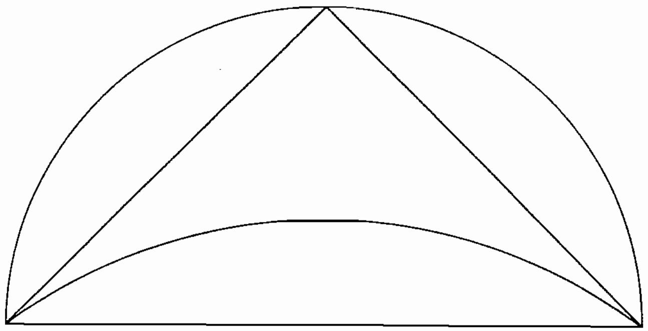
2.B Hippocrates' Quadrature of Lunes

A further extract from Eudemus's *History of Geometry* has been handed down by the commentator Simplicius (sixth century AD), concerning the quadrature of lunes—the area of crescent-like figures—by Hippocrates (late fifth century BC). 'This is one of the most precious sources for the history of Greek geometry before Euclid', in the judgement of Sir Thomas Heath (*History of Greek Mathematics*, Volume I, 1921, p.182), and is especially interesting as recording in detail so early an attempt on the area of figures with *curved* sides, from which the theorems that must have been known in order to establish this rigorously can be inferred. Simplicius said that he was copying out Eudemus 'word for word, adding only for the sake of clearness a few things taken from Euclid's *Elements*'. In the passage as given here, these additions (as far as they can be determined) have been removed, so what remains is believed to be close to the original Eudemus. How accurate an account Eudemus, writing a century after Hippocrates, gave of this work is hard to say.

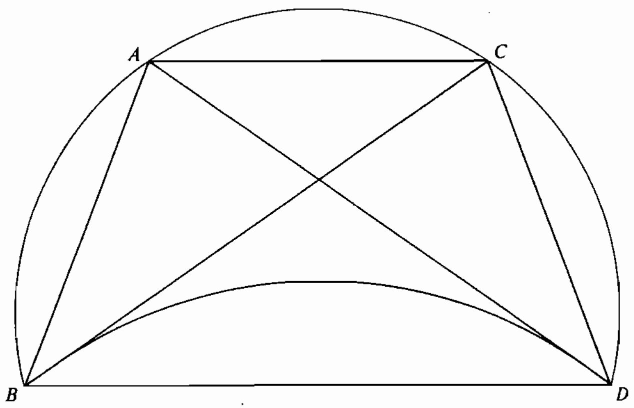
[1] The quadratures of lunes, which seemed to belong to an uncommon class of propositions by reason of the close relationship to the circle, were first investigated by Hippocrates, and seemed to be set out in correct form; therefore we shall deal with them at length and go through them. He made his starting-point, and set out as the first of the theorems useful to his purpose, that similar segments of circles have the same ratios as the squares on their bases. And this he proved by showing that the squares on the diameters have the same ratios as the circles.

[2] Having first shown this he described in what way it was possible to square a lune whose outer circumference was a semicircle. He did this by circumscribing about a right-angled isosceles triangle a semicircle and about the base a segment of a circle similar to those cut off by the sides. Since the segment about the base is equal to the sum of those about the sides, it follows that when the part of the triangle above the segment about the base is added to both the lune will be equal to the triangle. Therefore the lune, having been proved equal to the triangle, can be squared. In this way, taking a semicircle as the outer circumference of the lune, Hippocrates readily squared the lune.

[3] Next in order he assumes an outer circumference greater than a semicircle obtained by constructing a trapezium having three sides equal to one another while

