by their dimensions. Related to this is the fact that whereas Viète's *logistice speciosa* is a purely formal, uninterpreted calculus that can be interpreted either arithmetically or geometrically, Descartes' symbolic language is always already interpreted. Letters in Descartes' language, both those for known quantities and those for unknowns, are signs for line segments that are themselves to be understood as representative of arbitrary quantities. Whereas Viète's concern is ultimately with arithmetical and geometrical objects as traditionally conceived, Descartes' concern is with the relations among line segments that can be expressed algebraically. As the point has been put for the case of a right triangle, whereas for the ancients, and Viète following them, the Pythagorean theorem tells us something about a certain sort of triangle, that triangles of that sort have such and such a property, for Descartes, the triangle itself is of interest only as a particular configuration of line segments, one that is expressed algebraically as $a^2+b^2=c^2$ (where 'c' signifies the hypotenuse and 'a' and 'b' the sides). It is just this reorientation that we need to understand, and we can understand it by analogy with the shift from a bottom-up, object-oriented view of space to a top-down, object-independent, and essentially holistic view of it.

Imagine, again, a line drawing of a right triangle on a piece of paper. The example of two ways of regarding a map suggests that such a drawing too can be regarded in either of two very different ways—much as the familiar duck-rabbit display can be regarded in either of two ways, either as a duck or as a rabbit. First, one can see the drawing as a drawing of a two-dimensional object, an area of a particular shape, something that might be better represented by a piece of paper cut in the shape of a right triangle. In that case, the drawing is viewed literally as a picture of something. But one can also see it differently, namely, as a collection of line segments arranged (in the antecedently given space of the sheet of paper) in a particular way, that is, as bearing certain relations one to another. In that case, a piece of paper cut in the shape of a right triangle would not be a better representation of what is wanted but a worse one because to see what is wanted one would then have to imagine the cutout as part of a larger whole within which its three sides could be seen as mere line segments in a particular relationship. From this second perspective, the drawing is not a drawing of a thing, an object, at all but instead a representation of line segments in a certain relation; what is depicted is not a certain sort of thing but instead a relationship certain sorts of things can bear to one another. Viète, I am suggesting, would conceive the drawing in the first of these two ways. He would see in it a picture of a geometrical object traditionally conceived. Descartes would conceive the drawing in the second way. He would see three lines in a particular spatial configuration. That this is indeed Descartes' view is made explicit in his treatment of the Pappus problem: in choosing two lines as his principal lines to which all others are to be referred, Descartes generates a frame of reference for the space within which the given lines are positioned. He does not, as the ancients would, look for depictions of figures within the drawing; he looks instead for algebraically expressible relations among the given lines. Descartes' is a top-down view, achieved through a radical reorientation in his understanding of what a standard Euclidean diagram represents. Descartes *sees* diagrams differently from the way the ancients see them, and it is by virtue of this new way of seeing that he is able, in Book II of the *Geometry*, to develop his new theory of curves, one that is, just as he says (in a letter to Mersenne), "as far removed from ordinary geometry, as the rhetoric of Cicero is from a child's ABC" (AT I 479; CSM III 78). It is not the symbolic language of algebra itself that marks the emergence of modern mathematics but instead the radical reorientation that language enabled, a reorientation relative to which we can read the notation not as Viète did, as an uninterpreted calculus, but as Descartes does, as the language of mathematics itself.

4. Conclusion

The fact that the symbolic languages developed by Viète and Descartes are to be read very differently—Viète's as an uninterpreted calculus and Descartes' as a language proper expressing relations among line segments themselves representative of arbitrary quantities—should be understood, I have suggested, in terms of two essentially different intellectual orientations: Viète's object-based, or bottom-up, orientation, and Descartes' radically new, essentially holistic, top-down orientation. If that is right, we might expect to find two corresponding tendencies in the literature regarding the work of Viète and Descartes: on the one hand, a tendency to read back into Viète's symbolism the top-down holistic understanding that was achieved only by Descartes, and on the other, a tendency to read forward into Descartes' symbolism the kind of formalism that is characteristic of Viète's work but in fact quite foreign to Descartes'. No survey of the literature is possible here, but two representative discussions are worth mentioning.

