11.A3 The general method for solving any problem (Descartes, Geometry)

[1] If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction—to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.
[2] We must find as many such equations as there are supposed to be unknown lines; but if, after considering everything involved, so many cannot be found, it is evident that the question is not entirely determined. In such a case we may choose arbitrarily lines of known length for each unknown line to which there corresponds no equation.
[3] If there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines; and so we must combine them until there remains a single unknown line which is equal to some known line, or whose square, cube, fourth power, fifth power, sixth power, etc., is equal to the sum or difference of two or more quantities, one of which is known, while the others consist of mean proportionals between unity and this square, or cube, or fourth power, etc., multiplied by other known lines. I may express this as follows:

$$z = b,$$

$$\text{or} \quad z^2 = -az + b^2,$$

$$\text{or} \quad z^3 = az^2 + b^2z - c^3,$$

$$\text{or} \quad z^4 = az^3 - c^3z + d^4, \quad \text{etc.}$$

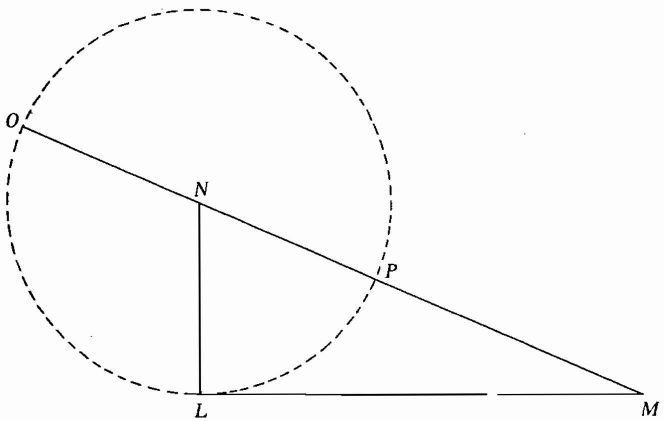
That is, z, which I take for the unknown quantity, is equal to b; or, the square of z is equal to the square of b diminished by a multiplied by z; or, the cube of z is equal to a multiplied by the square of z, plus the square of b multiplied by z, diminished by the cube of c; and similarly for the others.
[4] Thus, all the unknown quantities can be expressed in terms of a single quantity, whenever the problem can be constructed by means of circles and straight lines, or by conic sections, or even by some other curve of degree not greater than the third or fourth.
[5] But I shall not stop to explain this in more detail, because I should deprive you of the pleasure of mastering it yourself, as well as of the advantage of training your mind

by working over it, which is in my opinion the principal benefit to be derived from this science. Because, I find nothing here so difficult that it cannot be worked out by any one at all familiar with ordinary geometry and with algebra, who will consider carefully all that is set forth in this treatise.

[6] I shall therefore content myself with the statement that if the student, in solving these equations, does not fail to make use of division wherever possible, he will surely reach the simplest terms to which the problem can be reduced.

[7] And if it can be solved by ordinary geometry, that is, by the use of straight lines and circles traced on a plane surface, when the last equation shall have been entirely solved there will remain at most only the square of an unknown quantity, equal to the product of its root by some known quantity, increased or diminished by some other quantity also known. Then this root or unknown line can easily be found. For example, if I have $z^2 = az + b^2$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN, equal to $\frac{1}{2}a$, that is, to half the other known quantity which was multiplied by z, which I supposed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z. This is expressed in the following way:

$$z = \tfrac{1}{2}a + \sqrt{\tfrac{1}{4}a^2 + b^2}\,.$$



11.A4 Pappus on the locus to three, four or several lines (Pappus, Collection)

Apollonius, who completed the four books of Euclid's *Conics* and added another four, gave us eight books of *Conics*. Aristaeus, who wrote the still extant five books of *Solid Loci* supplementary to the *Conics*, called the three conics sections of an acute-angled, right-angled and obtuse-angled cone respectively. [...] Apollonius says in his third book that the 'locus with respect to three or four lines' had not been fully worked out by Euclid, and in fact neither Apollonius himself nor anyone else could have added anything to what Euclid wrote, using only those properties of conics which had been proved up to Euclid's time; as Apollonius himself bears witness when he says that the locus could not be fully investigated without the propositions that he had been compelled to work out for himself. Now Euclid regarded Aristaeus as deserving credit for his contributions to conics, and did not try to anticipate him or to overthrow his system; for he showed scrupulous fairness and exemplary kindness towards all who were able in any degree to advance mathematics, and was never offensive, but aimed at accuracy, and did not boast like the other. Accordingly he wrote so much about the locus as was possible by means of the *Conics* of Aristaeus, but did not claim finality for his proofs. If he had done so, we should have been obliged to censure him, but as things are he is in no wise to blame, seeing that Apollonius himself is not called to account, though he left the most part of his *Conics* incomplete. Moreover Apollonius was able to add the lacking portion of the theory of the locus through having become familiar beforehand with what had been written about it by Euclid, and through having spent much time with Euclid's pupils at Alexandria, whence he derived his scientific habit of mind.

