Euclidean Geometry and Physical Space

David E. Rowe

It takes a good deal of historical imagination to picture the kinds of debates that accompanied the slow process which ultimately led to the acceptance of non-Euclidean geometry a little more than a century ago. The difficulty stems mainly from our tendency to think of geometry as a branch of pure mathematics rather than as a science with deep empirical roots, the oldest natural science so to speak. For many of us, there is a natural tendency to think of geometry in idealized, Platonic terms. So to gain a sense of how late-nineteenth-century authorities debated over the true geometry of physical space, it may help to remember the etymological roots of the word geometry: *geo* plus *metria* literally meant to measure the earth. In fact, Herodotus reported that this was originally an Egyptian science; each spring the Egyptians were forced to remeasure the land after the Nile River flooded its banks, altering property lines. Among those engaged in this land survey were the legendary Egyptian rope-stretchers, the *harpedonaptai*, who were occasionally depicted in artwork relating to Egyptian ceremonials.

We are apt to smile when reading Herodotus's remarks, dismissing these as just another example of the Greek tendency to think of ancient Egypt as the fount of all wisdom. Herodotus was famous for repeating such lore, and here he was apparently confusing geometry with the science of geodesy, and the latter has little to do with the former; at least not anymore. We do not customarily think of circles, triangles, or the five Platonic solids as real figures: they are far too perfect, the products of the mind's eye. Of course, there is still plenty of room for disagreement. A formalist will stress that geometrical figures are mere conventions or, at best, images we attach to fictive objects that have no purpose other than to illustrate a system of ideas ultimately grounded in undefined terms and arbitrary axioms. A modern-day Platonist would vehemently object to this characterization, which puts too much emphasis on purely arbitrary constructions rather than conceiving of geometrical figures as idealized instantiations of perfect forms.

Those who might like to see what such a debate looked like around 1900 need only read the correspondence between the philosopher Gottlob Frege and the mathematician David Hilbert [Gabriel 1980]. Their dispute began when Frege wrote Hilbert after reading the opening pages of Hilbert's *Foundations of Geometry* [Hilbert 1899], the work that did so much to make the modern axiomatic approach fashionable. Although one of the founding fathers of modern logic, Frege simply could not accept Hilbert's contention that the fundamental concepts of geometry had no intrinsic meaning when seen *from a purely logical point of view*. For Frege, points, lines, and planes were not simply empty words. They were in some deep sense real; geometry was the science that studied the properties of real figures composed of them. Hilbert, to be sure, was by no means advocating a modern formalist approach to geometry that broke with the classical tradition. In essence, his axioms for Euclidean geometry were merely a refinement of those presented in Euclid's *Elements*. In 1905 he emphasized that "the aim of every science is . . . to set up a network of concepts based on axioms to which we are led naturally by *intuition and experience*" [Corry 2004, 124]. Hilbert thus recognized the empirical roots of geometrical knowledge, but he also emphasized that the question as to how and why Euclidean geometry conformed to our spatial perceptions lay outside the realm of mathematical and logical investiga-

Send submissions to **David E. Rowe,** Fachbereich 17—Mathematik, Johannes Gutenberg University, D55099 Mainz, Germany.

tions. For these, all that mattered at bottom was proving the consistency of a certain set of axioms. This reductionist viewpoint was sheer anathema in the eyes of Gottlob Frege.

Most mathematicians, to the extent that they grasped what was at stake, sided with Hilbert in this debate. Over the course of the next three decades the status of the continuum nevertheless played a major role in the larger foundations debates between formalists and intuitionists. Those rather esoteric discussions, however, left traditional realist assumptions about the nature of geometrical knowledge behind. For despite their strong differences, the proponents of formalism and intuitionism were both guided by their respective visions of pure mathematics, independent of its relevance to other disciplines, like astronomy and physics.

This suggests that a fundamental shift took place around 1900 regarding the status of geometrical knowledge. This reorientation was certainly profound, but it seems to have been rather quickly forgotten in the wake of other, even more dramatic developments. Soon afterward Einstein's general theory of relativity would lead to a flurry of new discussions about the interplay between space, time, and matter. Leading mathematicians like Hilbert and Hermann Weyl became strong proponents of Einstein's ideas, even as they sharply disagreed about epistemological issues relating to the mathematical continuum, a concept of central importance for the geometer.

Looking backward from the 1920s, it would seem that the opposing views of formalists and intuitionists actually reflect distinctly modern attitudes about the nature of geometrical knowledge that would have been scarcely thinkable prior to 1900. Up until then, geometry was always conceived as somehow wedded to a physical world that displayed discernible geometrical features. Take the developments that led to the birth of modern science in the seventeenth century: anyone who studies the works of Copernicus, Kepler, Galileo, or Newton cannot help but notice the deep impact of geometry on their conceptions of the natural world. But the same can be said of Gauss, whose career ought to make us rethink what Herodotus wrote about the Egyptian roots of the science of geometry.