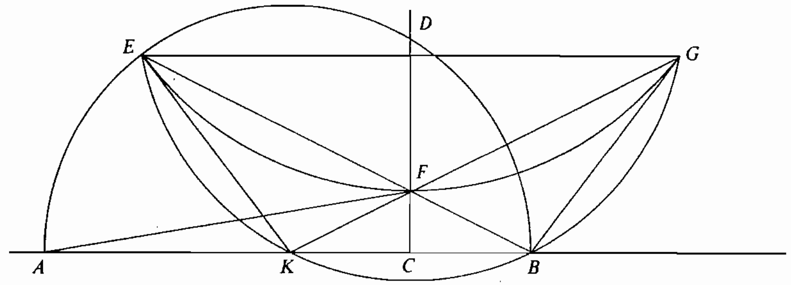


one, the greater of the parallel sides, is such that the square on it is three times the square on each of those sides, and then comprehending the trapezium in a circle and circumscribing about its greatest side a segment similar to those cut off from the circle by the three equal sides. That the said segment is greater than a semicircle is clear if a diagonal is drawn in the trapezium. For this diagonal, subtending two sides of the trapezium, must be such that the square on it is greater than double the square on one of the remaining sides. Therefore the square on *BC* is greater than double the square on either *BA*, *AC*, and therefore also on *CD*. Therefore the square on *BD*, the greatest of the sides of the trapezium, must be less than the sum of the squares on the diagonal and that one of the other sides which is subtended by the said greatest side together with the diagonal. For the squares on *BC*, *CD* are greater than three times, and the square on *BD* is equal to three times, the square on *CD*. Therefore the angle standing on the greatest side of the trapezium is acute. Therefore the segment in which it is is greater than a semicircle. And this segment is the outer circumference of the lune.



[4] If the outer circumference were less than a semicircle, Hippocrates solved this also, using the following preliminary construction. Let there be a circle with diameter *AB* and centre *K*. Let *CD* bisect *BK* at right angles; and let the straight line *EF* be placed between this and the circumference verging towards *B* so that the square on it is one-and-a-half times the square on one of the radii. Let *EG* be drawn parallel to *AB*, and from *K* let straight lines be drawn joining *E* and *F*. Let the straight line *KF* joined to *F* and produced meet *EG* at *G*, and again let straight lines be drawn from *B* joining *F* and *G*. It is then manifest that *EF* produced will pass through *B*—for by hypothesis *EF* verges towards *B*—and *BG* will be equal to *EK*.

This being so, I say that the trapezium *EKBG* can be comprehended in a circle.



Next let a segment of a circle be circumscribed about the triangle *EFG*; then clearly each of the segments on *EF*, *FG* will be similar to the segments on *EK*, *KB*, *BG*.

This being so, the lune so formed, whose outer circumference is *EKBG*, will be equal to the rectilineal figure composed of the three triangles *BFG*, *BFK*, *EKF*. For the segments cut off from the rectilineal figure, inside the lune, by the straight lines *EF*, *FG* are together equal to the segments outside the rectilineal figure cut off by *EK*, *KB*, *BG*. For each of the inner segments is one-and-a-half times each of the outer, because, by hypothesis, the square on *EF* is one-and-a-half times the square on the radius, that is, the square on *EK* or *KB* or *BG*. Inasmuch then as the lune is made up of the three segments and the rectilineal figure *less* the two segments—the rectilineal figure including the two segments but not the three—while the sum of the two segments is equal to the sum of the three, it follows that the lune is equal to the rectilineal figure. [. . .]

[5] Thus Hippocrates squared every lune, seeing that he squared not only the lune which has for its outer circumference a semicircle, but also the lune in which the outer circumference is greater, and that in which it is less, than a semicircle.

2.E Plato

The great renown attached to Plato (c. 427–348 BC) with regard to mathematics is clear from sources such as Eudemus' history (see 2.A1) and Aristoxenus' report of Plato's lecture on the Good (2.E8). Plato's own works not only spell this out further, but also provide some of our best evidence (in lieu of much else) for the mathematical concerns and activities of the early fourth century. The evidence provided is not always easy to interpret unequivocally, though; the passage from Theaetetus (2.E3), for example, has inspired much recent scholarly disputation on what it may tell us of the mathematical developments later found as Book X of Euclid's *Elements* (see 3.E1). The works of Plato are, of course, more than just a primary source for their own time. They are also of almost unparalleled influence on subsequent Western thought, nowhere more so than in his creation myth *Timaeus* (2.E5), whose vision of the creator-god as mathematician has underpinned a stream of cosmological belief down to our own time. And in the *Republic* (2.E2) we find the fullest early account of and rationale for the quadrivium subjects (even though in Plato's conception there are five of them, for reasons he explains in the text).

2.E1 Socrates and the slave boy

MENO: What do you mean when you say that we don't learn anything, but that what we call learning is recollection? Can you teach me that it is so?

