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Nathan Sidoli

Waseda University, School for International Liberal Studies
1-6-1 Nishi-Waseda, Shinjuku
Tokyo, Japan

sidoli@waseda.jp

Constructions and Foundations in the *Spherics* of Theodosios and Menelaos*

* I thank R.S.D. Thomas for permission to use translations of the Theodosios' *Spherics* and diagrams that we have jointly prepared for a book that we have written on Theodosius' treatise. This paper has also benefitted from critical remarks made on a previous draft by Athanase Papadopoulos.

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ABSTRACT | This paper explores the constructive and foundational methods of spherical geometry as introduced in Menelaos' *Spherics* and their relation to similar techniques in Theodosios' *Spherics*. In particular, it argues that, although Menelaos had the key insight to develop spherical analogs and divergences in relation to Euclid's propositions treating the plane triangle, and took the step of producing the first geometry of figures drawn on the surface of a sphere, the foundational mathematical methods as well as the lexical stylistic tropes that Menelaos used were available to him from a close reading of Theodosios' *Spherics*.

KEYWORDS | Theodosios, Menelaos, *Spherics*, spherical geometry, geometrical constructions, mathematical foundations

1. Introduction

This paper explores the foundations of Menelaos' spherical geometry by comparing his constructions and foundational theorems with related constructions and theorems in the *Spherics* of Theodosios. As is well known, Menelaos produced an intrinsic geometry of the properties of figures drawn on the surface of a sphere, and many of his propositions can be compared directly to plane propositions of Euclid's *Elements* I and VI. In the following, I will detail how, although Menelaos developed the new conception of an intrinsic geometry of the sphere and developed this to a considerable degree, the foundational methods that he used, and the basic analogies that he employed, were adumbrated in the work of his predecessor Theodosios. That is, although Menelaos took the crucial step of developing analogies for the propositions of Euclid's theories of plane triangles and sought to do this work as much as possible directly on the surface of the sphere, the underlying constructive methods and congruence relations, and even many of the linguistic tropes that he used, would have been available to him though a careful reading of Theodosios' *Spherics*.

There are various different approaches to studying the geometry of the sphere preserved in the Greco-Roman mathematical sources. We can group these broadly into a number of different categories. There are some investigations of the solid properties of the sphere in those texts that explore the constructions and properties of regular figures in the sphere, such as *Elements* XIII and XIV, by Euclid and Hypsikles respectively. There are works that treat the mensuration of various elements of the sphere, such as treatises by Archimedes and Heron. There is the group of texts such as Autolykos' *Moving Sphere* and *Risings and Settings*, Euclid's *Phenomena*, and the surviving works of Theodosios, which apply solid geometry to studying the properties of great and small, particularly parallel, circles in the sphere, and then apply these to topics in spherical astronomy. There are the analemma methods, as exemplified in Ptolemy's *Analemma*, which involve configurations that arise from orthogonal rotations and projections of circles on a sphere into a certain plane. Such constructions allow for nomographic and plane chord-table trigonometric computations. There are methods that involve constructions that could be produced by stereographic projection of objects on the surface of the sphere into a plane, as exemplified in Ptolemy's *Planisphere*. Such constructions also admit nomographic and plane trigonometric computations. Finally, there are studies of the intrinsic properties of figures produced on the surface of the sphere, which can also be developed into a trigonometry of spherical figures involving either spherical quadrilaterals or triangles, as found in Menelaos' *Spherics* and

Ptolemy's *Almagest*. In this paper, I will restrict my attention to certain aspects of the development of Menelaos' intrinsic geometry of figures drawn on the surface of a sphere, especially as related to the constructive methods of Theodosios' *Spherics*. I will refer to this particular intrinsic approach as *spherical geometry*, and to all of the various mathematical methods found in the ancient sources as *geometries of the sphere*.

There was no special terminology among Greek authors for referring specifically to either intrinsic spherical geometry, or the various geometries of the sphere on the whole. The word *spherics*, which is commonly used by modern authors, may be taken to derive from (1) the title of either of the works by Theodosios and Menelaos, τὰ σφαιρικά, meaning spherical things or matters, or (2) the term for an ancient field of the mathematical sciences, ἡ σφαιρική, which was used by later authors with Pythagorean or Platonist leanings.[1] In the works of Theodosios and Menelaos, however, we find that although the constructive methods are the same, the overall concept is different, because Theodosios does not attempt to systematically develop an intrinsic geometry of figures on the surface of the sphere, whereas this was clearly Menelaos' primary goal. What joins these two treatises together in a single project is that they are based on the same foundational mathematical methods, and they both culminate in theorems of significance for spherical astronomy, a discipline that would have been of cosmological relevance at the time. Moreover, when later authors, such as Nikomachos or Proklos, refer to a discipline of *spherics*, this is, in fact, a rhetorical expression for the field of astronomy as a whole.[2] Hence, the two Greek terms that we could directly understand as *spherics* can be taken to mean neither an intrinsic spherical geometry, nor all of the mathematical methods for studying the sphere. Thus, although we will use this terminology for our own convenience, we must acknowledge that it does not delineate an ancient conception that was articulated in our sources.

It used to be held that Theodosios' *Spherics* was more or less entirely the work of some earlier author,[3] and, indeed, many inferences in the *Moving*

1 Although the term *spherics* used to be attributed to the older Pythagoreans (for example, Heath, *A History*, 1921, p. 243; Bulmer-Thomas, "Theodosius of Bithynia", 1970, vol. 13, p. 319, col. 2), more recently scholars working on the Pythagorean sources have independently recognized the quotes attributed to Pythagoras that contain this word as later forgeries, and the attribution of this term to Archytas by Nikomachos is now understood to be an insertion into the text by Nikomachos himself (see Huffman, *Archytas of Terentum*, 2005, p. 103).

2 See Hoche, *Nicomachi Geraseni*, 1866, p. 6; Friedlein, *Procli Diadochi*, 1873, p. 37, 59.

3 See, for example, Heiberg, *Litterargeschichtliche Studien*, 1882, p. 44-52; Hultsch, "Autolykos und Euklid", 1886; Bjørnbo, *Studien über Menelaos*' Sphärik, 1902, p. 63.

Sphere of Autolykos and the *Phenomena* of Euclid read almost word-for-word as applications of propositions demonstrated in the *Spherics*.[4] More recently, however, scepticism has grown that we are now in a position to be certain about the details of work on the geometry of the sphere prior to the time of Autolykos and Euclid.[5] Furthermore, there are serious difficulties involved with the assumption that *Phenomena* 7 relies directly on Th.*Sph*. III.22, or that *Phenomena* 8, 12, and 13 rely on Th.*Sph*. III.5-III.8, as was made in the 19th and early 20th centuries.[6] Hence, while it is agreed by all that there must have been some deductive work on the geometry of the sphere prior to the time of Autolykos and Euclid, on which they could draw, we should admit that the contents of this work is now unknown to us in detail, and even in overall structure. Furthermore, it is likely that some of the verbal agreement that we now find between the treatises of Autolykos, Euclid, and Theodosios was introduced later when these works were grouped together and studied as a codicological and didactical unit, starting from the Roman Imperial period and continuing through the late ancient period up until the 9th century, when our earliest manuscripts were composed.[7] Whatever the case, such considerations need not concern us for the topic of this chapter, because we are here interested in the relationship between the two *Spherics* of Theodosios and of Menelaos. What is clear is that Menelaos read the former text and attributed it to Theodosios,[8] and, as I will argue below, he expected his readers to have mastered this text before they read his own. Hence, from the perspective of Menelaos, the only work on the geometry of the sphere that he acknowledged as being a significant precedent to his own was the *Spherics* attributed to Theodosios.

4 Aujac, "Le langage formulaire", 1984.

5 Schmidt, *On the Relation*, 1943, p. 11-12; Neugebauer, *A History*, 1975, p. 750; Berggren, "The relation", 1991, p. 241.

6 See the next section for a discussion of the abbreviations that I use for the texts of Euclid's *Elements*, Theodosios' *Spherics*, and Menelaos' *Spherics*.

7 Acerbi, "Types, function, and organization", p. 141-151.

8 Krause, *Die* Sphärik *von Menelaos*, 1936, p. 239 n. 1; Sidoli, Kusuba, "Al-Harawī's version", 2014, p. 167, 172; Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 697, 769.