Now this 'locus with respect to three and four lines', the theory of which he is so proud of having expanded—though he ought rather to acknowledge his debt to the original author—is of this kind. If three straight lines be given in position, and from one and the same point straight lines be drawn to meet the three straight lines at given angles, and if the ratio of the rectangle contained by two of the straight lines towards the square on the remaining straight line be given, then the point will lie on a solid locus given in position, that is on one of the three conic sections. And if straight lines be drawn to meet at given angles four straight lines given in position, and the ratio of the rectangle contained by two of the straight lines so drawn towards the rectangle contained by the remaining two be given, then in the same way the point will lie on a conic section given in position.

If from any point straight lines be drawn to meet at given angles five straight lines given in position, and the ratio be given between the volume of the rectangular parallelepiped contained by three of them to the volume of the rectangular parallelepiped contained by the remaining two and a given straight line, the point will lie on a curve given in position. If there be six straight lines, and the ratio be given between the volume of the aforesaid solid formed by three of them to the volume of the solid formed by the remaining three, the point will again lie on a curve given in position. If there be more than six straight lines, it is no longer permissible to say 'if the ratio be given between some figure contained by four of them to some figure contained by the remainder', since no figure can be contained in more than three dimensions. It is true that some recent writers have agreed among themselves to use such expressions, but they have no clear meaning when they multiply the rectangle contained by these straight lines with the square on that or the rectangle contained by those. They might, however, have expressed such matters by means of the composition of ratios, and have given a general proof both for the aforesaid propositions and for further propositions after this manner: *If from any point straight lines be drawn to meet at given angles straight lines given in position, and there be given the ratio compounded of that which one straight line so drawn bears to another, that which a second bears to a second, that which a third bears to a third, and that which the fourth bears to a given straight line—if there be seven, or, if there be eight, that which the fourth bears to the fourth—the point will lie on a curve given in position;* and similarly, however many the straight lines be, and whether odd or even. Though, as I said, these propositions follow the locus on four lines, [geometers] have by no means solved them to the extent that the curve can be recognized.

11.A5 Descartes to Marin Mersenne (1632) (Descartes, letter)

In my last letter I did not thank you for the demonstration of the two geometrical means which you sent me: but I had not yet received your letters, and I must tell you that Mr. Mydorge also found the demonstration for them, since you got me to make the construction for them, and I never judged it to be difficult. I would prefer you to have proposed the construction by trisecting the angle, a method which, if I am not mistaken, I gave you at the same time as the other; for it is a little less easy, and Mr. Mydorge admitted to me that he had not been able to demonstrate it. But I should like it even better if they would practice attempting Pappus's proposition, for it is said that Mr. Mydorge has put a solution to it in his *Conics*; but those who, like me, have examined it a little closely, cannot easily be persuaded of this. I do not think that they could persuade Mr. Golius of it either. He told me he had once proposed it to Mr. Mydorge, as you may easily ascertain, if you wish to write to him about it.

11.A6 Descartes's solution to the Pappus problem (Descartes, Geometry)

This led me to try to find out whether, by my own method, I could go as far as they had gone.

First, I discovered that if the question be proposed for only three, four, or five lines, the required points can be found by elementary geometry, that is, by the use of the ruler and compasses only, and the application of those principles that I have already explained, except in the case of five parallel lines. In this case, and in the cases where there are six, seven, eight, or nine given lines, the required points can always be found by means of the geometry of solid loci, that is, by using some one of the three conic sections. Here, again, there is an exception in the case of nine parallel lines. For this and the cases of ten, eleven, twelve, or thirteen given lines, the required points may be found by means of a curve of degree next higher than that of the conic sections. Again, the case of thirteen parallel lines must be excluded, for which, as well as for the cases of fourteen, fifteen, sixteen, and seventeen lines, a curve of degree next higher than the preceding must be used; and so on indefinitely.

Next, I have found that when only three or four lines are given, the required points lie not only all on one of the conic sections but sometimes on the circumference of a circle or even on a straight line.