Gauss, Measurement, and the Pythagorean Theorem

Gauss, after all, was not only a mathematician and astronomer, he was also a professional surveyor who at least occasionally waded through the marshy hinterlands of Hanover taking sightings in order to construct a net of triangles that would span this largely uncharted region [Bühler 1981, 95–103]. This work helped inspire a profound contribution to pure geometry: Gauss's study of the intrinsic geometry of surfaces, which helped launch a theory of measurement in geometry that opened the way to probing the geometry of space itself. Only about a half century earlier, two leading French mathematicians, Clairaut and Maupertuis, had studied the shape of the earth's surface, showing that it formed an oblate spheroid. As one moved northward, they discovered, the curvature of the earth flattened, just as Newtonian theory predicted. Maupertuis's celebrated expedition to Lapland brought him fame and the nickname of "the earth flattener." It also provided the French Academy with stunning proof that Descartes's theory of gravity could not be right, thereby overcoming the last major bastion of resistance against Newtonianism in France. Thus precise measurements of the earth's curvature had already exerted a deep impact on modern science.

In the 1820s Gauss took the measurement of the earth as his point of departure for an abstract theory of surfaces, asking whether and how a scientist could determine the curvature of an arbitrary surface through measurements made only along the surface itself, without knowing anything at all about the way in which the surface might be embedded in space. To talk about curvature as an intrinsic property of a surface requires a careful reconsideration of concepts like the measure of distances between points and angles between curves. So let's briefly review the historical role of measurement in geometry.

Consider the Pythagorean Theorem, a ubiquitous result familiar to many cultures and already found by Babylonian mathematicians more than a millennium before the time of Pythagoras. It tells us something fundamental about planar measurements in right triangles: the square on the hypotenuse equals the sum of the squares on the two other sides. Some have conjectured that the Egyptian *harpedonaptai*, whom Democritus once praised, used the converse of the Pythagorean Theorem to lay out right angles at the corners of temples and pyramids. The claimants suggest that these professional surveyors used a rope tied with 12 knots at equal distance from each other; by pulling the rope taut, they could form a 3-4-5 right triangle. It's a nice idea, but nothing more. Archaeologists can still measure the angles of Egyptian buildings, of course, but our access to the mathematical knowledge that lay behind the architectural splendours of ancient Egypt is highly limited. Papyri can easily disintegrate with time, and only two have been found that provide much insight into the mathematical methods of the time: the Rhind and Moscow papyri, which were presumably used as training manuals for Egyptian scribes. Neither contains anything close to the Pythagorean Theorem.

Historians of Mesopotamian mathematics have been luckier; they have had plenty of source material available ever since it became possible to decipher the clay cuneiform tablets archaeologists began turning up a little more than a century ago. Some of the Babylonian mathematical texts reveal not just a passing familiarity with the Pythagorean Theorem but even a masterful use of it for numerical computations, like approximating the value of $\sqrt{2}$ or calculating Pythagorean triples. Herodotus claimed that "the Greeks learnt the πολος, the *gnomon*, and the twelve parts of the day from the Babylonians" [Heath 1956, vol. 1, 370].

Still, justly or not, we tend to credit the Greeks with being the first to give a proof of this ancient theorem. But since nearly all information about early Greek mathematical texts is lost, we can only speculate about the context of discovery; we know nothing about the original proof itself. What we can observe is that right triangles play a central role in Euclid's *Elements* [Heath 1956, vol. 1]. Moreover, the Pythagorean Theorem and its converse appear as I.47

Figure 1. Returning from the Lapland Expedition, Maupertuis became famous as the "Earth Flattener" who confirmed Newton's theory.

and I.48, the two culminating propositions in Book I of the *Elements*. Thereafter Euclid makes use of it in several of the most important propositions of Books II and III.

Euclidean Traditions and *anschauliche Geometrie*

Today we talk about various models for all kinds of geometries and spaces, without realizing that this is a distinctly modern way of thinking. Classically, geometry was always about figures in space, whereas space itself was never the object of study. One did not go about thinking of different kinds of spaces, or even space in the plural. True, nineteenth-century mathematicians differentiated between the metrical and projective properties of curves and surfaces, and they used calculus to study their differential properties. By mid-century they had even begun to leap by analogy into higher dimensions, and above all they liked to use complex numbers in connection with a mathematical realm of four dimensions. But to the extent they identified themselves as geometers, mathematicians drew their inspiration from phenomena they could somehow visualize or imagine in ordinary Euclidean space.

In the German tradition one spoke of *anschauliche Geometrie*, a term that does not really translate well into English. The popular textbook with this title by Hilbert and Cohn-Vossen, published in the United States as *Mathematics and the Imagination*, serves as a reminder that Hilbert by no means held to a narrow formalist view, despite the influence of his work on axiomatization. In the preface, he wrote:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward intuitive understanding fosters a more immediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their relations. [Hilbert and Cohn-Vossen 1965, preface].