1.1. Notations, diagrams, abbreviations, sources

I will use two different notations for denoting mathematical objects. For referring closely to the ancient and medieval texts, I will use letter-names drawn from the text, in which specific points, or unspecified parts of the letter-names, are denoting using italics. I will use bold letters to denote geometric objects that are named by letter, such as $\mathbf{C}(ABC)$ for circle *ABC*, $\mathbf{gC}(ABC)$ for great circle *ABC*, $\mathbf{T}(ABC)$ for triangle *ABC*, $\mathbf{sphT}(ABC)$ for spherical triangle *ABC*, and so on. When summarizing the mathematical concepts, I will also use italic letters to denote geometric objects, such as *C* for any circle, *gC* for a great circle, *sC* for a small circle, *pC* for a parallel circle, and so on. In this situation, it is sometimes useful to differentiate between an object in the sphere and others on the Euclidean plane, such as sC_S for a small circle in the sphere, and l_E for a line in the plane.

For a number of the propositions discussed below, I give reconstructed diagrams drawn using principles of linear perspective to display the objects under investigation. These diagrams are often strikingly different from the diagrams in our medieval evidence for the texts under discussion. For the purposes of this paper nothing depends on the diagrams themselves; hence, I believe that there is no danger of conceptual misunderstanding by taking this approach. Furthermore, because I believe that the ancient mathematicians expected their texts to be read in conjunction with the use of a solid or armillary sphere, on which the diagrams could actually be drawn, the use of perspective diagrams allows us, in written communication, to discuss something which was probably closer to what the ancient authors expected their readers to see.[9] Finally, the most important exemplars of the texts that I will read are now available in color images online – Vat.gr. 204, at the Biblioteca Apostolica Vaticana, and

9 There are a number of indications from literary and art-historical sources that teachers of the mathematical sciences often used solid spheres for instruction. For example, Strabon states that anyone who has seen a globe, and understood the things taught in "the first course of the mathematical sciences" will be able to understand his *Geography* (Strabon, *Geography*, I.1.21 [C13]). There are many images of philosophers, or goddesses, holding a pointer and a globe or armillary sphere, as a representation of a teacher. Such as one of the figures in a tomb from around 300 BCE at Pella, Central Macedonia, Greece, now in the Ephorate of Antiquities of Pella, or the central figure in the famous mosaic from the Villa of Titus Siminius Stephanus, Pompei, 1st c. BCE-1st c. CE, now in the Museo Archeologico Nazionale, Naples, inv. 124545. In the 5th-century allegorical fiction of Martianus Capella, the *Marriage of Philology and Mercury*, the lady Geometry appears with a rod and a globe, which is a miniature cosmos (Martianus Capella, *Marriage of Philology and Mercury*, 575-585). Hence, over many centuries, we see the trope of a master of the mathematical sciences depicted with a rod and a sphere.

Leiden or. 930, at the Universiteitsbibliotheek Leiden – hence, readers who are interested in the medieval diagrams can consult them directly.

I often refer to propositions of certain standard texts of Greek mathematical works by book and proposition number, sometimes without giving a summary of the proposition in question – particularly in the case of propositions of Euclid's *Elements*. My expectation is that readers who expect to follow the details of the mathematical methods of Greco-Roman authors will be willing to consult these works, or their translations, directly. For example, *Elem.* I. post.1 refers to the first postulate of the *Elements*, found in the first book; and Th.*Sph.* I.1 refers to the first proposition of the *Spherics* of Theodosios, while M.*Sph.* III.1 refers to the first proposition of the third part of Menelaos' *Spherics*.[10]

In this paper I use the most recent edition of the Greek text by Claire Czinczenheim as a base text for reading Theodosios' *Spherics*.[11] The manuscript evidence for Menelaos' *Spherics* is rather involved.[12] The Greek text of this treatise has only been preserved in fragments.[13] While the Latin text of Gerard of Cremon has been studied but not published,[14] two different Arabic versions have been critically edited, each of which is explicitly referred to as a correction, or restitution (*iṣlāḥ*), in the medieval scholarship: (1) the 10th-century Arabic edition of al-Harawī, made on the basis of a partial correction by

10 For Menelaos' *Spherics*, I use the proposition numbering in the version of Abū Naṣr ibn ʿIrāq; see Krause, *Die* Sphärik *von Menelaos*, 1936. For a discussion of the evidential basis for the Menelaos' text, and my reasons for using Ibn ʿIrāq's numbering system, see note 51, below.

11 Currently, the most accessible translation of the Greek text of this treatise into a modern language is the French translation of Ver Eecke, *Les* Sphériques, 1927, which was based on the edition of Nizze, *Theodosii Tripolitae*, 1852, and checked against a preprint manuscript of Heiberg's 1927 critical edition of the text (see Ver Eecke, *Les* Sphériques, 1927, p. lii). There is also a German translation by Nizze, *Theodosius Von Tripolis*, 1826, and an English translation by Stone, *Clavius's Commentary*, 1721, of the Latin version of Clavius, *Theodosii Tripolitae Sphaericorum*, 1586, which was itself based on the Arabo-Latin tradition of the text. Czinczenheim, Sphériques *de Théodose*, 2000, corrects a number of errors introduced by Heiberg's reliance on a 14th-century Byzantine recension of the treatise, and provides a French translation.

12 An overview of the medieval transmission of Menelaos' *Spherics* is given in Sidoli, "Review of Menelaus' Spherics", 2020, p. 16-20. The brief remarks here follow the explanation presented there.

13 Bjørnbo, *Studien über Menelaos'* Sphärik, 1902, p. 22-25; Acerbi, "Traces of Menelaus' Sphaerica", 2015.

14 Bjørnbo, *ibid.*

al-Māhānī in the 9th-century,[15] and (2) the late-10th or early-11th-century Arabic recension of Abū Naṣr Manṣūr ibn ʿIrāq, based on a different translation, probably that of Isḥāq ibn Hunayn.[16] Although al-Harawī's edition is earlier, there are a number of concerns about using it as a base text for our understanding of Menelaos' treatise. In the first place, it was based on a poor translation, and then only partially edited by al-Māhānī, who wisely stopped working on the text when serious mathematical difficulties were found in his source. Al-Harawī attempted to carry this further, but he made mathematical errors in this process,[17] and introduced a number of linguistic ambiguities, so that it is not clear how well he understood the source material. Ibn ʿIrāq's text, on the other hand, was based on a better translation and is mathematically sound, but it still contains a number of differences with the known Greek fragments that may be the result of his own revisions or improvements.[18] Nevertheless, while still awaiting critical editions of the Latin and Hebrew versions of the text before we can fully assess the medieval transmission, for the time being, I prefer to use Ibn ʿIrāq's revision over that of al-Harawī, because of the clear problems in the source material for the latter. In the passages discussed in this paper, where the two texts diverge, I will mention the evidence of al-Harawī's version as well.

2. Theodosios' Adumbrations of Spherical Geometry

Theodosios' *Spherics* uses the plane and solid geometry of Euclid's *Elements* to develop a geometry of the sphere, which is then used to address topics that, while stated in ostensibly geometric terms, were almost certainly motivated by considerations of the ancient celestial sphere, probably as modeled on a solid or armillary sphere. The treatise is divided into three books:

15 Rashed, Papadopoulos, *Menelaus*' Spherics, 2017.

16 Krause, *Die* Sphärik *von Menelaos*, 1936.

17 Sidoli, Kusuba, "Al-Harawī's version", 2014, p. 178-179, 185-190; Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 277-278.

18 See the example discussed by Acerbi, "Traces of Menelaus' *Sphaerica*", 2015, p. 96-97.

– Theodosios' *Spherics* I introduces the properties of small and great circles in the sphere, treats various relationships between them, and develops some *problems*[19] that are used in the rest of the treatise.

– Theodosios' *Spherics* II begins with theories of tangency and parallelism relating to small and great circles on the sphere, and then uses these to study a configuration involving a bundle of parallel circles and a set of great circles that either pass through the poles of the parallel circles or are tangent to a pair of equal parallels, as well as a bundle of parallel circles cut by a single oblique great circle. These geometric configurations can be understood to represent the relationship between the local horizon at different times of the day and a bundle of small circles parallel to the celestial equator as well as the instantaneous positions of the ecliptic relative to the local horizon. Along the way, Theodosios includes two final *problems*, which show how to produce great circles tangent to a given small circle.

– Theodosios' *Spherics* III starts with some geometric lemmas and then uses the same two-part configuration treated in the previous book to develop a number of theorems that treat various relations concerning arcs of another inclined great circle and arcs cut off by the bundle of parallel circles or the great circles either passing through their poles or parallel to a pair of equal parallels. These propositions can be interpreted as making claims about the rising times of arcs of the ecliptic over the horizon. There are then some theorems that lead to a bound on the ratio between arcs of the ecliptic and their rising times, which is itself related to the obliquity of the ecliptic.

The astronomical topics that can be treated by the theorems of Theodosios' *Spherics* II and III belong to what we call spherical astronomy, and such subjects were also treated in astronomical texts by earlier authors such as Autolykos, Euclid, Hipparchos, and, then later by Menelaos and Ptolemy.

