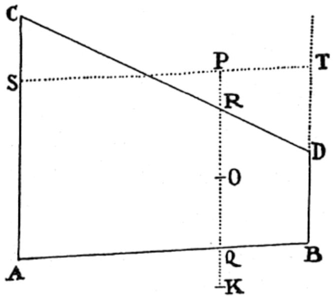
From: I. B. Cohen, A. Whitman, Isaac Newton: The Principia, 1993.

SECTION 5

To find orbits when neither focus is given

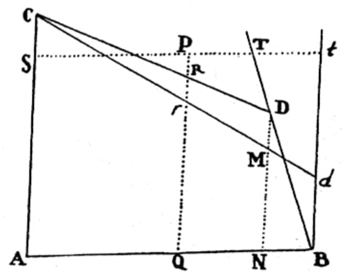
Lemma 17 *If four straight lines* PQ, PR, PS, *and* PT *are drawn at given angles from any point* P *of a given conic to the four indefinitely produced sides* AB, CD, AC, *and* DB *of some quadrilateral* ABDC *inscribed in the conic, one line being drawn to each side, the rectangle* PQ × PR *of the lines drawn to two opposite sides will be in a given ratio to the rectangle* PS × PT *of the lines drawn to the other two opposite sides.*

CASE 1. Let us suppose first that the lines drawn to opposite sides are parallel to either one of the other sides, say PQ and PR parallel to side AC, and PS and PT parallel to side AB. In addition, let two of the opposite sides, say AC and BD, be parallel to each other. Then the straight line which bisects those parallel sides will be one of the diameters of the conic and will bisect RQ also. Let O be the point in which RQ is bisected, and PO will be an ordinate to that diameter. Produce PO to K so that OK is equal to PO, and OK will be the ordinate on the opposite side of the diameter. Therefore, since points A, B, P, and K are on the conic and PK cuts AB at a given angle, the rectangle PQ × QK will be to the rectangle AQ × QB in a given ratio (by book 3, props. 17, 19, 21, and 23, of the *Conics* of Apollonius). But QK and PR are equal, inasmuch as they are differences of the equal lines OK and OP, and OQ and OR, and hence also the rectangles PQ × QK and PQ × PR are equal, and therefore the rectangle PQ × PR is to the rectangle AQ × QB, that is, to the rectangle PS × PT, in a given ratio. Q.E.D.

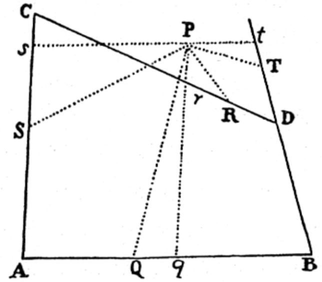


CASE 2. Let us suppose now that the opposite sides AC and BD of the quadrilateral are not parallel. Draw B*d* parallel to AC, meeting the straight line ST in *t* and the conic in *d*. Join C*d* cutting PQ in *r*; and draw DM parallel to PQ, cutting C*d* in M and AB in N. Now, because triangles BT*t* and DBN are similar, B*t* or PQ is to T*t* as DN to NB. So also R*r* is to AQ or PS as DM to AN. Therefore, multiplying the antecedents by the

antecedents and the consequents by the consequents, the rectangle PQ × R*r* is to the rectangle PS × T*t* as the rectangle ND × DM is to the rectangle AN × NB, and (by case 1) as the rectangle PQ × P*r* is to the rectangle PS × P*t*, and by separation [or dividendo] as the rectangle PQ × PR is to the rectangle PS × PT. Q.E.D.

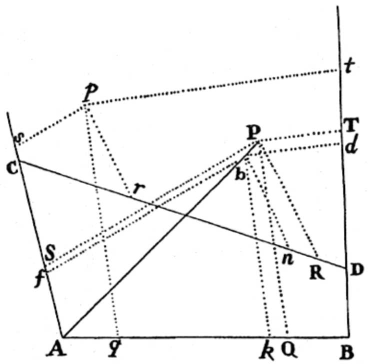


CASE 3. Let us suppose finally that the four lines PQ, PR, PS, and PT are not parallel to the sides AC and AB, but are inclined to them in any way whatever. In place of these lines draw P*q* and P*r* parallel to AC, and P*s* and P*t* parallel to AB; then because the angles of the triangles PQ*q*, PR*r*, PS*s*, and PT*t* are given, the ratios of PQ to P*q*, PR to P*r*, PS to P*s*, and PT to P*t* will be given, and thus the compound ratios of PQ × PR to P*q* × P*r*, and PS × PT to P*s* × P*t*. But, by what has been demonstrated above, the ratio of P*q* × P*r* to P*s* × P*t* is given, and therefore also the ratio of PQ × PR to PS × PT. Q.E.D.