It would be exaggerated to call Hilbert–Cohn-Vossen a "picture-book" approach to geometry, but without the visuals their text would certainly lose its effectiveness. Throughout much of Western history, the discipline of geometry was closely linked with logic. But many experts appreciated that the *source* of geometrical knowledge owed nothing to logical rigor. Hilbert and Cohn-Vossen make this abundantly clear: they mainly just describe and explain what the reader is supposed to *see*. This kind of seeing, though, requires imagination, and to be imagined a figure or configuration must have some relation to objects in real space, which takes us back to Euclid again.

Euclidean geometry, in its original garb, was traditionally regarded not as *one model* among many for geometry, but rather as *the model* for a rigorous scientific system based on deductive argument. As late as the 1880s, Charles Dodgson, better known today as Lewis Carroll, was a valiant spokesman for this conservative approach to teaching geometry. The tightly constructed arguments in the first two books of Euclid's *Elements* left an indelible impression on the logician Carroll.

In *Euclid and his Modern Rivals* [Carroll 1885], one of the oddest dramatic works ever written (and surely never performed), Carroll brought Euclid's ghost back to life to face his challengers. This play without a plot quickly turns into a bizarre mathematical dialogue in which Euclid defends his text and leaves it to the judges in Hades to decide whether any of the thirteen other modernized treatments of plane geometry deserved to take its place. The showdown that ensues has all parties citing the *Elements*, chapter and verse, scurrying to discover which textbook most effectively honed the minds of young Englishmen. Just to be sure that you have the right picture here: the authors whose books come under discussion are all respectable English gentlemen, among the leading mathematicians of their day. So these rival texts obviously did not breathe a word about the new-fangled non-Euclidean geometries that had since found their way across the channel from the Continent. Such monstrosities clearly had no place in the college curriculum. Euclid's rivals merely sought to upgrade the very same body of knowledge one found in the *Elements*. Needless to say, the author of *Alice in Wonderland* gave Euclid's ghost full satisfaction, routing the thirteen rivals with scholarly acumen and witty jibes. Nor should we be surprised that the meatiest arguments on both sides were reserved for Euclid's controversial fifth postulate concerning parallel lines:

> That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the straight lines, if produced indefinitely, will meet on that side on which the angles are less than two right angles.

Historically, mathematicians had long focused attention on this parallel postulate as the crux of what made Euclid's presentation of geometry Euclidean.

Some of Euclid's modern rivals preferred Playfair's more elegant formulation:

Playfair's Axiom: Given a line l and a point P not on it, there exists no more than one line through P parallel to l.

> *"The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn." [Newton 1726]*

But Lewis Carroll clearly disagreed with them [Carroll 1885, 40–47, 77–84]. His arguments seem to me both sound and convincing. Playfair's version of the Parallel Postulate is in the spirit of an existence theorem in modern mathematics; it lacks the constructive character that makes Euclid's rendition so useful. Euclid's Fifth Postulate thus enables the geometer to know in advance not merely that two lines, when extended, will intersect, but also *where the point of intersection will occur*, namely on the side where the angle sum is smaller than two right angles. Euclid makes crucial use of this property, for example, in Prop. I.44 while executing a parabolic application of area for a given triangle. Carroll clearly understood the distinction between theorems and problems in geometry. He even has Euclid's ghost address the proposal by the British Committee for the Improvement of Geometrical Teaching that called for presenting theorems and problems separately.

What about Spherical Geometry?

Over time, mathematicians came to realize that the Pythagorean Theorem is mathematically equivalent to the parallel postulate, and both are equivalent to the proposition that the sum of the angles in a triangle equals 180°. But why did the Greeks feel compelled to limit geometry to what we now see as one special case? Clearly they were familiar with another alternative, since we can still read ancient works on spherics, or what came to be known as spherical geometry. The shortest distance between two points on the surface of a sphere forms the arc of a great circle. Taking such geodesics as the counterparts to straight lines in the plane, it is easy to see that the sum of the angles in a spherical triangle exceeds 180° (this angle sum varies: the smaller the triangle, the smaller the sum of its angles) But if the Greeks already knew several fundamental results of spherical geometry, why did it take so long for mathematicians to accept the validity of non-Euclidean geometries? Why did they not realize that the geometry on the surface of a sphere already provided a counterexample to Euclid's theory of parallels? The answer to this puzzle deserves serious scrutiny as it not only tells us a good deal about the intellectual context of geometrical investigations in antiquity but also how the discipline of geometry was perceived for many centuries afterward.