When there are five, six, seven, or eight lines, the required points lie on a curve of degree next higher than the conic sections, and it is impossible to imagine such a curve that may not satisfy the conditions of the problem; but the required points may possibly lie on a conic section, a circle, or a straight line. If there are nine, ten, eleven, or twelve lines, the required curve is only one degree higher than the preceding, but any such curve may meet the requirements, and so on to infinity.

Finally, the first and simplest curve after the conic sections is the one generated by the intersection of a parabola with a straight line in a way to be described presently.

I believe that I have in this way completely accomplished what Pappus tells us the ancients sought to do, and I will try to give the demonstration in a few words, for I am already wearied by so much writing.

Let $AB, AD, EF, GH, \ldots$ be any number of straight lines given in position, and let it be required to find a point C, from which straight lines $CB, CD, CF, CH, \ldots$ can be drawn, making given angles $CBA, CDA, CFE, CHG, \ldots$ respectively, with the given lines, and such that the product of certain of them is equal to the product of the rest, or at least such that these two products shall have a given ratio, for this condition does not make the problem any more difficult.

First, I suppose the thing done, and since so many lines are confusing, I may simplify matters by considering one of the given lines and one of those to be drawn (as, for example, AB and BC) as the principal lines, to which I shall try to refer all the others. Call the segment of the line AB between A and B, x, and call BC, y. Produce all the other given lines to meet these two (also produced if necessary) provided none is parallel to either of the principal lines. Thus, in the figure, the given lines cut AB in the points A, E, G, and cut BC in the points R, S, T.

Now, since all the angles of the triangle ARB are known, the ratio between the sides AB and BR is known. If we let $AB{:}BR = z{:}b$, since $AB = x$, we have $RB = \frac{bx}{z}$; and since B lies between C and R, we have $CR = y + \frac{bx}{z}$. (When R lies between C and B, CR is equal to $y - \frac{bx}{z}$, and when C lies between B and R, CR is equal to $-y + \frac{bx}{z}$.) Again, the three angles of the triangle DRC are known, and therefore the ratio between the sides CR and CD is determined. Calling this ratio $z{:}c$, since $CR = y + \frac{bx}{z}$, we have $CD = \frac{cy}{z} + \frac{bcx}{z^2}$. Then, since the lines AB, AD, and EF are given in position, the distance from A to E is known. If we call this distance k, then $EB = k + x$; although $EB = k - x$ when B lies between E and A, and $E = -k + x$ when E lies between A and B. Now the angles of the triangle ESB being given, the ratio of BE to BS is known. We may call this ratio $z{:}d$. Then $BS = \frac{dk + dx}{z}$ and $CS = \frac{zy + dk + dx}{z}$. When S lies between B and C we have $CS = \frac{zy - dk - dx}{z}$, and when C lies between B and S we have $CS = \frac{-zy + dk + dx}{z}$. The angles of the triangle FSC are known, and hence, also the ratio of CS to CF, or $z{:}e$. Therefore, $CF = \frac{ezy + dek + dex}{z^2}$. Likewise, AG or l is given, and $BG = l - x$. Also, in triangle BGT, the ratio of BG to BT, or $z{:}f$, is known. Therefore, $BT = \frac{fl - fx}{z}$ and $CT = \frac{zy + fl - fx}{z}$. In triangle TCH, the ratio of TC to CH, or $z{:}g$, is known, whence $CH = \frac{gzy + fgl - fgx}{z^2}$.

And thus you see that, no matter how many lines are given in position, the length of any such line through C making given angles with these lines can always be expressed by three terms, one of which consists of the unknown quantity y multiplied or divided by some known quantity; another consisting of the unknown quantity x multiplied or divided by some other known quantity; and the third consisting of a known quantity. An exception must be made in the case where the given lines are parallel either to AB (when the term containing x vanishes), or to CB (when the term containing y vanishes). This case is too simple to require further explanation. The signs of the terms may be either $+$ or $-$ in every conceivable combination.

You also see that in the product of any number of these lines the degree of any term containing x or y will not be greater than the number of lines (expressed by means of x and y) whose product is found. Thus, no term will be of degree higher than the second if two lines be multiplied together, nor of degree higher than the third, if there be three lines, and so on to infinity.

Furthermore, to determine the point C, but one condition is needed, namely, that the product of a certain number of lines shall be equal to, or (what is quite as simple), shall bear a given ratio to the product of certain other lines. Since this condition can be expressed by a single equation in two unknown quantities, we may give any value we please to either x or y and find the value of the other from this equation. It is obvious that when not more than five lines are given, the quantity x, which is not used to express the first of the lines can never be of degree higher than the second.

Assigning a value to y, we have $x^2 = \pm ax \pm b^2$, and therefore x can be found with ruler and compasses, by a method already explained. If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn.