16.A2 From Gerolamo Saccheri's *Euclides Vindicatus* (1733)

Preface to the reader

Of all who have learned mathematics, none can fail to know how great is the excellence and worth of Euclid's *Elements*. As erudite witnesses here I summon Archimedes, Apollonius, Theodosius, and others almost innumerable, writers on mathematics even to our times, who use Euclid's *Elements* as foundations long established and wholly unshaken. But this so great celebrity has not prevented many, ancients as well as moderns, and among them distinguished geometers, maintaining they had found certain blemishes in these most beauteous nor ever sufficiently praised *Elements*. Three such flecks they designate, which now I name.

The first pertains to the definition of parallels and with it the axiom which in Clavius is the thirteenth of the First Book, where Euclid says:

If a straight line falling on two straight lines lying in the same plane, make with them two internal angles toward the same parts less than two right angles, these two straight lines infinitely produced toward those parts will meet each other.

No one doubts the truth of this proposition; but solely they accuse Euclid as to it, because he has used for it the name axiom, as if obviously from the right understanding of its terms alone came conviction. Whence not a few (withal retaining Euclid's definition of parallels) have attempted its demonstration from those propositions of Euclid's First Book alone which precede the twenty-ninth, wherein begins the use of the controverted proposition. [...]

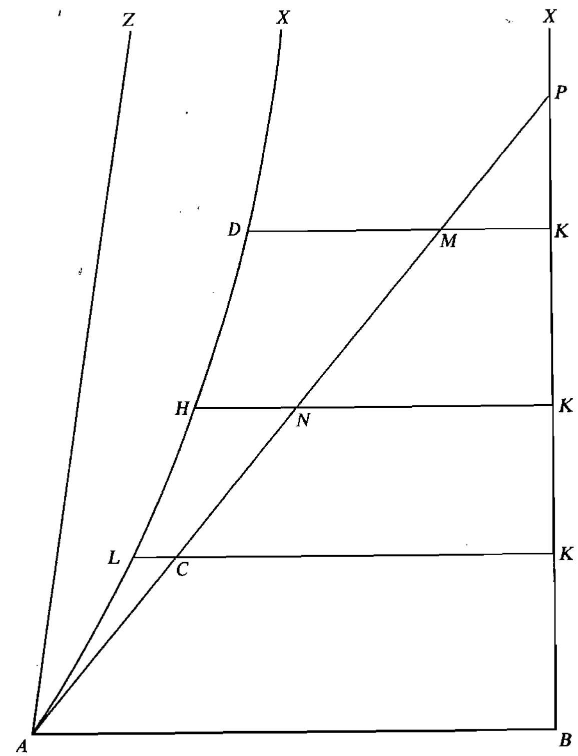
And this is enough to indicate to the reader what will be the material of the First Book of this work of mine: for a more complete explication of all that has been said will be given in the scholia after the twenty-first proposition of this Book.

I divide this Book into two parts. In the First Part I will imitate the antique geometers, and not trouble myself about the nature or the name of that line which at all its points is equidistant from a certain line supposed straight; but merely undertake without any *petitio principii* clearly to demonstrate the disputed Euclidean axiom. Therefore never will I use from those prior propositions of Euclid's First Book, not merely the twenty-seventh or the twenty-eighth, but not even the sixteenth or the seventeenth, except where clearly it is question of a triangle every way restricted.

Then in the Second Part for a new confirmation of the same axiom, I shall demonstrate that the line which at all its points is equidistant from an assumed straight line can only be a straight line. But every one sees that on this occasion the very first principles of all geometry are to be subjected to a rigid examination. [...]

Proposition XXXII

Now I say there is (in the hypothesis of acute angle) a certain determinate acute angle BAX drawn under which AX only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX.



Proof First it holds (from Cor. II to P. XXIX) that no determinate acute angle will be the greatest of all drawn under which a straight from the poind A meets the aforesaid BX at a finite distance.