2.1. Analogies between Theodosios' Spherics *and Euclid's* Elements *III*

Most of the purely geometric propositions of this treatise – that is, much of Theodosios' *Spherics* I and the opening theorems and *problems* of his *Spherics* II – have direct analogs among the propositions of Euclid's *Elements* III, Euclid's treatment of the circle. The analogies between the geometric objects in Theo-

19 Throughout this paper, I use *problem*, in italics, to refer to the type of proposition in a Greek mathematical text, enunciated in the infinitive, in which the mathematician sets out some constructive procedure, and then proves that it has been successfully carried out.

dosios' *Spherics* and their analogs in the Euclid's *Elements*, however, go through a three-stage transition. Initially, the analogy is (1) between the sphere itself and some base circle, $S \leftrightarrow C_E$, a diameter of the sphere and a diameter of the base circle, $d_S \leftrightarrow d_E$, and a solid small circle in the sphere and a chord in the base circle, $sC_S \leftrightarrow Crd_E$. In the middle of Theodosios *Spherics* I, the analogy shifts to that (2) between the sphere and a base circle, $S \leftrightarrow C_E$, a great circle on the surface of the sphere and a diameter of the base circle, $gC_S \leftrightarrow d_E$, and a small circle on the surface of the sphere and a chord in the base circle, $sC_S \leftrightarrow Crd_E$. Finally, in the *problems*, the analogy becomes (3) between a great circle on the surface of the sphere and a line in the plane, $gC_S \leftrightarrow l_E$, and a small circle on the surface of the sphere and circle in the plane, $sC_S \leftrightarrow C_E$. We recognise this final analogy as that at the crux of later conceptions of spherical geometry. It is not clear, however, that Theodosios himself recognized the importance of this analogy, because he did not further develop it in his treatise. Indeed, we should recognize that, for Theodosios, this fundamental analogy may have arisen as a sort of unintentional result of his need for certain constructive propositions and his strategy of modeling his propositions on those in Euclid's *Elements* III.

In order to understand how this key analogy was suggested by Theodosios and then fully developed by Menelaos, we need to explore the role of constructions and constructive *problems* in each of their *Spherics*. In this section we consider the development of the *problems* of the Theodosios' *Spherics*, and conjecture about some possible motivations for the form that these took in Theodosios' treatise.

2.2. *Theodosios'* Problems

In all but one of his *problems*, Theodosios assumes two constructive procedures without discussion or postulation, namely (1) the ability to draw a circle around a given point on the surface of the sphere, as pole, with a given distance, or span, that may be carried from anywhere in the assumed or constructed configuration, and (2) the ability to cut off an arc of a given circle on the sphere, or a length in a plane, equal to a given distance that may be carried from anywhere in the configuration.[20] Both of these operations are clearly related to Euclid's circle postulate, *Elem.* I.post.3, but it should be observed that, in practical terms on a solid sphere, these actions can be carried out with a normal

20 These two constructions are discussed in Sidoli, Saito, "The role", 2009, p. 587–588. In *Sph.* I.2, Theodosios also assumes the ability to pass a cutting plane through a sphere, but this is part of a fully solid construction, like those in Euclid's books on solid geometry, *Elements* IX–XIII, and we will ignore it for the purposes of this paper.

compass – that is, a compass that carries a fixed span, as all real compasses do. Although we could denote these two operations separately, as related to the different ways that *Elem*. I.post.3 and I.3 function in the *Elements*, for our purposes in this paper, I will simply refer to both of these operations as *Elem*. I.post.3(compass).

At the end of Theodosios' *Spherics* I, we encounter a run of four *problems* that show (1) how to set out the diameter of a given circle in a sphere (Th.*Sph.* I.18), (2) how to set out the diameter of a given sphere (Th.*Sph.* I.19), (3) how to draw a great circle between two given points on the surface of a sphere (Th.*Sph.* I.20), and (4) how to find the pole of a given circle on the surface of a sphere (Th.*Sph.* I.21).[21] The most important proposition of this group is Th.*Sph.* I.20 – "Draw a great circle through two given points that are on a spherical surface."[22] If we consider this as one of the main goals of this group of propositions, we can use an analytical argument following the style of ancient geometrical analysis to motivate Th.*Sph.* I.18-I.20 as found in the received text.

It is well known that we can often understand the motivation for the constructions in an ancient Greek *problem* by working backwards from the assumption that what we seek has already been produced.[23] In this case, we begin with the analytical assumption that a great circle, gC_1, passes through the two given points, say P_1 and P_2. Then, if we could determine a pole of gC_1, at say P_3, and if we could use *Elem*. I.post.3(compass) to draw a great circle around P_3 as a given point, then we could complete Th.*Sph.* I.20 (see Fig. 1 (left)). But if we could use *Elem*. I.post.3(compass) to draw great circles gC_2 and gC_3 around P_1 and P_2 as poles, then they would meet at P_3, by an argument analogous to that of *Elem*. I.1. Hence, the key to completing Th.*Sph.* I.20 comes down to whether or not we can draw a great circle around a given point as pole.

21 In the medieval manuscripts of the Greek text there are two redundant propositions that conclude Theodosios' *Spherics* I (for example, Vat.gr. 204, f. 9*r*-9*v*), but these are missing in Thābit's Arabic edition, and are clearly spurious interpolations; see Kunitzsch, Lorch, *Theodosius* Spherica, 2010, p. 84.

22 Czinczenheim, Sphériques *de Théodose*, 2000, p. 77.

23 Knorr W., *The Ancient Tradition*, 1986.

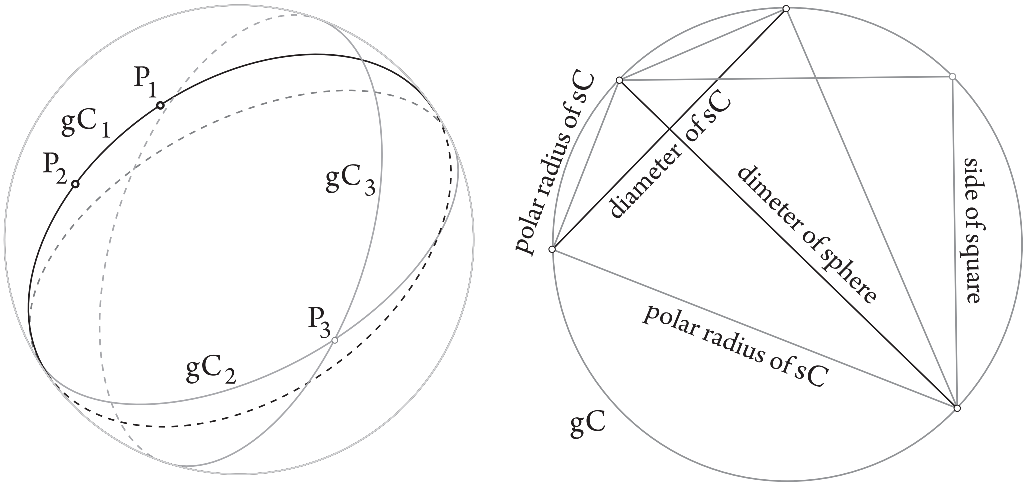

Fig. 1: (left) Perspective diagram of the primary objects in Th.*Sph.* I.20. For the medieval manuscript figure, see Vat.gr. 204, f. 8*v*. (right) The diameter of the sphere and related objects, in the plane.

By Th.*Sph.* I.16 and I.17, we know that the distance between the pole of a great circle and its circumference – what we can call its *polar radius* – is equal to the side of a square inscribed in a great circle, *gC*. Hence, if we could set out the diameter of the sphere, we could use the postulates and *problems* of the *Elements* to drawn a circle around it and inscribe a square (see Fig. 1 (right)). Moreover, since the diameter of any small circle in the sphere, *sC*, will be perpendicular to the diameter of the great circle *gC* through the poles of *sC*, by Th.*Sph.* I.10 and I.6, and since joining the endpoints of both diameters by the polar radii of *sC* will result in congruent right triangles on either side of the diameter of *gC*, if we could set out the diameter of an arbitrary *sC*, we could then work backwards to complete the construction necessary for Th.*Sph.* I.20, because we could produce *sC* using *Elem.* I.post.3(compass) and, hence, have its polar radius determined. Indeed, these two constructions are provided by the two foregoing propositions. Th.*Sph.* I.19 shows how to set out the diameter of a given sphere, and it crucially relies on Th.*Sph.* I.18, which shows how to set out the diameter of a given circle on sphere.[24]

All three of these *problems* are solid constructions and there is, as of yet, little indication of any analogy between great circles and lines, $gC_S \leftrightarrow l_E$. One hint toward this conception, however, can be found in the treatment of cases in Th.*Sph.* I.20 itself. In Th.*Sph.* I.20, the problem of drawing a great circle through two given points is addressed in two cases. First, it is simply remarked