If we widen the scope of our inquiry into the ancient sciences, it becomes clear that spherics was studied for many centuries as a foundation for the higher disciplines of astronomy and astrology. For example, we learn from Aristotle that the early fourth-century B.C. geometer Eudoxus of Cnidus developed a theory of homocentric spheres to study heavenly bodies, including the retrograde motion of planets like Mars. About a century later a contemporary of Euclid, Autolycus of Pitane, wrote *On the Moving Sphere*, one of the oldest extant mathematical texts. Another related work by the same author, *On Risings and Settings*, describes the paths of the sun and stars throughout the year. A more poetic account of the geometry of the heavens can be found near the beginning of Plato's *Timaeus*, where he describes how the cosmos was constructed by the Demiurge using two great circles. These works stood at the beginning of a long tradition, typified by the work of the thirteenth-century Augustinian cleric John of Holywood, better known as Johannes de Sacrobosco, whose *Sphaera* was widely studied by European astronomers up until the time of Christoph Clavius, an older contemporary of Galileo. By studying

Sacrobosco, one could learn the rudiments of spherical geometry, enough for an ambitious reader to proceed on to Ptolemy's *Almagest.* Yet there is something fundamentally wrong with this formulation. For by combining *spherics* with *geometry* to describe a tradition that extended from Autolycus to Sacrobosco, I have invoked a hybrid concept—spherical geometry—that makes no sense within that historical setting.

In the ancient world, the motions of heavenly bodies were thought to be perfectly circular, unlike the natural motions of terrestrial objects, which rise or fall as they seek their natural place in the world. Physical objects, as we know them here on earth, move about in the space surrounding us. The traditional purpose of plane and solid geometry was to study the properties of simple, idealized figures in this terrestrial realm. Whether or not the roots of this science were Egyptian, it surely drew on centuries of human experience with physical objects in space. To fill up parts of space, builders used bricks of a uniform size and shape, rectangular solids, not round ones or solids with curved surfaces. Straightness and flatness were the primary spatial qualities one imagined in everyday life. A solid, such as a sphere, was obviously in some basic sense a more complex object than a solid figure bounded by plane figures. Both are treated in the last three books of Euclid's *Elements*, but it is clear that the latter figures, particularly the five Platonic solids, were regarded as fundamental. Some of these regular solids occur in nature in crystalline forms, and Plato identified four of them as the shapes of the four primary elements: earth, water, air, and fire. For the Greeks, the sphere had a deep cosmic significance: the planets and fixed stars were conceived as carried about on giant invisible spheres. The natural rotational motion of spheres could be simulated here on earth, of course, but the truly natural motions of terrestrial objects were rectilinear: straight up and down. Of course the earth itself was not seen as flat, but it was mainly on a cosmic scale that the sphere came forcefully into play in Greek science.

These distinctions were described in detail by Aristotle, who drew heavily on earlier authors. By the period of Euclid, who lived shortly after Aristotle, around 300 B.C., they had become firmly established categories. Astronomy and physics had virtually nothing to do with each other. Whereas spherics belonged to the former, geometry had close ties to ancient mechanics, which was not a branch of natural science at all, but rather was synonymous with ancient technology. Mechanicians constructed machines just as geometers constructed figures and diagrams. Among the Greeks, Archimedes was a virtuoso in both disciplines. Indeed, his work successfully bridged the gap between mechanics and geometry, as he relied heavily on the law of the lever to "weigh" geometrical figures before proving theorems about their areas and volumes. Geometry's close links to technology, machines, and the science of mechanics became even stronger after the works of Archimedes were taken up by the mathematicians of the Renaissance. Only with the Copernican Revolution did this bifurcated worldview come to an end. Galileo, the mechanician, sought to refute both Ptolemy and Aristotle, but it took the even more daring ideas of Kepler and Newton to finish the job. In the course of doing so, the prospect of an infinite space emerged for the first time.

On the Shoulders of Ancient Giants

Newton effectively created the discipline of celestial mechanics in his *Principia* [Newton 1687]. In the forty years remaining to him, he had ample time to comment on his legendary success, and his assessments have been repeated countless times. It is tempting to discount his famous pronouncement that