Mahoney argues in "The Beginnings of Algebraic Thought" that the modern algebraic mode of thought has three main characteristics: it has an operative symbolism, it focuses on mathematical relations rather than on mathematical objects, and it is independent of "the intuitive ontology of the physical world." Greek mathematical thought, by contrast, involves essentially no symbolism, is focused (as we have seen) on mathematical objects, and is "strongly dependent on physical ontology."[49] According to Mahoney, Viète's work manifests the modern mode of thought, that is, as it has been characterized here, the top-down, essentially holistic understanding that is manifested in Descartes' geometry. Now Viète did of course have an operative symbolism. But he was also firmly rooted in the ancient tradition. In particular, he follows the ancients in focusing on mathematical objects, and it is because he does that his symbolism must be read as an uninterpreted calculus. That Viète has an operative symbolism is simply not sufficient to show that he has a modern concern with mathematical relations rather than the ancient's concern with mathematical objects. Consider a different case, that of the sign '0'. The fact that one has the sign '0' in one's arithmetic is not by itself enough to show that one has the modern number concept and thereby the notion of the number zero. Indeed, for centuries after its introduction, the sign '0' was treated not as a sign for a number (again, if numbers are collections of units then there can be no *number* zero because there can be no such collection of units) but instead as a mark to indicate the absence of number. That sign could be read as a sign for a number, as a *numeral*, only in light of the modern number concept according to which numbers are nodes in the antecedently given whole of computational space. Viète's symbolism similarly can be read as a language focusing on relations rather than on objects only in light of the modern conception of mathematics, a conception that there is no reason to believe Viète himself achieved.

Gaukroger's discussion of Descartes in *Cartesian Logic*, although not concerned with the relationship of Viète and Descartes, manifests the opposite tendency, the tendency to read Descartes as a formalist. According to Gaukroger, algebraic thinking just is a species of formal reasoning. It never occurs to him that Descartes was not a formalist in mathematics, and as a result he cannot help but see "a fundamental problem in Descartes's account": "How can Descartes hold up algebra as a model on the one hand, and deride attempts to provide a formal account of inference on the other?"[50] As Gaukroger reads him, Descartes performed just the sort of abstractive move in algebra that we have suggested is performed, for good reason, in Viète's work, but then strangely failed to recognize that it could be made again in the domain of logic. There is, however, no reason to hold that Descartes was a formalist in mathematics. The "fundamental problem" that Gaukroger finds in Descartes' account lies not in that account but in a reading of it that fails to take into account the transformation Descartes effects in our understanding of the most fundamental nature of the subject matter of geometry.

I have suggested that Descartes was able to achieve a modern conception of quantity and a modern understanding of curves in virtue of a radical reorientation in his thinking, that Viète *could have* achieved the results Descartes achieves in the *Geometry* only if he too had made the transition from a bottom-up to a top-down conception of mathematical space. History has bequeathed us a figure who can seem to provide a clear counterexample to the claim, namely, Fermat, at once a faithful follower of Viète and the co-discoverer, with Descartes, of the fundamental principle of analytic geometry: that a curve is expressed by an equation in two unknowns. I conclude with a brief remark, a conjecture as it were, regarding the orientation of Fermat's thinking as it bears on the question at issue here.