24 Sidoli, Saito, "The role", 2009, p. 589–592.

that if the given points, say P_1 and P_2, lie on a diameter of the sphere, then it is clear that unlimited great circles will be drawn through them. Second, if P_1 and P_2 are not on a diameter of the sphere, great circles, say gC_2 and gC_3 are drawn around them as poles meeting in say P_3(Fig. 1 (left)). Then a great circle, say gC_1, is drawn around P_3, which will be the unique great circle passing through P_1 and P_2.[25] If we think of these two cases in terms of the $gC_S \leftrightarrow l_E$ analogy, the first would correspond to passing a line through one point, whereas the second would be related to passing a line through two points. That is, the first case does not result in a definite object, whereas the second does. In fact, we do find Greek geometers passing indeterminate lines through points,[26] mostly involving circles, but they do not seem to have seen the need to explicitly postulate such a construction. The second case, on the other hand, results in a unique great circle, and this can be understood as analogous to the postulate for joining a straight line in the plane, *Elem*. I.post.1. Whether or not Theodosios was guided by such considerations in developing Th.*Sph*. I.20 is unclear, but as we will see the analogy between great circles and lines, $gC_S \leftrightarrow l_E$, becomes more pronounced when it is contrasted with the analogy between small circles and circles in the plane, $sC_S \leftrightarrow C_E$, as happens in the next proposition.

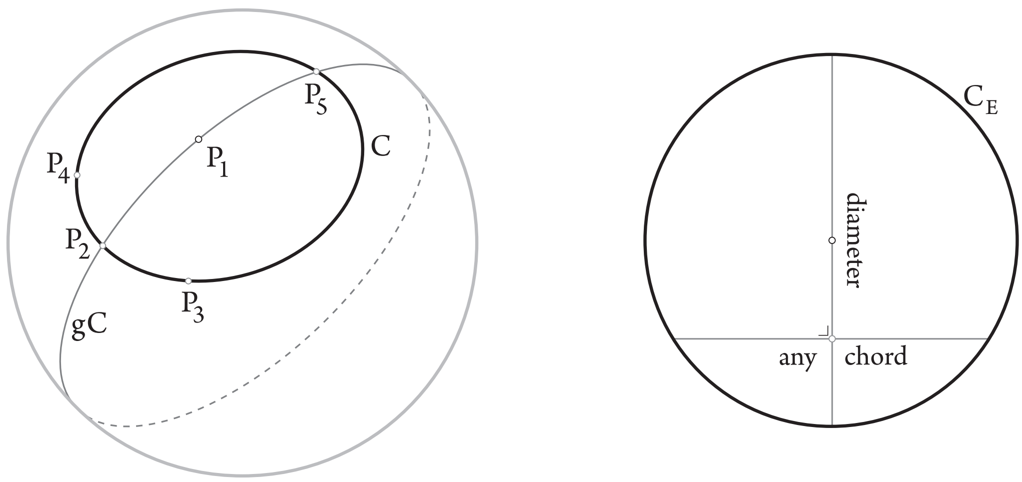

Fig. 2: (left) Perspective diagram of the objects in Th.*Sph*. I.21. For the medieval manuscript figure, see Vat.gr. 204, f. 9*r*. (right) The primary objects of *Elem.* III.1.

25 It is not explicitly stated that gC_1 is unique, but this follows from the facts that great circles diametrically bisect one another, Th.*Sph*. I.11, I.12, and all polar radii of every great circle in a sphere are equal, Th.*Sph*. I.16, I.17.

26 For some examples, see the constructions in *Elem*. III.8, IV.3, Euclid's *Data* 92, Th.*Sph*. I.7, I.8, and so on.

The *enunciation*[27] of Th.*Sph.* I.21 reads: "Find the pole of a given circle in a sphere."[28] That is, given some circle on the sphere, C_S, it is necessary to find one of its poles, say P_1(see Fig. 2 (left)). Conceptually and linguistically this proposition is an analog to *Elem.* III.1, which shows how to find the center of a given circle. In fact, however, it is the second analog to *Elem.* III.1 in the Theodosios' *Spherics* – Th.*Sph.* I.2 shows how to find the center of a sphere,[29] using the first set of analogies of a sphere to a circle and a diameter of the sphere to a diameter of the circle, $S \leftrightarrow C_E$ and $d_S \leftrightarrow d_E$. Consideration of the construction in *Elem.* III.1, along with the fact that we can now draw a great circle through two points, provides the key to the construction in Th.*Sph.* I.21.

In *Elem.* III.1, the *construction* proceeds by producing a diameter and then bisecting it, using *Elem.* I.10. This is done by taking an arbitrary chord, bisecting that, *Elem.* I.10, then erecting a perpendicular chord (a diameter) at the midpoint of the original chord, *Elem.* I.11, and bisecting that, *Elem.* I.10 (see Fig. 2 (right)). In *Elements* III, however, it is not shown that a diameter of a circle bisects a chord until *Elem.* III.3 and III.4; hence, in the earlier *Elem.* III.1, the proof that the center of the circle has actually been found must be indirect. Nevertheless, this configuration provides the key to Th.*Sph.* I.21 – namely, if we can draw a great circle that passes through the pole of our given circle, then we can bisect it to find the pole.

In Th.*Sph.* I.13-I.15 it is shown that a great circle that passes through the pole of another circle will be perpendicular to the circle, and will bisect it. Hence, if we could bisect the given circle, we could pass a great circle through the points of bisection and bisect the arc cut off on that.[30] Moreover, such a construction would allow a direct proof that the construction had been carried through. Indeed, in Th.*Sph.* I.21, Theodosios proceeds, in the first case, by taking an arbitrary point on a given small circle C, say P_2, then laying off two arbitrary but equal arcs on either side of P_2, say $\mathbf{Arc}(P_3P_2) = \mathbf{Arc}(P_2P_4)$, using *Elem.* I.post.3(compass), and next bisecting $\mathbf{Arc}(P_3P_5P_4)$ at point P_5, perhaps with

27 Throughout this paper I use *italics* to refer to the well-known parts of a mathematical proposition of Greek geometry: *enunciation* (πρότασις), *exposition* (ἔκθεσις), *specification* (διορισμός), *construction* (κατασκευή), *demonstration* (ἀπόδειξις), and *conclusion* (συμπέρασμα); see Friedlein, *Procli Diadochi*, 1873, p. 203; Netz, "Proclus' division", 1999; Acerbi, *La sintassi logica*, 2011, 1-117.

28 Czinczenheim, Sphériques *de Théodose*, 2000, p. 78.

29 Sidoli And Saito, "The role", 2009, p. 589.

30 Indeed, a more direct analog of the construction in *Elem.* III.1, as passing an arbitrary great circle through the given circle and then bisecting its arc and erecting a perpendicular, would require that we be able to construct a great circle perpendicular, or at an arbitrary angle, to another. But such a construction is not known to have been established before Menelaos.

Elem. III.31 (see Fig. 2 (left)).[31] Finally, a great circle, gC, is passed through P_2 and P_5, Th.*Sph.* I.20, and **Arc**(P_2P_5) is bisected at P_1, *Elem.* III.31.[32] From this construction, it follows directly and almost immediately that P_1 is a pole of C, by the properties established in Th.*Sph.* I.13-I.15 and Th.*Sph.* I.def.5, which asserts that the polar radii joining a circle with each of its poles are all equal.

This configuration, and its analogy with *Elem.* III.1, allows us to see a bit more clearly the final set of analogies that are developed in Theodosios' *Spherics* – namely, $gC_S \leftrightarrow l_E$ and $sC_S \leftrightarrow C_E$. The second case of Th.*Sph.* I.21, however, again makes it unclear whether this analogy was actually motivating Theodosios' approach or was simply a consequence of the patterns found in the mathematical objects themselves. In particular, Th.*Sph.* I.21 Case 2 shows how to find the pole of C when C is taken to be a great circle. The construction involves a slight modification in which either **Arc**($P_2P_3P_5$) or **Arc**($P_2P_4P_5$) is bisected and then gC is drawn around that bisection point as a pole (not shown in Fig. 2). Since such a construction involves two great circles intersecting at right angles on the sphere, it seems unlikely that Theodosios, or indeed any ancient mathematician, would have regarded it as being analogous to a plane configuration.[33] Indeed, although the final set of analogies are suggested by the final two *problems* of Theodosios' *Spherics* I, it is not until the *problems* of *Spherics* II that they come into better focus. We turn to those *problems* now.