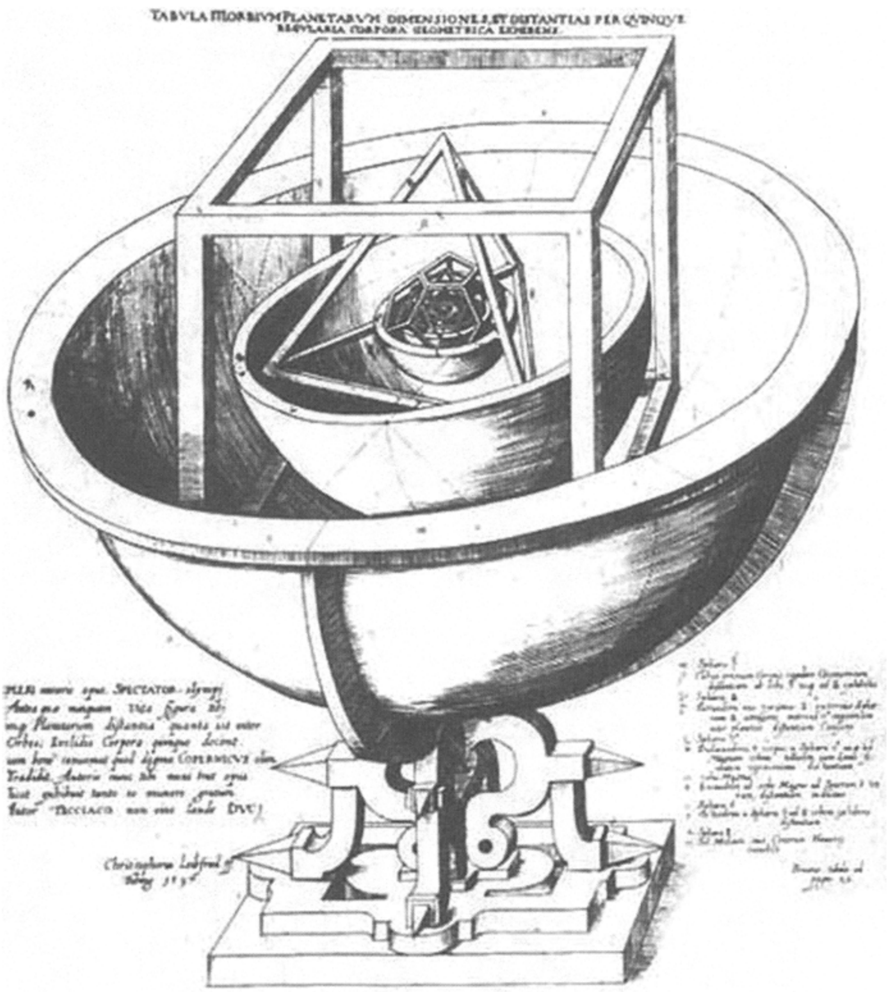

Figure 2. In Kepler's *Mysterium Cosmographicum*, the five Platonic solids structured the Universe.

The distinction between Gaussian and Eulerian curvature is easily illustrated by considering ordinary analytic plane geometry and geometry on the surface of a cylinder. In the former case, all the cutting planes that pass through a normal to the surface are simply lines, which are flat, so that $\kappa_1 = \kappa_2 = 0$; thus the Gaussian curvature is zero, as expected. In the case of a cylinder, the two principal directions at any point lie parallel and perpendicular to the base, since the circles parallel to the base have the greatest curvature among all plane sections of the surface, whereas the vertical lines cut by planes perpendicular to the base have the smallest curvature, namely zero. So again the Gaussian curvature, given by the product, turns out to be zero, which means that the plane and cylinder have equivalent metrical properties, at least locally. Their global, or topological, properties are of course different, but if we cut the cylinder along a vertical generator then the resulting surface can be folded out and mapped onto a plane without distorting any figures on the surface of the cylinder. So it makes no difference whether we roll up a map or flatten it out so long as the surface it represents is intrinsically flat.

Cartographers had realized long before that it was impossible to find an isometric mapping that projected a sphere—a surface of constant positive curvature—onto a plane. Thus something had to be sacrificed in representing the surface of the earth on a flat map, either by distorting the distances between points or the angles between curves. On a sphere the surface normal at each point passes through the sphere's center and each plane section determines a great circle. So $\kappa_1 = \kappa_2 = (1/r)$, and the Gaussian curvature is therefore just $\kappa = \kappa_1 \cdot \kappa_2 = (1/r^2)$. Since this scalar is an intrinsic invariant, two spheres admit the same geometry if and only if their radii are equal. In 1839 Gauss's student Ferdinand Minding proved that two surfaces with the same constant curvature are locally isometric. He also introduced the notion of geodesic curvature for the paths connecting points on a surface. Measuring along the surface, the path of shortest distance between two points has geodesic curvature zero. As an example, the great circles on the surface of a sphere minimize the distance between two points; they therefore correspond to the straight lines in a Euclidean plane. In general, one speaks of those paths that minimize distance as the geodesics of the surface. But these notions were just natural ways of talking about the geometries on surfaces embedded in Euclidean 3-space.

he "stood on the shoulders of giants"—on most occasions Newton was not prone to such modesty—but perhaps we can make sense of this by asking just whose shoulders he might have had in mind.

We might start by reading the preface to the *Principia*: "The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn" [Newton 1726]. Newton went on to mention briefly the conception of mechanics set forth by Pappus of Alexandria around 300 A.D. In his *Collection* Pappus described five types of machines designed to save work, a concept that Newton's system of mechanics would help quantify. This is the tradition of terrestrial mechanics represented by Archimedes and later by Leonardo da Vinci and Galileo. None of these Renaissance figures ever dreamed, of course, that the heavens themselves might be understood as a giant machine. That was Newton's grand vision, made famous by the image of a Deity that designed the world like an intricate clock.