As Viète did, Fermat understood his work to be directly continuous with that of the ancients, and yet he was able to extend Viète's analytic method to the case of equations in two (or more) unknowns, employing, as Descartes did, the idea of two principal lines forming a coordinate system relative to which an equation in two unknowns could be seen to express a curve. Fermat also, in his practice though not explicitly, seems to have solved the problem of the dimensionality of geometric operations in just the way Descartes did, by appeal to an arbitrary unit.[51] My conjecture is that we will be able to understand these apparently modern developments, despite Fermat's apparently classical orientation, in terms of the Eulerian idea that one's pencil can surpass one's geometrical understanding in intelligence. I conjecture, in other words, that Fermat saw that, at least on paper, Viète's symbolism could be extended to cases involving two (or more) unknowns but could give no intuitive content to that extension. Just as in the case of '0' and, as we have seen, in the case of the symbolic notation of algebra, a coordinate system of lines relative to which a curve can be described does not by itself settle the question how it is to be regarded; the mere fact that one uses a coordinate system does not by itself show that one has the modern conception of a curve. For as we have seen, just as an ordinary map showing the relative locations of landmarks can be read in either of two ways—either bottom-up commencing with landmarks, as showing their relative positions one to another, or top-down commencing with a conception of space as a given irreducible whole, as showing the absolute positions of landmarks in that space—so a depiction of a curve within a coordinate system can be read in either of two ways, either bottom-up or top-down. Descartes has the latter, top-down, conception of a curve in space. My conjecture is that Fermat had the former, bottom-up, conception, that despite his technical achievements he did not win through to the modern understanding of a curve, and as a result, that his findings were strangely opaque to him, opaque because they remained situated within the ancient orientation. (Might this explain, at least in part, his resistance to the publication of his results?) The overall structure of Fermat's *Tripartite dissertation*, written in response to Descartes' *Geometry*, suggests just such a view. That treatise is characterized by (in Bos' words) a "concentration on technical algebraic results and disregard for methodological aspects" of Descartes' program—an attitude, Bos suggests, that "was to be repeated by many seventeenth-century readers of the book."[52] Such a pattern of response would be expected of one who had achieved the relevant algebraic results but without the transformed understanding that informs Descartes' program in the *Geometry* and renders those results transparent to thought. If that is right, and of course much more would need to be said to show that it is, then Fermat's work does not contradict but instead reinforces the claim that has been made here, that modern mathematics emerges (for the first time) in Descartes' *Geometry* out of a radical and thoroughgoing transformation, a metamorphosis in our understanding of the subject matter of mathematics.

NOTES

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1. Henk J.M. Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* (New York: Springer-Verlag, 2001), p. 154.
2. Michael S. Mahoney, "The Beginnings of Algebraic Thought in the Seventeenth Century," in *Descartes: Philosophy, Mathematics and Physics*, ed. Stephen Gaukroger (New Jersey: Barnes and Noble Books, 1980), p. 144.
3. I.G. Bashmakova and G.S. Smirnova, "The Literal Calculus of Viète and Descartes," *The American Mathematical Monthly* 106:3 (1999), pp. 260-3, 261.
4. Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*, trans. Eva Brann (Cambridge: MIT Press, 1968), p. 183.
5. François Viète, *The Analytic Art*, trans. T. Richard Witmer (Kent: Kent State University Press, 1983), Introduction, chap. 7, p. 29. Witmer's translation omits both the dedication and the notes by van Schooten that are included in the original. The dedication, and the Introduction with van Schooten's notes, as translated by J. Winfree Smith, are included as an appendix to Klein's *Greek Mathematical Thought*, pp. 313-53.
6. François Viète, *Opera Mathematica*, edited with notes by Frans van Schooten (Leyden, 1646). A facsimile reprint has been issued (Hildesheim: Georg Olms Verlag, 1970).
7. A Latin translation of the *Collection* appeared in 1588-1589. It is likely that Viète had access to the recovered original before then (see Klein, *Greek Mathematical Thought*, p. 259n. 214).
8. The first six books of the *Arithmetica* were rediscovered in 1462 and published (in Latin) around 1560.
9. The *Elements* was first printed in Latin in 1482.
10. I here follow Michael S. Mahoney's translation in "Another Look at Greek Geometrical Analysis," *Archive for History of Exact Sciences* 5 (1968), pp. 318-48, 322.
11. See I.G. Bashmakova, *Diophantus and Diophantine Equations*, trans. Abe Shenitzer and updated by Joseph Silverman (Washington, D.C.: The Mathematical Association of America, 1997), p. 8.
12. On the question of the notion of negative number in play in Diophantus, see Bashmakova, *Diophantus and Diophantine Equations*, pp. 5-6.
13. Klein, *Greek Mathematical Thought*, pp. 330-1n. 22.