After establishing the concept of parallelism between circles in a sphere in Th.*Sph.* II.1-II.2, developing a theory of the tangency of circles in a sphere in Th.*Sph.* II.3-II.8, and introducing a bundle of parallels and a set of great circles that is either perpendicular to, or similarly inclined on, the greatest of the parallel circles, Theodosios provides two final *problems*, showing how to draw a great circle that is tangent to a given small circle, passing through a given point. The configurations of these two propositions are closely related to those of *Elem.* III.16 and III.17.

31 More specifically, since Th.*Sph.* I.18 allows one to set out a diameter of a given circle, we may set out the diameter of C in the plane as, say d_E, about which we draw a circle in the plane, $C_E = C$, using *Elem.* I.10 and I.post.3. Hence, we can use *Elem.* I.post.3(compass) to carry Arc(P_3P_4) and lay it off on C_E. This plane arc can be bisected using *Elem.* III.31. The result of this bisection can then be carried back and laid off on the circle in the sphere, C, using *Elem.* I.post.3(compass).

32 As in the previous note.

33 From our perspective we may think of this as analogous to two perpendicular lines, where the lines are regarded as circles of infinite diameter – but such a conception is not known to have been articulated in any ancient text.

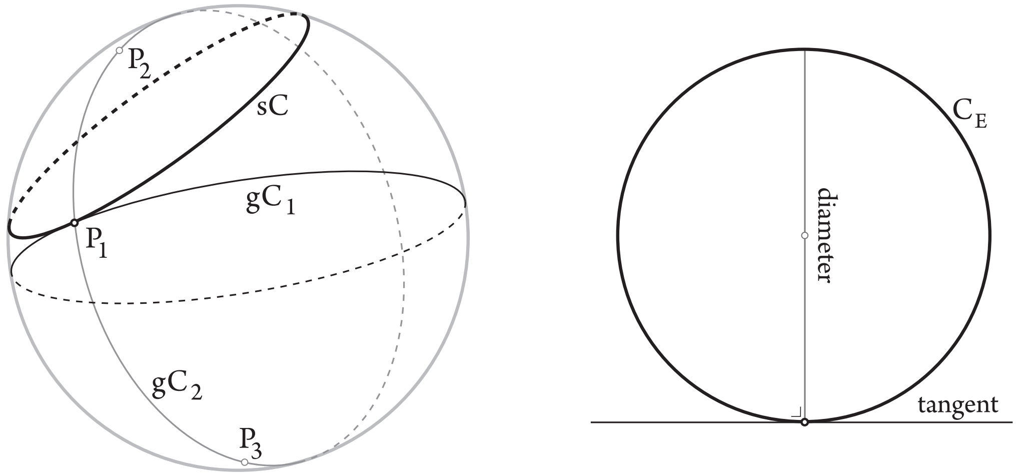

Fig. 3: (left) Perspective diagram of the objects in Th.*Sph*. II.14. For the medieval manuscript figure, see Vat.gr. 204, f. 16*r*. (right) The primary objects of *Elem*. III.16.cor.

Th.*Sph*. II.14 begins, "Given a circle less than great in a sphere and some point on its circumference, draw through the [given] point a great circle touching the given circle."[34] This means that if we have some given small circle, sC, and a given point, P_1, on it, then we are to produce a great circle, gC_1, passing through P_1 that is tangent to sC (see Fig. 3 (left)). Since, when two circles are tangent a unique great circle will pass through their poles and the point of tangency, by Th.*Sph*. II.3-II.5, the construction needed for Th.*Sph*. II.14 can be effected by first drawing the great circle that will pass through these three points, or rather any pair of them. This can be done, fairly straightforwardly, by first taking the pole of *sC*, as say P_2, Th.*Sph*. I.21,[35] and drawing great circle gC_2 passing through P_2 and P_1, Th.*Sph*. I.20. Then, if P_3 is found by cutting off Arc(P_1P_3) subtending the side of a square inscribed in a great circle, with *Elem*. I.post.3(compass),[36] a great circle, gC_1, can be drawn around P_3 as pole, with the same polar radius.

The simplicity of both the diagram and the text of Th.*Sph*. II.14 makes it clear that this proposition follows almost immediately from the concepts that Theodosios has developed in *Spherics* II up to this point. Nevertheless, it is

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34 Czinczenheim, Sphériques *de Théodose*, 2000, p. 101.

35 Notice that when the pole P_2 is taken, it simply appears in the figure. That is, none of the auxiliary objects that would have been used to construct it, such as a great circle through it, are found in the figure, and since such a great circle is later needed, it must be constructed independently in the next step. This follows the standard practice of Euclid's *problem-constructions*; see Sidoli, "Uses of construction", 2018, p. 432-442.

36 The details of finding the side of a square inscribed on a great circle are discussed above in the treatment of Th.*Sph*. I.20, Section 2.2, paragraph 3.

worth pointing out that this *problem* is closely related to *Elem*. III.16. This theorem, which is one of the more interesting propositions in the early geometrical books of the *Elements*, demonstrates the various properties of a tangent to a circle, and in particular shows that it is perpendicular to the diameter through the point of tangency. Then, in a corollary, it is remarked that "the [straight line] produced at an upright to the diameter of the circle at an extremity touches the circle."[37] Although *Elem*. III.16.cor. is expressed as a statement of fact and does not use the idiom usually employed for a *problem* or construction, it clearly has a constructive implication. Indeed, since tangents are often constructed through a given point on a circle, especially in Euclid's *Elements* IV, this corollary was almost certainly introduced to justify such constructions, whether by Euclid or a later editor. The construction implied by *Elem*. III.16.cor. would be to take the center of the given circle, *Elem*. III.1, join a diameter through the center and the given point, *Elem*. I.post.1, and then erect a perpendicular to it at the given point, *Elem*. I.11 (see Fig. 3 (right)). If we consider such a configuration as an analog to Th.*Sph*. II.14, we see that it involves the final set of analogies discussed above – namely, $gC_S \leftrightarrow l_E$ and $sC_S \leftrightarrow C_E$. Furthermore, in this analogy, we see that the spherical construction that is the analog to erecting a line perpendicular to a given line, *Elem*. I.11, is drawing a great circle whose pole lies on a given great circle.

Th.*Sph*. II.15 presents the clearest example of a proposition in Theodosios' *Spherics* that develops an analogy between the properties of objects on the surface of the sphere with similarly configured objects in the plane.[38] Indeed, considering the analogy between Th.*Sph*. II.14 and *Elem*. III.16.cor. and the fact that many of the propositions in the *Spherics* have analogs in *Elements* III, it seems likely that Theodosios intentionally modeled the construction in Th.*Sph*. II.15 on that in *Elem*. III.17. The enunciation of Th.*Sph*. II.15 is: "Given a circle less than great in a sphere and some point on the surface of the sphere between it and the [circle] equal and parallel to it, draw through the [given] point a great circle touching the given circle."[39] That is, given a small circle, sC_1, and a point, P_1, between sC_1 and the other parallel circle that is equal

37 Heiberg, Stamatis, *Euclidis* Elementa, 1969-1977, vol. I, p. 119.

38 Bjørnbo, *Studien über Menelaos'* Sphärik, 1902, p. 47; Schmidt, On the Relation, 1943, p. 12-13.

39 Czinczenheim, Sphériques *de Théodose*, 2000, p. 102.

to it,[40] it is necessary to drawn a great circle, say gC_1 (or gC_1'), passing through P_1 and tangent to sC_1(see Fig. 4 (left)).

In order to understand how Theodosios proceeds in Th.*Sph*. II.15, we will first consider the analogous *problem* of *Elem*. III.17, which begins, "From a given point produce a straight line touching a given circle."[41] That is, with a given a circle, C_{E_1}, and a given point, P_1, outside of C_{E_1}, it is necessary to find a line passing through P_1 and tangent to C_{E_1} (see Fig. 4 (right)). The construction proceeds as follows: (1) The center of C_{E_1} is taken as P_2, *Elem*. III.1, and (2) P_2 and P_1 are joined, *Elem*. I.post.1, meeting C_{E_1} at P_3. Then, (3) with P_2 as center and P_2P_1 as distance, a circle, C_{E_2}, is drawn, *Elem*. I.post.3. (4) A perpendicular, P_3P_4, is erected from point P_3 on line P_2P_1, *Elem*. I.11, meeting C_{E_1} at P_4. (5) P_4P_2 is joined, *Elem*. I.post.1, meeting C_{E_1} at P_5. Finally, (6) P_1P_5 is joined, *Elem*. I.post.1. That this line is tangent to C_{E_1} is shown by demonstrating that $\angle P_2P_5P_1$ is right, so that P_1P_5 is tangent, by *Elem*. III.16. Although the Euclid's *Elements* never mentions this possibility, it is clear that another tangent, P_1P_5', could be found by initially erecting P_3P_4' in the other direction. The fact that only one of these two possible tangents is produced in the *Elements* follows the standard procedures of that text.[42]

40 The fact that there are no more than two equal parallel circles in a sphere is assumed in the proof of Th.*Sph*. II.7, but it can be shown directly by appealing to Th.*Sph*. II.1, II.2, I.8 and I.6.