Secondly, it holds in like manner that (in the hypothesis of acute angle) no acute angle will be the least of all drawn, under which a straight has a common perpendicular in two distinct points with BX; since indeed (from what precedes) there can be no determinate limit, such that there cannot be found, under a lesser angle constituted at the point A, a common perpendicular in two distinct points, which is less than any assignable length R.

And hence follows thirdly, that (in this hypothesis) there must be a certain determinate acute angle BAX, drawn under which AX so approaches ever more to BX, that only at an infinite distance does it meet it.

But further that this AX is a limit in part from within in part from without of each of the aforesaid classes of straights is proved thus. First, it agrees with those straights which meet BX at a finite distance since it also finally meets: but it differs, because it meets only at an infinite distance.

But secondly it also agrees with, and at the same time differs from those straights which have a common perpendicular in two distinct points with BX; because it also has a common perpendicular with BX; but in one and the same point X infinitely distant. But this latter ought to be considered demonstrated in P. XXVIII, as I point out in its corollary.

Therefore it holds, that (in the hypothesis of acute angle) there will be a certain determinate acute angle BAX, drawn under which AX only at an infinite distance meets BX, and thus is a limit in part from within, in part from without; on the one hand of all those which under lesser acute angles meet the aforesaid BX at a finite distance; on the other hand also of the others which under greater acute angles, even to a right angle inclusive, have a common perpendicular in two distinct points with BX.

Quod erat etc.

Proposition XXXIII

The hypothesis of acute angle is absolutely false; because repugnant to the nature of the straight line.

Proof From the foregoing theorem may be established, that at length the hypothesis of acute angle inimical to the Euclidean geometry has as outcome that we must recognize two straights AX, BX, existing in the same plane, which produced *in infinitum* toward the parts of the points X must run together at length into one and the same straight line, truly receiving, at one and the same infinitely distant point a common perpendicular in the same plane with them.

But since I am here to go into the very first principles, I shall diligently take care, that I omit nothing objected almost too scrupulously, which indeed I recognize to be opportune to the most exact demonstration.

16.B3 Nicolai Lobachevskii's theory of parallels (1840)

(a) Opening remarks

In geometry I find certain imperfections which I hold to be the reason why this science, apart from transition into analytics, can as yet make no advance from that state in which it has come to us from Euclid.

As belonging to these imperfections, I consider the obscurity in the fundamental concepts of the geometrical magnitudes and in the manner and method of representing the measuring of these magnitudes, and finally the momentous gap in the theory of parallels, to fill which all efforts of mathematicians have been so far in vain.

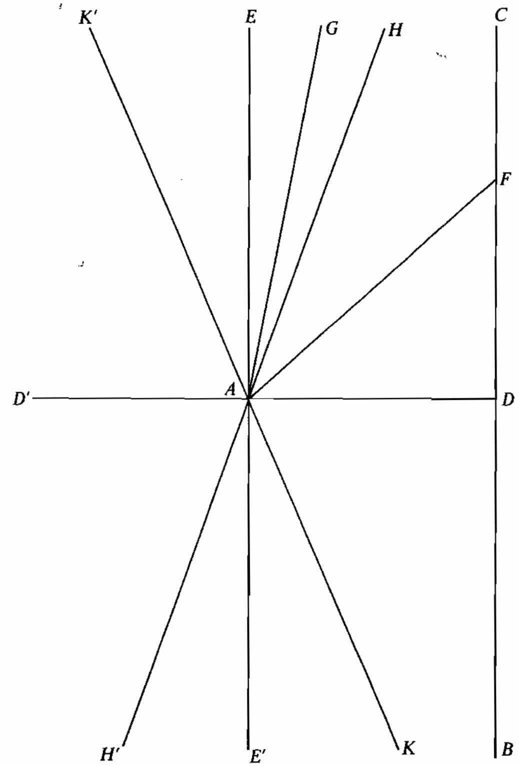
For this theory Legendre's endeavours have done nothing, since he was forced to leave the only rigid way to turn into a side path and take refuge in auxiliary theorems which he illogically strove to exhibit as necessary axioms. My first essay on the foundations of geometry I published in the Kasan *Messenger* for the year 1829. In the hope of having satisfied all requirements, I undertook hereupon a treatment of the whole of this science, and published my work in separate parts in the *Gelehrten Schriften der Universität Kasan* for the years 1836, 1837, 1838, under the title 'New Elements of Geometry, with a Complete Theory of Parallels'. The extent of this work perhaps hindered my countrymen from following such a subject, which since Legendre had lost its interest. Yet I am of the opinion that the Theory of Parallels should not lose its claim to the attention of geometers, and therefore I aim to give here the substance of my investigations, remarking beforehand that contrary to the opinion of Legendre, all other imperfections—for example, the definition of a straight line—show themselves foreign here and without any real influence on the theory of parallels. [...]