SOCRATES: I have just said that you're a rascal, and now you ask me if I can teach you, when I say there is no such thing as teaching, only recollection. Evidently you want to catch me contradicting myself straight away.

MENO: No, honestly, Socrates, I wasn't thinking of that. It was just habit. If you can in any way make clear to me that what you say is true, please do.

SOCRATES: It isn't an easy thing, but still I should like to do what I can since you ask me. I see you have a large number of retainers here. Call one of them, anyone you like, and I will use him to demonstrate it to you.

MENO: Certainly. (*To a slave boy.*) Come here.

SOCRATES: He is a Greek and speaks our language?

MENO: Indeed yes—born and bred in the house.

SOCRATES: Listen carefully then, and see whether it seems to you that he is learning from me or simply being reminded.

MENO: I will.

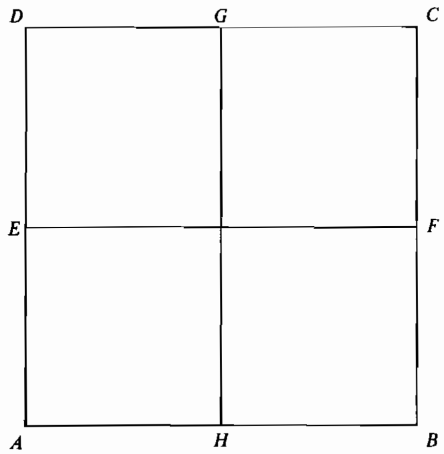
SOCRATES: Now boy, you know that a square is a figure like this?

(*Socrates begins to draw figures in the sand at his feet. He points to the square ABCD.*)

BOY: Yes.

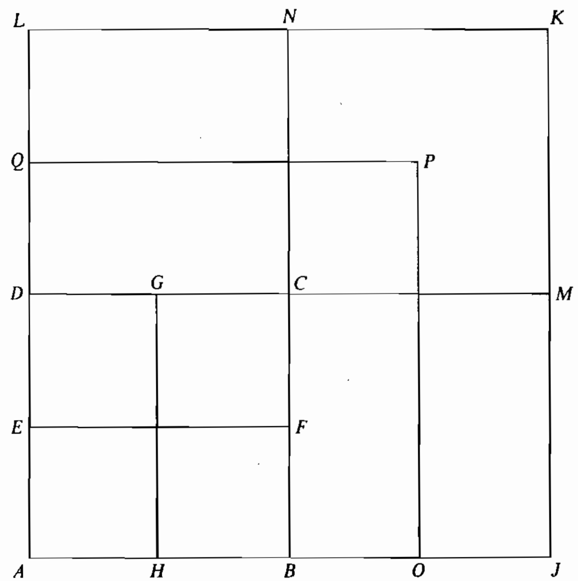
SOCRATES: It has all these four sides equal?

BOY: Yes.



SOCRATES: And these lines which go through the middle of it are also equal? (*The lines EF, GH.*)
BOY: Yes.
SOCRATES: Such a figure could be either larger or smaller, could it not?
BOY: Yes.
SOCRATES: Now if this side is two feet long, and this side the same, how many feet will the whole be? Put it this way. If it were two feet in this direction and only one in that, must not the area be two feet taken once?
BOY: Yes.
SOCRATES: But since it is two feet this way also, does it not become twice two feet?
BOY: Yes.
SOCRATES: And how many feet is twice two? Work it out and tell me.
BOY: Four.
SOCRATES: Now could one draw another figure double the size of this, but similar, that is, with all its sides equal like this one?
BOY: Yes.
SOCRATES: How many feet will its area be?
BOY: Eight.
SOCRATES: Now then, try to tell me how long each of its sides will be. The present figure has a side of two feet. What will be the side of the double-sized one?
BOY: It will be double, Socrates, obviously.
SOCRATES: You see, Meno, that I am not teaching him anything, only asking. Now he thinks he knows the length of the side of the eight-feet square.
MENO: Yes.
SOCRATES: But does he?
MENO: Certainly not.
SOCRATES: He thinks it is twice the length of the other.
MENO: Yes.