41 Heiberg, Stamatis, *Euclidis* Elementa, 1969-1977, vol. I, p. 119.

42 In fact, when a *problem*, such as *Elem*. I.11, is applied in the *Elements* it produces a single object, so that applying *Elem*, I.11 once and then following through with the stated construction, would give only a single tangent, P_1P_5, which was probably Euclid's intention.

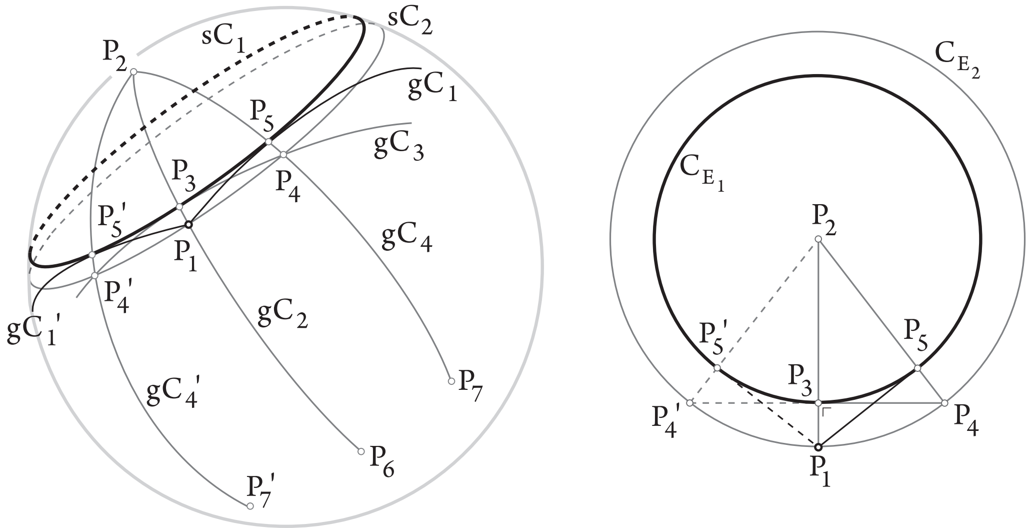

Fig. 4 : (left) Perspective diagram of the primary objects in Th.*Sph*. II.15. (right) The objects of *Elem*. III.17, with the other possible solution added in dotted lines. For the medieval manuscript figures, see Vat.gr. 204, f. 17*r*, and Vat.gr. 190, f. 57*v*.

We will now go through the construction in Th.*Sph*. II.15, noting the similarities and differences between it and that of *Elem*. III.17.[43] The *exposition* of Th.*Sph*. II.15 states that there is a given small circle, say sC_1, and a given point, P_1, between sC_1 and the parallel circle that is equal to it (see Fig. 4 (left)). The construction proceeds as follows. (1) The pole of sC_1 is taken as P_2, Th.*Sph*. I.21, and then (2) a great circle, gC_2, is drawn through P_1 and P_2, Th.*Sph*. I.20, meeting sC_1 at P_3, and (3) a small circle, sC_2, is drawn around pole P_2, with distance P_2P_1. Next, (4) **Arc**(P_3P_6) is cut off subtending the side of a square inscribed in a great circle, with *Elem*. I.post.3(compass),[44] and a great circle, gC_3, drawn around P_6 as a pole, *Elem*. I.post.3(compass), passing through P_3 and meeting sC_2 at P_4 and P_4'.[45] (5) Great circles, gC_4 and gC_4',

43 The editions of Heiberg, *Theodosius* Sphaerica, 1927, and Czinczenheim, Sphériques *de Théodose*, 2000, are rather different for this proposition. Here I use Czinczenheim's text, because Heiberg followed a 14th-century Byzantine recension (Par.gr. 2448). Furthermore, I drop the extra case that is bracketed by Czinczenheim at the end of the proposition (p. 105, ll.8-16), agreeing that it is a clear interpolation. This leaves us with just the case explicitly mentioned in the *enunciation*. The other cases are simpler than that given in the text, so Theodosios himself probably omitted them, following a common practice in Greek mathematical texts.

44 For the details of this construction, see the discussion above concerning Fig. 1 (right), Section 2.2, paragraph 3.

45 It is clear that gC_3 must intersect sC_2 because P_3 is on one side of sC_2, while P_6 is on the other.

are joined through points P_2 and P_4 as well as P_2 and P_4', Th.*Sph*. I.20 (twice). Finally, (6) **Arc**(P_5P_7) and **Arc**($P_5'P_7'$) are cut off subtending the side of a great square, *Elem*. I.post.3(compass), and then a somewhat involved argument concerning solid objects shows that if a great circle, gC_1, is drawn around P_7 as pole, it will pass through both P_1 and P_5, and likewise if a great circle, gC_1', is drawn around pole P_7', it will pass through P_1 and P_5'. The *demonstration* that the two great circles gC_1 and gC_1' complete the *problem*, follows immediately from the construction.

Remembering that drawing a great circle whose pole lies on another great circle is analogous to erecting a line perpendicular to another in the plane, as we saw in comparing Th.*Sph*. II.14 with *Elem*. III.16, when we compare steps (1)-(6) of Th.*Sph*. II.15 with those of *Elem*. III.17, we see that the numbered operations correspond, step-by-step, between the two constructions. Furthermore, this comparison makes it clear why there are two solutions for Th.*Sph*. II.15 but only one for *Elem*. III.17. Specifically, in step (4) of *Elem*. III.17 a perpendicular is erected, whereas in step (4) of Th.*Sph*. II.15 a great circle is drawn. Since in the syntax of a *problem-construction* new objects are brought in by relying on previously established *problems*,[46] and since in *Elem*. I.11 and I.12 a perpendicular is produced as a segment, or ray, on one side of a given line, only one perpendicular will be produced when *Elem*. I.11 is applied. On the other hand, when a great circle is drawn about a pole passing through a point, the whole circle will be introduced. Hence, simply following through the syntax of the *constructions* in each proposition will lead to one solution for *Elem*. III.17 but two solutions for the analogous Th.*Sph*. II.15. Finally, it should be noted that although step (4) of the *construction* of Th.*Sph*. II.15 could be carried through by simply applying Th.*Sph*. II.14 to produce gC_3 tangent at P_3, this would not produce the pole P_6, since the tangent would simply appear in the figure.[47] But since the pole is needed for the demonstration, and hence would have to be taken anyway, nothing would be gained by applying Th.*Sph*. II.14 directly.

In working through the details of the constructions of Th.*Sph*. I.20, I.21, II.14 and II.15, and in comparing them step-by-step with their analogs, *Elem*. I.post.1, III.1, III.16, and III.17, we see a strong relationship between the two sets of propositions, and it seems almost certain that Theodosios designed his approach by modeling his propositions on those of Euclid's theory of the circle, *Elements* III. This resulted in the analogies between a great circle on the surface of the sphere and a line in the plane, $gC_S \leftrightarrow l_E$ and, and between a small circle

46 Sidoli, "Uses of construction", 2018, p. 418-431.

47 *Ibid.*, p. 432-442.

on the surface of the sphere and circle in the plane, $sC_S \leftrightarrow C_E$. Whether or not Theodosios explicitly intended to develop these analogies, however, remains unclear, and as I will argue in the next section, he does not seem to have considered the possible implications of these analogies for producing analogs to the propositions of Euclid's theory of plane rectilinear figures, such as those in *Elements* I and VI.

2.3. Limitations of Spherical Geometry in Theodosios' Approach

There are three theorems in Theodosios' *Spherics* that establish what we may call congruence relations.[48] Th.*Sph.* II.11 and II.12, which are solid configurations that may occur in a sphere, involve perpendicular circles, one of which may be great or small, and are stated in terms of internal straight lines. Th.*Sph.* III.3 is described as occurring in a sphere, mathematically involves three great circles, but is again stated in terms of internal lines. If we use the analogies developed in the *problems* of Theodosios' *Spherics*, we find that Th.*Sph.* II.11 and II.12 do not have specific analogs in the propositions of Euclid's *Elements*, although their plane analogs can be readily shown using the key propositions of the *Elements*. Nevertheless, it is unlikely that Theodosios used such an approach in motivating Th.*Sph.* II.11 and II.12. From a mathematical perspective, Th.*Sph.* III.3 is related to *Elem.* I.4 and to M.*Sph.* I.4, but conceptually it seems rather far removed from either.