All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes—into *cutting* and *not-cutting*.

The *boundary lines* of the one and the other class of those lines will be called *parallel to the given line*.

From the point A let fall upon the line BC the perpendicular AD, to which again draw the perpendicular AE.

In the right angle EAD either will all straight lines which go out from the point A meet the line DC, as for example AF, or some of them, like the perpendicular AE, will not meet the line DC. In the uncertainty whether the perpendicular AE is the only line which does not meet DC, we will assume it may be possible that there are still other lines,



for example AG, which do not cut DC, how far soever they may be prolonged. In passing over from the cutting lines, as AF, to the not-cutting lines, as AG, we must come upon a line AH, parallel to DC, a boundary line, upon one side of which all lines AG are such as do not meet the line DC, while upon the other side every straight line AF cuts the line DC.

The angle HAD between the parallel HA and the perpendicular AD is called the parallel angle (angle of parallelism), which we will here designate by $\Pi(p)$ for $AD = p$.

If $\Pi(p)$ is a right angle, so will the prolongation AE' of the perpendicular AE likewise be parallel to the prolongation DB of the line DC, in addition to which we remark that in regard to the four right angles, which are made at the point A by the perpendiculars AE and AD, and their prolongations AE' and AD', every straight line which goes out from the point A, either itself or at least its prolongation, lies in one of the two right angles which are turned toward BC, so that except the parallel EE' all others, if they are sufficiently produced both ways, must intersect the line BC.

If $\Pi(p) < \frac{1}{2}\pi$, then upon the other side of AD, making the same angle $DAK = \Pi(p)$ will lie also a line AK, parallel to the prolongation DB of the line DC, so that under this assumption we must also make a distinction of *sides in parallelism*.

All remaining lines or their prolongations within the two right angles turned toward BC pertain to those that intersect, if they lie within the angle $HAK = 2\Pi(p)$ between the parallels; they pertain on the other hand to the non-intersecting AG, if they lie upon the other sides of the parallels AH and AK, in the opening of the two angles $EAH = \frac{1}{2}\pi - \Pi(p)$, $E'AK = \frac{1}{2}\pi - \Pi(p)$, between the parallels and EE' the perpendicular to AD. Upon the other side of the perpendicular EE' will in like manner the prolongations AH' and AK' of the parallels AH and AK likewise be parallel to BC; the remaining lines pertain, if in the angle $K'AH'$, to the intersecting, but if in the angles $K'AE$, $H'AE'$ to the non-intersecting.

In accordance with this, for the assumption $\Pi(p) = \frac{1}{2}\pi$, the lines can be only intersecting or parallel; but if we assume that $\Pi(p) < \frac{1}{2}\pi$, then we must allow two parallels, one on the one and one on the other side; in addition we must distinguish the remaining lines into non-intersecting and intersecting.

For both assumptions it serves as the mark of parallelism that the line becomes intersecting for the smallest deviation toward the side where lies the parallel, so that if AH is parallel to DC, every line AF cuts DC, how small soever the angle HAF may be.