14. Viète, *The Analytic Art*, Introduction, chap. 2, p. 15.

15. Where this difference does not need to be marked, as in setting out the rules governing addition, subtraction, multiplication, and division of "species," Viète uses the two sorts of letters indifferently.

16. See Mahoney, "Beginnings," p. 148.

17. See Michael S. Mahoney, *The Mathematical Career of Pierre de Fermat 1601-1665* (Princeton: Princeton University Press, 1973), pp. 32-3; also his Ph.D. dissertation, "The Royal Road: the development of algebraic analysis from 1550 to 1650, with special reference to the work of Pierre de Fermat" (Ph.D. diss., Princeton University, 1967), chap. 3.

18. This is widely (if at times only implicitly) recognized. See, for example, Bos, *Redefining Geometrical Exactness*, p. 151; Mahoney, *The Mathematical Career of Pierre de Fermat*, p. 149; Carl B. Boyer, *A History of Mathematics*, second edition, revised by Uta C. Merzbach (New York: John Wiley and Sons, 1989), p. 305; and David Eugene Smith, *History of Mathematics*, vol. 2 (Boston: Ginn and Co., 1925), pp. 449, 465. In "The Literal Calculus of Viète and Descartes," p. 261, Bashmakova and Smirnova instead take these annotations to function just as Viète's '*quadratum*' and '*cubum*' do, to raise a root to a power.

19. Viète introduces the law of homogeneity in chap. 3 of the Introduction to *The Analytic Art*, claiming that "much of the fogginess and obscurity of the old analysts is due to their not having been attentive" to it and its consequences (p. 16).

20. See, for example, chap. 12 of the first of the "Two Treatises on the Understanding and Amendment of Equations." (In Theorem II, 'B *plani-quad.*' should be 'E *plani-quad.*'; the error is corrected in Witmer's translation, p. 192.)

21. Aristotle, *Posterior Analytics* I.5, 74a4-5.

22. Mahoney, *The Mathematical Career of Pierre de Fermat*, p. 39.

23. Bos, *Redefining Geometrical Exactness*, p. 148.

24. Viète, *The Analytic Art*, Introduction, chap. 8, p. 32.

25. The discussion of Descartes' geometry in this section and the next has been substantially improved by very helpful and generous comments from Kenneth Manders. Any errors that remain are my own.

26. Sextus Empiricus, *Outlines of Scepticism*, trans. Julia Annas and Jonathan Barnes (Cambridge: Cambridge University Press, 1994), book II, §196.

27. All references to Descartes' works are to the Adam-Tannery (AT) edition of the *Œuvres de Descartes* (Paris: J. Vrin, 1964-76), and, except for those to the *Geometry*, to the translations by John Cottingham, Robert Stoothoff, and Dugald Murdoch (and in the case of the correspondence, Anthony Kenny) in *The Philosophical Writings of Descartes* (Cambridge:

Cambridge University Press, 1984-91); hereafter cited as CSM followed by volume and page numbers.

28. Descartes remarks in the *Discourse* that dialectic is “of less use for learning things than for explaining to others the things one already knows” (AT VI 17; CSM I 119).

29. Sextus Empiricus, *Outlines*, book II, §165.

30. J.S. Mill, *A System of Logic Ratiocinative and Inductive*, eighth edition, ed. J.M. Robson (Toronto: University of Toronto Press, 1973), book 2, chap. 3, §4.

31. The precise nature of this entailment is subtle because the first truth that Descartes knows is not ‘I think’ but ‘I exist’. Janet Broughton, in *Descartes’ Method of Doubt* (Princeton: Princeton University Press, 2002), chaps. 6 and 7, cogently argues that Descartes comes at this point in his meditations to see that his existence is a necessary condition of his meditating, that is, of his thinking. It is by reflecting on the significance of what one is doing, on what must be true given what one is doing, that one comes to recognize indubitably that one (oneself) exists. The entailment is governed by the constitutive rational relation between one’s activity and one’s existence.