Newton was ahead of his time in so many respects that it is easy to overlook how steeped he was in the traditions of the past. Unlike Descartes, he revered Euclid and the ancients, and this surely accounts for the arcane geometric style of the *Principia*. Not that he was merely paying homage to the ancients or trying to accommodate modern readers, as some historians of physics have suggested. Since most physicists learn celestial mechanics based on the analytic methods first developed by Euler, D'Alembert, and others, they have a hard time squaring what they find in the *Principia* with their image of Newton as co-founder of the calculus. Thus some have occasionally argued that Newton must have originally derived the results in his *Principia* by using the calculus. He then supposedly chose to couch the whole thing in the language of traditional geometry in order not to overwhelm readers with mathematical terminology and techniques that were new and unfamiliar. There seems not to be a scintilla of evidence to support this claim, nor do I know of any leading Newton scholar who thinks that Newton just dressed up the *Principia* in geometry to make it easier to swallow (if he had done so, we would have to conclude that he failed pretty miserably, since his contemporaries found it a very tough read, too). What we do find in Newton's published and unpublished writings are numerous explicit statements and arguments expressing why he preferred to use geometry as the natural mathematical language for treating problems in mechanics. And by geometry, he meant traditional synthetic geometry in the tradition of Euclid and the Greeks.

Newton believed that space had an absolute reality that endowed it with physical properties and geometrical structure. He attributed inertial effects like the centrifugal forces that accompany rotations to the effort required to move against the grain of space, so to speak. His first law, describing the nature of force-free motion, tells us what it means to move with the grain of absolute space, namely in a straight line with uniform speed. The principle of relativity, in its original form, is then a simple consequence of Newton's first two laws. Because the laws of mechanics all deal with forces acting on bodies, and because these forces are directly linked to accelerations by Newton's second law, no physical experiment can distinguish between two inertial frames of reference. Both move with uniform velocity with respect to absolute space and hence with respect to each other. So no accelerations arise, unlike Newton's famous example of the rotating water bucket.

Leibniz and others objected vociferously to Newton's quasi-theological doctrines regarding absolute space and time. But toward the end of the eighteenth century Immanuel Kant gave them a central place in his epistemology. Kant's *Critique of Pure Reason* was widely regarded as a *tour de force* that tamed the excesses of Continental rationalism and metaphysics while overcoming the scepticism of Humean empiricism. Newtonian space and time provided Kant with the keys that led to a new synthesis. He argued that our knowledge of space and time had an utterly different character than all other forms of knowing: it was neither analytic nor *a posteriori*—meaning that we cannot know the properties of space and time by means of deductive reasoning nor by appealing to sense experience. Nevertheless, we can formulate true synthetic *a priori* propositions about them because they provide the foundations for all other forms of knowledge. Thus, according to Kant, space and time are the necessary preconditions for knowing; they supply the transcendent categories that give mankind the ability to know. (This was heady stuff, of course, but Kant's views were enormously influential throughout the nineteenth century, an era when professional philosophers were still widely read.)

Gauss and the Intrinsic Geometry of Surfaces

In different ways Euclid, Newton, and Kant were still massive authorities during the nineteenth century, and each reinforced the established view that space carried a geometrical structure that was Euclidean. That position seemed invulnerable throughout most of the century, in part because no other mathematical alternative seemed conceivable. It was not until the 1860s that mathematicians began to take the possibility of a non-Euclidean geometry seriously, this despite the fact that Carl Friedrich Gauss had entertained this idea throughout much of his career. In 1817 Gauss wrote to a colleague: "I am coming more and more to the conviction that the necessity of our geometry cannot be proved. . . . Geometry should be ranked not with arithmetic, which is purely aprioristic, but with mechanics." [Dunnington 2004, 180]

Ten years later, the "Prince of Mathematicians" published his pioneering work on the intrinsic geometry of surfaces in which he introduced the notion we today call Gaussian curvature [Gauss 1828]. In one sense, this notion was a refinement of the classical notion of curvature introduced by Leonhard Euler in the eighteenth century. In Euler's theory there are two principal curvatures associated with each point of a surface. These are obtained by taking the surface normal at each point as the axis for a pencil of planes. Euler proved that by rotating these planes about this axis there will be two particular ones that cut the surface in curves with a maximum and minimum plane curvature at the given point. Moreover, these two special planes will always be perpendicular to one another. Thus they determine two principal directions with plane curvatures κ_1, κ_2. These, however, are not intrinsic invariants of the surface since they depend on knowing how it sits in the surrounding space. Remarkably, however, the product $\kappa_1 \cdot \kappa_2 = \kappa$ turns out to be an intrinsic invariant, as Gauss was able to prove in his Theorema Egregium:

> If two surfaces are isometric, then they have the same Gaussian curvature at corresponding points.