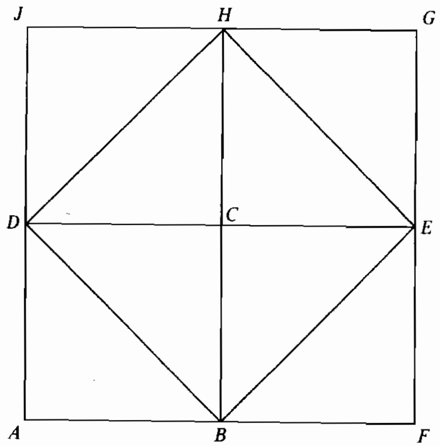
SOCRATES: Now watch how he recollects things in order—the proper way to recollect. You say that the side of double length produces the double-sized figure? Like this I mean, not long this way and short that. It must be equal on all sides like the first figure, only twice its size, that is eight feet. Think a moment whether you still expect to get it from doubling the side.
BOY: Yes, I do.
SOCRATES: Well now, shall we have a line double the length of this (*AB*) if we add another the same length at this end (*BJ*)?



BOY: Yes.
SOCRATES: It is on this line then, according to you, that we shall make the eight-feet square, by taking four of the same length?
BOY: Yes.
SOCRATES: Let us draw in four equal lines (*i.e. counting AJ, and adding JK, KL, and LA made complete by drawing in its second half LD*), using the first as a base. Does this not give us what you call the eight-feet figure?
BOY: Certainly.
SOCRATES: But does it contain these four squares, each equal to the original four-feet one?
(*Socrates has drawn in the lines CM, CN to complete the squares that he wishes to point out.*)
BOY: Yes.
SOCRATES: How big is it then? Won't it be four times as big?
BOY: Of course.

SOCRATES: And is four times the same as twice?
BOY: Of course not.
SOCRATES: So doubling the side has given us not a double but a fourfold figure?
BOY: True.
SOCRATES: And four times four are sixteen, are they not?
BOY: Yes.
SOCRATES: Then how big is the side of the eight-feet figure? This one has given us four times the original area, hasn't it?
BOY: Yes.
SOCRATES: And a side half the length gave us a square of four feet?
BOY: Yes.
SOCRATES: Good. And isn't a square of eight feet double this one and half that?
BOY: Yes.
SOCRATES: Will it not have a side greater than this one but less than that?
BOY: I think it will.
SOCRATES: Right. Always answer what you think. Now tell me: was not this side two feet long, and this one four?
BOY: Yes.
SOCRATES: Then the side of the eight-feet figure must be longer than two feet but shorter than four?
BOY: It must.
SOCRATES: Try to say how long you think it is.
BOY: Three feet.
SOCRATES: If so, shall we add half of this bit (*BO*, *half of BJ*) and make it three feet? Here are two, and this is one, and on this side similarly we have two plus one; and here is the figure you want.
(*Socrates completes the square AOPQ.*)
BOY: Yes.
SOCRATES: If it is three feet this way and three that, will the whole area be three times three feet?
BOY: It looks like it.
SOCRATES: And that is how many?
BOY: Nine.
SOCRATES: Whereas the square double our first square had to be how many?
BOY: Eight.
SOCRATES: But we haven't yet got the square of eight feet even from a three-feet side?
BOY: No.
SOCRATES: Then what length will give it? Try to tell us exactly. If you don't want to count it up, just show us on the diagram.
BOY: It's no use, Socrates, I just don't know.
SOCRATES: Observe, Meno, the stage he has reached on the path of recollection. At the beginning he did not know the side of the square of eight feet. Nor indeed does he know it now, but then he thought he knew it and answered boldly, as was appropriate—he felt no perplexity. Now however he does feel perplexed. Not only does he not know the answer; he doesn't even think he knows.
MENO: Quite true.
SOCRATES: Isn't he in a better position now in relation to what he didn't know?
MENO: I admit that too.

SOCRATES: So in perplexing him and numbing him like the sting-ray, have we done him any harm?
MENO: I think not.
SOCRATES: In fact we have helped him to some extent towards finding out the right answer, for now not only is he ignorant of it but he will be quite glad to look for it. Up to now, he thought he could speak well and fluently, on many occasions and before large audiences, on the subject of a square double the size of a given square, maintaining that it must have a side of double the length.
MENO: No doubt.
SOCRATES: Do you suppose then that he would have attempted to look for, or learn, what he thought he knew (though he did not), before he was thrown into perplexity, became aware of his ignorance, and felt a desire to know?
MENO: No.
SOCRATES: Then the numbing process was good for him?
MENO: I agree.
SOCRATES: Now notice what, starting from this state of perplexity, he will discover by seeking the truth in company with me, though I simply ask him questions without teaching him. Be ready to catch me if I give him any instruction or explanation instead of simply interrogating him on his own opinions.