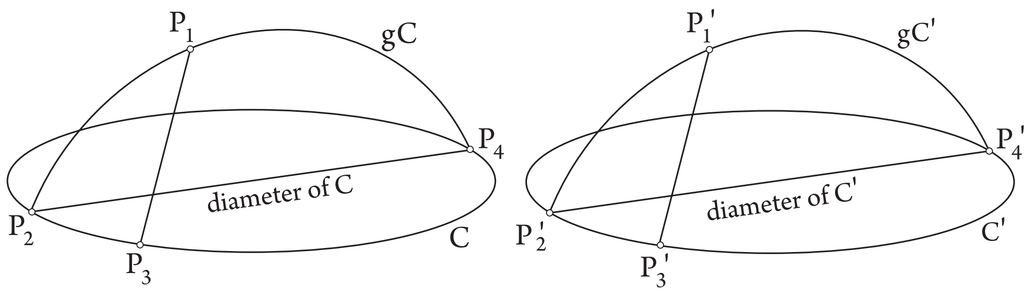

Fig. 5: Perspective diagram of the primary objects in Th.*Sph.* II.11 and II.12. For the medieval manuscript figures, see Vat.gr. 204, f. 14*r*–14*v*.

Since Th.*Sph.* II.11 and II.12 will be used below it may further the discussion to outline them here. These propositions show that if two equal segments,

48 Bjørnbo, *Studien über Menelaos'* Sphärik, 1902, p. 32-35; Schmidt, *On the Relation*, 1943, p. 17-18.

say $gC=gC'$,[49] are perpendicular on two equal circles, $C = C'$, and if $\mathbf{Arc}(P_1P_2) = \mathbf{Arc}(P_1'P_2') \neq \mathbf{Arc}(P_2P_4)/2 = \mathbf{Arc}(P_2'P_4')/2$,[50] are cut off of gC and gC', then (see Fig. 5)

$$\mathbf{Arc}(P_2P_3)=\mathbf{Arc}(P_2'P_3') <=> P_1P_3 = P_1'P_3'.$$

If these configurations are found in a sphere, then the segments will be arcs of great circles, and since great-circle arcs may be drawn subtending lines P_1P_3 and $P_1'P_4'$, these theorems establish congruence relations between a pair of spherical figures that have two sides that are great circles and one side that may be either a great or a small circle – and this is, in fact, how they are commonly used in both *Spherics*.

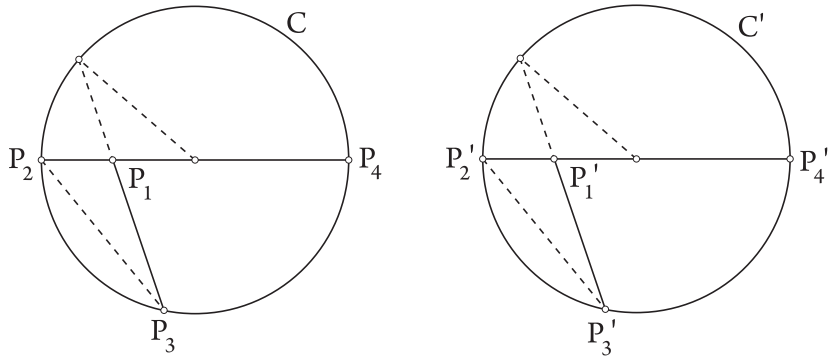

Fig. 6: Diagram for a possible plane analog to Th.*Sph*. II.11 and II.12. Not found in our sources.

Using the analogies developed in the problems – namely, $gC_S \leftrightarrow l_E$ and $sC_S \leftrightarrow C_E$ – it is fairly straightforward to see that the plain analogs of Th.*Sph*. II.11 and II.12 can be shown using propositions of the *Elements*, such as *Elem*. I.4, I.8, III.27, III.28 and III.35. That is, assuming two equal circles, $C = C'$, such that $P_2P_4 = P_2'P_4'$ (see Fig. 6), it can be shown that where $P_1P_2 = P_1'P_2'$, then

$$\mathbf{Arc}(P_2P_3) = \mathbf{Arc}(P_2'P_3') <=> P_2P_3 = P_2'P_3' <=> P_1P_3 = P_1'P_3'.$$

49 These segments are treated as *any* segments in Th.*Sph*. II.11 and II.12 themselves, but when these theorems are applied, they will always be great circles, hence I denote them with gC and gC'.

50 These theorems explicitly state that $\mathbf{Arc}(P_1P_2) = \mathbf{Arc}(P_1'P_2') < \mathbf{Arc}(P_2\ P_4)/2 = \mathbf{Arc}(P_2'P_4')/2$, but since for any assumed $\mathbf{Arc}(P_2P_3)$ and $\mathbf{Arc}(P_2'P_3')$ or P_1P_3 and $P_1'P_3'$ the same claims hold by symmetry at the other ends of the diameters of C and C', at P_2' and P_5', it is clear that this implies that $\mathbf{Arc}(P_1P_2) = \mathbf{Arc}(P_1'P_2') \neq \mathbf{Arc}(P_2P_4)/2 = \mathbf{Arc}(P_2'P_4')/2$. (This point is made by Ver Eecke, *Les* Sphériques, 1927, p. 45, n.2.) We will see below that this is, in fact, how the proposition was understood.

In the plane, however, these are not particularly useful congruence relations, and they would be shown using more the general congruence theorems *Elem.* I.4 and I.8. Hence, it is likely that the existence of this plane analog to Th.*Sph.* II.11 and II.12 is simply a geometrical fact and does not provide us with any historical insight into the motivation for Theodosios' approach.

It remains to examine Theodosios' final congruence theorem. In order to compare Th.*Sph.* III.3 with M.*Sph.* I.4, which we will later read in detail, the proposition can be sketched as follows. Th.*Sph.* III.3 shows that if two great circles, say gC_1 and gC_2, intersect at, say, P_1, and if $\mathbf{Arc}(P_1P_2) = \mathbf{Arc}(P_1P_3)$ are cut off of gC_1, while $\mathbf{Arc}(P_1P_4) = \mathbf{Arc}(P_1P_5)$ are cut off of gC_2, then the straight lines joining the endpoints of the arcs so cut off are equal, $P_2P_4 = P_3P_5$ (see Fig. 7). The proof, which we do not need to follow here, uses an argument of solid geometry.

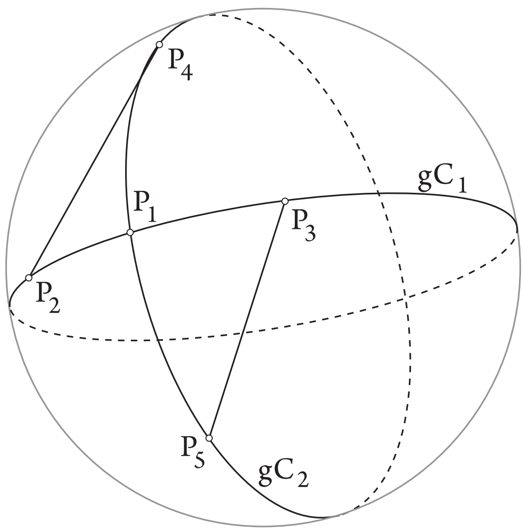

Fig. 7: Perspective diagram of the primary objects in Th.*Sph.* III.3. For the medieval manuscript figure, see Vat.gr. 204, f. 28*r*.

From this configuration, it can be seen immediately that if a great circle is drawn through P_2 and P_4 and another through P_3 and P_5, then the two great-circle arcs between those points will be equal, *Elem.* III.28. Indeed, when Th.*Sph.* III.3 is applied in Th.*Sph.* III.13, and when a related proposition had been applied in Euclid's *Phenomena* 12, the goal is to show that these great-circle arcs are equal. Nevertheless, when Th.*Sph.* III.3 is applied, the argument proceeds by first asserting that the lines are equal, and then that the corresponding great-circle arcs are equal. In Th.*Sph.* III.6 and III.8, the only other two propositions that use Th.*Sph.* III.3 in the Theodosios' *Spherics*, only the equality of the lines is necessary for the argument.

When we consider this configuration, especially in the light of Menelaos' later conception of a *spherical figure*, we can recognize that Th.*Sph*. III.3 is mathematically equivalent to a special case of side-angle-side congruence for spherical triangles – that is, it is related to *Elem*. I.4. In Theodosios' *Spherics*, however, this proposition is only articulated for the special case in which the two spherical triangles share vertical angles, and it is only used in propositions that geometrically describe the configuration that obtains for rising times of arcs of the ecliptic on the inclined sphere. Hence, it is clear that Theodosios regarded Th.*Sph*. III.3, not as an important theorem in a new conception of spherical geometry, but rather as a special lemma useful to his project of establishing the geometry of spherical astronomy. As we will see in the next section, this orientation changed with the work of Menelaos.