> *"I am coming more and more to the conviction that the necessity of our geometry cannot be proved. . . ." [C. F. Gauss]*

There seemed to be no reason to regard these purely mathematical ideas as a threat to Euclidean geometry so long as space itself was taken to be flat. Gauss, however, thought differently about this matter. Publicly he said nothing, but privately he made several allusions to the possibility that the parallel postulate might actually fail to hold in physical space. In a letter from 1830 to the astronomer Wilhelm Bessel, Gauss wrote: "We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds, so that we cannot completely prescribe its properties *a priori*." Eight years later, Bessel successfully measured stellar parallax under the assumption that light travelled along Euclidean geodesics. But what if it did not?

On the surface of a sphere, the sum of the angles in a geodesic triangle exceeds 180°, but Gauss recognized there was a second distinct possibility. In a letter from November 1824 he wrote:

> There is no doubt that it can be rigorously established that the sum of the angles of a rectilinear triangle cannot exceed 180°. But it is otherwise with the statement that the sum of the angles cannot be less than 180°; this is the real Gordian knot, the rocks which cause the wreck of all. . . . I have been occupied with the problem over thirty years and I doubt if anyone has given it more serious attention, though I have never published anything concerning it [Gauss 1900, 187].

Gauss clearly ruled out spherical geometry, presumably because its geodesics have finite length, contradicting Euclid's second postulate that a straight line can always be extended. But though he dismissed the possibility of triangles with an angle sum exceeding 180° he went on to note how

> The assumption that the sum of the three angles of a triangle is less than 180° leads to a special geometry, quite different from ours [i.e. Euclidean geometry], which is absolutely consistent and which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of the determination of a constant, which cannot be fixed *a priori*. The larger one assumes this constant, the closer one approaches Euclidean geometry and an infinitely large value makes the two coincide. If non-Euclidean geometry were the true one, and that constant in some relation to such magnitudes as are in the domain of our measurements here on earth or in the heavens, then it could be found out *a posteriori*. [Dunnington 2004, 181–182].

Soon after Gauss's death in 1855, his friend Sartorius von Waltershausen wrote that he had actually attempted to test the hypothesis that space might be curved by measuring the angles in a large triangle that he used in his survey of Hanover. The triangle had vertices located on the mountain peaks of Hohenhagen, Inselberg, and Brocken, which served as a reference system for the system of smaller triangles. Several writers have argued that this famous story is just a myth, but even if true, Gauss apparently concluded that the deviation of the sum of the angle measurements from 180° was smaller than the margin of error. So this test would have merely confirmed that the geometry of space is either flat or else its curvature was too small to be detected.

Slow Acceptance of Non-Euclidean Geometry

If one is tempted to speak of revolutions in the history of mathematics (a debatable point), then one might well regard the advent of non-Euclidean geometry in the nineteenth century as a striking example [Bonola, Rosenfeld]. Mathematicians tinkered for centuries trying to find a completely elementary proof that Euclid's fifth postulate was true. Most, including the eighteenth-century Jesuit Giovanni Girolamo Saccheri, were convinced that it was not a postulate at all, but rather a theorem. By the 1820s and 1830s, the Russian mathematician Nicholas Lobachevsky and a young Hungarian named Janos Bolyai showed that one could develop an exotic system of geometry in which the fifth postulate was false and, instead of having only one line in the plane that passes through a given point without meeting a given line, there will be infinitely many.

It would take over three decades before the publications of Lobachevsky and Bolyai gained belated recognition; not before the 1860s did mathematicians begin to take the new theory seriously. A major obstacle for this new non-Euclidean geometry was the lack of a "real-world model" in Euclidean 3-space comparable to the sphere. The eighteenth-century Alsatian mathematician J. L. Lambert had found an analytic model by studying the properties of a sphere with imaginary radius, but neither he nor any of his contemporaries seem to have regarded this as any more than a curiosity. Not until 1866 when Eugenio Beltrami obtained a surface of constant negative curvature by revolving a tractrix curve about its axis, did it become possible to visualize such a non-Euclidean geometry.

This lack of *Anschaulichkeit* had something to do with the slow reception of the work of Lobachevsky and Bolyai, but the delay was also due to a lack of intellectual courage on the part of the leading mathematical minds of Europe. Surely the history of non-Euclidean geometry would have unfolded quite differently had Gauss made public his views on the theory of parallels.