16.D1 Fyodor Dostoevsky, from *The Brothers Karamazov* (1881)

'Joking? They said I was joking at the elder's yesterday. You see, my dear chap, there was an old sinner in the eighteenth century who delivered himself of the statement that if there were no God, it would have been necessary to invent him: *S'il n'existait pas Dieu il faudrait l'inventer*. And, to be sure, man has invented God. And what is so strange, and what would be so marvellous, is not that God actually exists, but that such an idea—the idea of the necessity of God—should have entered the head of such a savage and vicious animal as man—so holy it is, so moving and so wise and so much does it rebound to man's honour. So far as I'm concerned, I made up my mind long ago not to speculate whether man has created God or God has created man. Nor, of course, am I going to analyse all the modern axioms laid down by the Russian boys on that subject, all of them based on European hypotheses; for what is only an hypothesis there, becomes at once an axiom with a Russian boy, and not only with the boys but, I suppose, also with their professors, for Russian professors are quite often just the same Russian boys. And for this reason I'm going to disregard all the hypotheses. For what is it you and I are trying to do now? What I'm trying to do is to attempt to explain to you as quickly as possible the most important thing about me, that is to say, what sort of man I am, what I believe in and what I hope for—that's it, isn't it? And that's why I declare that I accept God plainly and simply. But there's this that has to be said: if God really exists and if he really has created the world, then, as we all know, he created it in accordance with the Euclidean geometry, and he created the human mind with the conception of only the three dimensions of space. And yet there have been and there still are mathematicians and philosophers, some of them indeed men of extraordinary genius, who doubt whether the whole universe, or, to put it more widely, all existence, was created only according to Euclidean geometry and they even dare to dream that two parallel lines which, according to Euclid, can never meet on earth, may meet somewhere in infinity. I, my dear chap, have come to the conclusion that if I can't understand even that, then how can I be expected to understand about God? I humbly admit that I have no abilities for settling such questions. I have a Euclidean, an earthly mind, and so how can I be expected to solve problems which are not of this world! And I advise you too, Alyosha, my friend, never to think about it, and least of all about whether there is a God or not. All these are problems which are entirely unsuitable to a mind created with the idea of only three dimensions. And so I accept God, and I accept him not only without reluctance, but, what's more, I accept his divine wisdom and his purpose—which are completely beyond our comprehension. I believe in the underlying order and meaning of life. I believe in the eternal harmony into which we are all supposed to merge one day. I believe in the Word to which the universe is striving and which itself was "with God" and which was God, and, well, so on and so forth, *ad infinitum*. Many words have been bandied about on this subject. So it would seem I'm on the right path—or am I? Anyway, you'd be surprised to learn, I think, that in the final result I refuse to accept this world of God's, and though I know that it exists, I absolutely refuse to admit its existence. Please understand, it is not God that I do not accept, but the world he has created. I do not accept God's world and I refuse to accept it. Let me put it another way: I'm convinced like a child that the wounds will heal and their traces will fade away, that all the offensive and comical spectacle of human contradictions will vanish like a pitiful mirage, like a horrible and odious invention of the feeble and infinitely puny Euclidean mind of man, and that in the world's finale, at the moment of eternal harmony, something so precious will happen and come to pass that it will suffice for all hearts, that it will allay all bitter resentments, that it will atone for all men's crimes, all the blood they have shed. It will suffice not only for the forgiveness but also for the justification of everything that has ever happened to men. Well, let it, let it all be and come to pass, but I don't accept it and I won't accept it! Let even the parallel lines meet and let me see them meet, myself—I shall see and I shall say that they've met, but I still won't accept it. That is the heart of the matter, so far as I'm concerned, Alyosha. That is where I stand. I'm telling you this in all seriousness. I deliberately began our talk as stupidly as I could, but I finished it with my confession, because that's all you want. You didn't want to hear about God, but only to find out what your beloved brother lived by. And I've told you.'