(*Socrates here rubs out the previous figures and starts again.*)
Tell me, boy, is not this our square of four feet? (*ABCD.*) You understand?
BOY: Yes.
SOCRATES: Now we can add another equal to it like this? (*BCEF.*)
BOY: Yes.
SOCRATES: And a third here, equal to each of the others? (*CEGH.*)
BOY: Yes.
SOCRATES: And then we can fill in this one in the corner? (*DCHJ.*)
BOY: Yes.

SOCRATES: Then here we have four equal squares?
BOY: Yes.
SOCRATES: And how many times the size of the first square is the whole?
BOY: Four times.
SOCRATES: And we want one double the size. You remember?
BOY: Yes.
SOCRATES: Now does this line going from corner to corner cut each of these squares in half?
BOY: Yes.
SOCRATES: And these are four equal lines enclosing this area? (*BEHD*.)
BOY: They are.
SOCRATES: Now think. How big is this area?
BOY: I don't understand.
SOCRATES: Here are four squares. Has not each line cut off the inner half of each of them?
BOY: Yes.
SOCRATES: And how many such halves are there in this figure? (*BEHD*.)
BOY: Four.
SOCRATES: And how many in this one? (*ABCD*.)
BOY: Two.
SOCRATES: And what is the relation of four to two?
BOY: Double.
SOCRATES: How big is this figure then?
BOY: Eight feet.
SOCRATES: On what base?
BOY: This one.
SOCRATES: The line which goes from corner to corner of the square of four feet?
BOY: Yes.
SOCRATES: The technical name for it is 'diagonal'; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area.
BOY: That is so, Socrates.
SOCRATES: What do you think, Meno? Has he answered with any opinions that were not his own?
MENO: No, they were all his.
SOCRATES: Yet he did not know, as we agreed a few minutes ago.
MENO: True.
SOCRATES: But these opinions were somewhere in him, were they not?
MENO: Yes.
SOCRATES: So a man who does not know has in himself true opinions on a subject without having knowledge.
MENO: It would appear so.
SOCRATES: At present these opinions, being newly aroused, have a dream-like quality. But if the same questions are put to him on many occasions and in different ways, you can see that in the end he will have a knowledge on the subject as accurate as anybody's.
MENO: Probably.

SOCRATES: This knowledge will not come from teaching but from questioning. He will recover it for himself.
MENO: Yes.
SOCRATES: And the spontaneous recovery of knowledge that is in him is recollection, isn't it?
MENO: Yes.
SOCRATES: Either then he has at some time acquired the knowledge which he now has, or he has always possessed it. If he always possessed it, he must always have known; if on the other hand he acquired it at some previous time, it cannot have been in this life, unless somebody has taught him geometry. He will behave in the same way with all geometrical knowledge, and every other subject. Has anyone taught him all these? You ought to know, especially as he has been brought up in your household.
MENO: Yes, I know that no one ever taught him.
SOCRATES: And has he these opinions, or hasn't he?
MENO: It seems we can't deny it.
SOCRATES: Then if he did not acquire them in this life, isn't it immediately clear that he possessed and had learned them during some other period?
MENO: It seems so.
SOCRATES: When he was not in human shape?
MENO: Yes.
SOCRATES: If then there are going to exist in him, both while he is and while he is not a man, true opinions which can be aroused by questioning and turned into knowledge, may we say that his soul has been for ever in a state of knowledge? Clearly he always either is or is not a man.
MENO: Clearly.
SOCRATES: And if the truth about reality is always in our soul, the soul must be immortal, and one must take courage and try to discover—that is, to recollect—what one doesn't happen to know, or (more correctly) remember, at the moment.
MENO: Somehow or other I believe you are right.
SOCRATES: I think I am. I shouldn't like to take my oath on the whole story, but one thing I am ready to fight for as long as I can, in word and act: that is, that we shall be better, braver and more active men if we believe it right to look for what we don't know than if we believe there is no point in looking because what we don't know we can never discover.
MENO: There too I am sure you are right.