In December 1853, Bernhard Riemann submitted his post-doctoral thesis to the Göttingen philosophical faculty along with three proposed topics for the final lecture required of all new members. The elderly Gauss presided on this occasion and requested that Riemann speak about the third topic on the list: "On the Hypotheses that Lie at the Foundations of Geometry." Riemann was undoubtedly surprised by that decision and none too pleased about it. He complained to his brother that this was the only topic he had not properly prepared at the time he submitted his thesis. The lecture took place the following June, and according to Richard Dedekind it made a deep impression on Gauss, as it "surpassed all his expectations. In the greatest astonishment, on the way back from the faculty meeting he spoke to Wilhelm Weber about the depth of the ideas presented by Riemann, expressing the greatest appreciation and an excitement rare for him" [Riemann 1892, 517]

This was the famous lecture in which Riemann explained how the notion of Gaussian curvature could be extended beyond surfaces to manifolds with an arbitrary number of dimensions. In particular, this meant that one could study the intrinsic geometry of three-dimensional spaces. Riemann began with this assessment of how little progress had been made in clarifying the foundations of geometrical research:

> It is well known that geometry presupposes not only the concept of space but also the first fundamental notions for constructions in space as given in advance. It only gives nominal definitions for them, while the essential means of determining them appear in the form of axioms. The relationship of these presumptions is left in the dark; one sees neither whether nor how far their connection is necessary or *a priori* even possible. From Euclid to Legendre, to name the most renowned of modern writers on geometry, this darkness has been lifted neither by the mathematicians nor the philosophers who have labored upon it [Riemann 1854, 133].

Riemann's approach abandoned reliance on a theory of parallels, turning instead to a theory of distance based on a generalization of the Pythagorean Theorem. On a human scale, he noted that the metric properties of space accorded well with Euclidean geometry. However, "the empirical concepts on which the metric determinations of space are based—the concepts of a rigid body and a light ray—lose their validity in the infinitely small; it is therefore quite likely that the metric relations of space in the infinitely small do not agree with the assumptions of geometry, and in fact one would have to accept this as soon as the phenomena can thereby be explained in a simpler way" [Riemann 1854, 149].

As for geometry in the large, Riemann emphasized that our intuition makes it hard to conceive of physical space as bounded, whereas a space of infinite extent poses real difficulties for cosmology. This suggested the possibility that our cosmos might have the structure of a 3-dimensional manifold of constant positive curvature. Riemann said all this and much more in 1854, but he took these thoughts with him to the grave. Neither Gauss nor anyone else urged him to publish his manuscript or pursue their consequences further.

Nearly unapproachable during his lifetime, Gauss passed from the scene without so much as once publicly addressing the dogma that the geometry of space had to be Euclidean. After Riemann's death in 1866, Dedekind was appointed editor of his Collected Works, and it was he who stumbled upon the manuscript on the foundations of geometry among his deceased friend's papers.

Its publication in 1867 sparked immediate interest not only in Germany, but in Italy and Great Britain as well. Still, it would take several more years before non-Euclidean geometry found widespread acceptance among mathematicians, many of whom remained convinced that Euclid still reigned supreme.

REFERENCES

Roberto Bonola, 1955. *Non-Euclidean Geometry: A Critical and Historical Study of its Development*. New York: Dover.

Walter Kaufmann-Bühler, 1981. *Gauss: A Biographical Study*, New York: Springer-Verlag.

Lewis Carroll, 1885. *Euclid and his Modern Rivals*. London: Macmillan, repr. New York: Dover, 1973.

Leo Corry, 2004. *David Hilbert and the Axiomatization of Physics (1898–1918)*, Dordrecht: Kluwer, 2004.

G. Waldo Dunnington, 2004. *Carl Friedrich Gauss, Titan of Science*, New York: Haffner, 1955; repr. Mathematical Association of America, 2004.

G. Gabriel, *et al.* eds. 1980. *Gottlob Frege—Philosophical and Mathematical Correspondence*. Chicago: University of Chicago Press.

C. F. Gauss, 1828. *Disquisitiones generales circa superficies curvas*. Göttingen: Dieterich.

——. 1900. Gauss Werke Bd. VIII, Göttingen: Königliche Gesellschaft der Wissenschaften.

T. L. Heath, ed., 1956. *The Thirteen Books of Euclid's Elements*. New York: Dover. 3 vols.

David Hilbert, 1899. "Die Grundlagen der Geometrie," in *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals*. Lepzig: B.G. Teubner. Pp. 3–92.

D. Hilbert and S. Cohn-Vossen, 1965. *Geometry and the Imagination.* New York, Chelsea.

Isaac Newton, 1687. *Philosophiae Naturalis Principia Mathematica*. London: Joseph Streater.

Isaac Newton, 1726. *Philosophiae Naturalis Principia Mathematica*, 3rd ed., Alexandre Koyré, I. Bernard Cohen, and Anne Whitman, eds. Cambridge, Mass.: Harvard University Press, 1972.

Moritz Pasch, 1882. *Vorlesungen über neueren Geometrie.* Leipzig: Teubner.

Bernhard Riemann, 1854. "Über die Hypothesen, welche der Geometrie zugrunde liegen," *Abhandlungen der Kgl. Gesellschaft der Wissenschaften zu Göttingen*. **13** (1867): 133–152.

——. 1892. *Bernard Riemann's Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass, hrsg. unter Mitwirkung von R. Dedekind und H. Weber, 2. Aufl.* Leipzig: Teubner.

B. A. Rosenfeld, 1988. *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*. Trans. Abe Shenitzer. New York: Springer-Verlag.