32. This passage has been interpreted in a variety of ways. See, for instance, Stephen Gaukroger, *Cartesian Logic: An Essay on Descartes’ Conception of Inference* (Oxford: Clarendon Press, 1989), chap. 3; and E.M. Curley, “Analysis in the *Meditations*: The Quest for Clear and Distinct Ideas,” in *Essays on Descartes’* Meditations, ed. A.O. Rorty (Berkeley: University of California Press, 1986), pp. 153-76.

33. In the *Discourse*: “of all those who have hitherto sought after truth in the sciences, mathematicians alone have been able to find any demonstrations—that is to say, certain and evident reasonings” (AT VI 19; CSM I 120).

34. According to Descartes, the analysis of the ancients “is so closely tied to the examination of figures that it cannot exercise the intellect without greatly tiring the imagination,” and the algebra of the moderns (including Viète’s?) “so confined to certain rules and symbols that the end result is a confused and obscure art which encumbers the mind, rather than a science which cultivates it” (AT VI 17-8; CSM I 119-20).

35. Descartes, *The Geometry of René Descartes*, trans. David Eugene Smith and Marcia L. Latham (New York: Dover, 1954), p. 4 (AT VI 298); hereafter cited as G followed by page number. The diagram is Descartes’.

36. Again, the diagram is Descartes’ (AT VI 302; G 12).

37. That Euclid’s geometry, at least parts of it, can be recast in algebraic form is crucial, as we will see; but it is equally important to recognize that this *is* a recasting.

38. Euclid, *The Thirteen Books of the Elements*, trans. Sir Thomas Heath (New York: Dover, 1956).

39. The diagram is that provided in the *Elements*, p. 241.

40. That is not to say that homogeneity has no role to play in Descartes' geometry. The point is only that there is no general need for homogeneity in an equation of Descartes' geometry as there is in an equation of Viète's.

41. Descartes assumes that the ratio of the products is unity although, as he notes, the problem is no more difficult if any other ratio is assumed.

42. The diagram is Descartes' (AT VI 309; G 27).

43. Tobias Dantzig, *Number: The Language of Science*, fourth edition (New York: The Free Press, 1954), p. 88.

44. Ibid., p. 85.

45. As Descartes notes, "there is not always a definite quantity corresponding to each root so conceived of"; some are "imaginary" (AT VI 380; G 175).

46. I owe this point to Kenneth Manders. It is made in his "Euclid or Descartes? Representation and Responsiveness," unpublished.

47. Unsurprisingly, Descartes also rejects the traditional distinction between mechanical and properly geometrical curves. According to him, mechanical curves are those that "must be conceived of as described by two separate movements whose relation does not admit of exact determination" (AT VI 317; G 44). Because the relationship between the two separate movements cannot be exactly determined, Descartes thinks, the curves themselves cannot be exactly and completely known.

48. Further evidence of this difference in orientation is indicated by the fact that whereas Viète emphasizes the continuity of his thought with that of the ancients, Descartes emphasizes instead the distance between his geometry and that of the ancients: "my intention has . . . [been] to pass beyond the ancients in every respect" (AT I 491; quoted in Timothy Lenoir, "Descartes and the Geometrization of Thought: The Methodological Background of Descartes' *Géométrie*," *Historia Mathematica* 6 [1979], pp. 355-79, 366). It is also, I think, significant that Descartes is emphatically not providing a textbook for the student in his *Discourse on Method*, but only a "history" or "fable" of the path of his own thinking. Although I cannot defend the claim here, Descartes' thought seems to be that one must achieve the required reorientation on one's own—perhaps with some guidance; it is not something that can be learned from a textbook of the sort Viète aimed to produce. (That the *Meditations* are written as meditations is obviously relevant to this point.)

49. Mahoney, "Beginnings," p. 142.

50. Gaukroger, *Cartesian Logic*, p. 72; see also p. 87.

51. See Mahoney, *The Mathematical Career of Pierre de Fermat*, p. 44.

52. Bos, *Redefining Geometrical Exactness*, p. 420.