Lemma 18 *With the same suppositions as in lem. 17, if the rectangle* PQ × PR *of the lines drawn to two opposite sides of the quadrilateral is in a given ratio to the rectangle* PS × PT *of the lines drawn to the other two sides, the point* P *from which the lines are drawn will lie on a conic circumscribed about the quadrilateral.*

Suppose that a conic is described through points A, B, C, D, and some one of the infinite number of points P, say *p*; I say that point P always lies on this conic. If you deny it, join AP cutting this conic in some point other than P, if possible, say in *b*. Therefore, if from these points *p* and *b* the straight lines *pq*, *pr*, *ps*, *pt* and *bk*, *bn*, *bf*, *bd* are drawn at given angles to the sides of the quadrilateral, then *bk* × *bn* will be to *bf* × *bd* as (by lem. 17) *pq* × *pr* is to *ps* × *pt*, and as (by hypothesis) PQ × PR is to PS × PT. Also, because the quadrilaterals *bk*A*f* and PQAS are similar, *bk* is to *bf* as PQ to PS. And therefore, if the terms of the previous

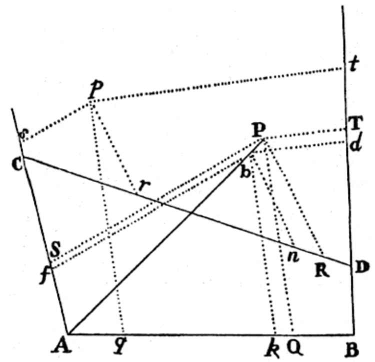


proportion are divided by the corresponding terms of this one, *bn* will be to *bd* as PR to PT. Therefore the angles of the quadrilateral D*nbd* are respectively equal to the angles of quadrilateral DRPT and the quadrilaterals are similar, and consequently their diagonals D*b* and DP coincide. And thus *b* falls upon the intersection of the straight lines AP and DP and accordingly coincides with point P. And therefore point P, wherever it is taken, falls on the assigned conic. Q.E.D.

Corollary. Hence if three straight lines PQ, PR, and PS are drawn at given angles from a common point P to three other straight lines given in position, AB, CD, and AC, one line being drawn to each of the other lines, and if the rectangle PQ × PR of two of the lines drawn is in a given ratio to the square of the third line PS, then the point P, from which the straight lines are drawn, will be located in a conic which touches lines AB and CD at A and C, and conversely. For let line BD coincide with line AC, while the position of the three lines AB, CD, and AC remains the same, and let line PT also coincide with line PS; then the rectangle PS × PT will come to be PS^2, and the straight lines AB and CD, which formerly cut the curve in points A and B, C and D, can no longer cut the curve in those points which now coincide, but will only touch it.

Scholium The term "conic" [or "conic section"] is used in this lemma in an extended sense, so as to include both a rectilinear section passing through the vertex of a cone and a circular section parallel to the base. For if point *p* falls on a straight line which joins points A and D or C and B, the conic section will turn into twin straight lines, one of which is the straight line on which point *p* falls and the other the straight line which joins the other two of the four points.

If two opposite angles of the quadrilateral, taken together, are equal to two right angles, and the four lines PQ, PR, PS, and PT are drawn to its



sides either perpendicularly or at any equal angles, and the rectangle PQ × PR of two of the lines drawn is equal to the rectangle PS × PT of the other two, the conic will turn out to be a circle. The same will happen if the four lines are drawn at any angles and the rectangle PQ × PR of two of the lines drawn is to the rectangle PS × PT of the other two as the rectangle of the sines of the angles S and T, at which the last two lines PS and PT are drawn, is to the rectangle of the sines of the angles Q and R, at which the first two lines PQ and PR are drawn.

In the other cases the locus of point P will be some one of the three figures that are commonly called conic sections [or conics]. In place of the quadrilateral ABCD, however, there can be substituted a quadrilateral whose two opposite sides decussate each other as diagonals do. But also, one or two of the four points A, B, C, and D can go off to infinity, and in this way the sides of the figure which converge to these points can turn out to be parallel, in which case the conic will pass through the other points and will go off to infinity in the direction of the parallels.