3. Spherical Geometry in Menelaos' *Spherics*

Menelaos' *Spherics* introduces the concept of a *spherical figure*, the sides of which are composed of great-circle arcs, to develop an intrinsic geometry of *spherical triangles* (literally, *trilaterals*), which is then applied to geometrical configurations that can be understood as treating the same astronomical topics that Theodosios handled in Th.*Spherics* III, and then to developing a series of propositions that could be useful to spherical trigonometry, especially as applied to spherical astronomy. Menelaos mentions Theodosios by name and expected his readers to be familiar with the former's treatise, as can be seen from the fact that he relies on a number of *problems* and theorems that are established in that text. Menelaos' *Spherics* is divided into the following three parts:[51]

51 Each of the various medieval versions of this treatise enumerate the propositions differently. A table of concordance for the proposition numbers in the main Arabic versions is given by Sidoli, Kusuba, "Al-Harawī's version", 2014, p. 196. The least likely numbering system is that adopted by Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, in their edition of al-Harawī's version. This numbering system is anomalous, because while the text itself makes multiple internal references to different books, the propositions are numbered continuously – which is not known in any other treatise of the Greek mathematical sciences. (For the references to the different books see by Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 683, 693, 697, 769, 777.) Furthermore, of the four manuscripts that witness this version, only two are numbered continuously, and these often also contain the discontinuous numbers written in the margins. Even if al-Harawī may have numbered his edition continuously, which is not certain, there is no possibility that Menelaos himself originally numbered the propositions in this way. Although the versions of Ibn ʿIrāq and Gerard number the propositions somewhat differently, they divide up the books in the

– Menelaos' *Spherics* I begins by defining spherical figures as contained by the arcs of great circles on the sphere, gives a *problem* for constructing an angle equal to a given angle, and then develops a theory of spherical triangles, in particular establishing conditions of congruence and various relations between elements of two triangles or among elements of one triangle.

– Menelaos' *Spherics* II begins with some constructive theorems related to the possibility of constructing angles under certain circumstances. It then develops a number of theorems that treat a configuration in which equal arcs are cut off of the side of a spherical triangle and joined to the base with great circles, making equal angles to the base. This arrangement results in figures that have the same geometry as certain configurations treated in Theodosios' *Spherics* II, and is used to treat some of the same configurations dealt with in Theodosios' *Spherics* III, namely those that can interpreted as pertaining to the rising times of arcs of the ecliptic over the local horizon.

– Menelaos' *Spherics* III begins with a proposition unlike anything else in the treatise, involving straight lines that have actually been constructed and named, and establishing a compound ratio between chords that subtend the double arcs of a cross-quadrilateral made up of great-circle arcs – the so-called Sector (or Menelaos) Theorem.[52] The Sector Theorem is then applied to produce propositions of spherical trigonometry, many of which have direct applications to spherical astronomy.[53]

Menelaos uses the same constructive procedures as are used in the Theodosios' *Spherics* and requires a number of propositions used in that text. Almost all of the objects studied by Menelaos are produced by great circles drawn on the surface of the sphere. The only exceptions are the Sector Theorem, which constructs and names lines, and the final theorems of Menelaos' *Spherics* II and III, which introduce parallel small circles to show properties already established by Theodosios but using the new methods of Menelaos.

same overall way. (See Krause, *Die* Sphärik *von Menelaos*, 1936; Bjørnbo, *Studien über Menelaos'* Sphärik, 1902.) I follow that division here, since both of these were based, in some way or another, on a better translation of the Greek text.

52 Sidoli, "The sector theorem", 2006.

53 Nadal, Taha, Pinel, "Le contenu astronomique", 2004.

3.1. Analogies between Menelaos' Spherics *and Euclid's* Elements *I, VI*

In the initial geometric portion of the his *Spherics*, Menelaos develops a theory of the spherical triangle that can be directly compared with the theories of the triangle established by Euclid in *Elements* I and VI. Hence, in Menelaos' approach there is a clear and sustained analogy between a great circle on the surface of the sphere and a line in the plane, $gC_S \leftrightarrow l_E$. This analogy is made clear already in the definition of a spherical triangle:

> Let that [object] contained by three circumferences in a spherical surface, each of which is less than a semicircle of a great circle, be called a trilateral figure.[54]

In order to develop this analogy, Menelaos produces propositions that are direct analogs to propositions treating properties of triangles in the plane, as well as propositions that clearly exhibit features of spherical triangles that are unlike those of plane triangles. Before we read examples of each of these, however, we should consider Menelaos' use of construction.

3.2. Menelaos' Single Problem

Because Menelaos seeks to develop an intrinsic geometry of the spherical surface, he adopts the constructive methods of Theodosios, applying the *problems* developed in Theodosios' *Spherics*, *Elem.* I.post.3(compass), and other constructions used by Theodosios. He also introduces one *problem* of his own, M.*Sph.* I.1, and a number of constructive theorems, M.*Sph.* II.1-II.3, that establish conditions under which M.*Sph.* I.1 can be applied.

M.*Sph.* I.1 shows how to draw a great circle standing at a given point on a given great circle such that the two great circles contain a given angle between one another. Hence, the subject matter of the proposition is related to that of *Elem.* I.23; but M.*Sph.* I.1 is not a direct analog of *Elem.* I.23. In fact, the construction provided in M.*Sph.* I.1 is exactly analogous to an alternative construction for *Elem.* I.23 that Proklos tells us was due to Apollonios. Proklos also mentions that Menelaos wrote an alternate version of *Elem.* I.25,[55] so it is clear that Menelaos was interested in questions regarding the foundations of

54 For the Greek text see Acerbi, "Traces of Menelaus' *Sphaerica*", 2015, p. 100. It may be worth noting that *Elem.* I.def.19 and I.def.20 also specifically refer to plain triangles as trilateral figures.

55 Friedlein, *Procli Diadochi*, 1873, p. 345-346.

mathematics, as such a topic was understood among Greek mathematicians.[56] Furthermore, Menelaos elsewhere mentions the relationship between some of his work and that of Apollonios,[57] so we know that Menelaos was familiar with Apollonios' work and likely read the latter's reworking of Euclid's *Elements* I.

After explaining that Apollonios' version of *Elem*. I.23 is not to be preferred to Euclid's, because it depends on material that is demonstrated in *Elements* III, Proklos says:

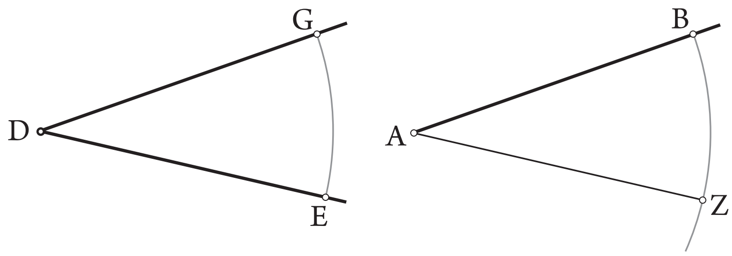

Fig. 8: Interpretive diagram of the objects in Apollonios' version of *Elem*. I.23. For the medieval manuscript figure, see Mon.gr. 427, f. 184*r*.

> For, the latter [Apollonius] – [1] having taken an arbitrary (τυχοῦσαν) angle, angle *DGE*, and a straight line, *AB* – [2] with a center, *G*, and a distance, *GE*, draws circumference *GE*, and [3] in the same way (ὡσαύτως) with a center, *A*, and a distance, *AB*, [draws circumference] *BZ*. And – [4] having taken *ZB* equal to *GE* – [5] he joins *AZ*. And [6] he declares *A* and *G* equal angles, as standing on equal circumferences.[58]

Following this, Proklos correctly points out that Apollonios must assume that *AB* is equal to *DG*, and then objects that the whole construction necessarily depends on later material. Although Proklos' account of Apollonios' procedure is somewhat looser than we generally expect from a Greek mathematical text, it should be possible to make sense of Apollonios' construction. Indeed, since I believe that the claim that *AB* = *DG* was part of the original argument, and that Apollonios had probably carefully thought through the issues involved in this construction, I will go through the details of the argument. In fact, if we assume that Apollonios intended his construction to work through an application of *Elem*. I.post.3(compass), then we will see that no use of material from *Elements* III is necessary to establish the validity of the construction.

56 For a discussion of two of the ways in which Greek mathematicians approached foundations of mathematics see Acerbi, "Two Approaches", 2010.

57 Sidoli, Kusuba, "Al-Harawī's version", 2014, p. 172-173.

58 For the Greek text see Friedlein, *Procli Diadochi*, 1873, p. 335-336.

Apollonios' construction is as follows. In [1], he begins with **Ang**(GDE) and line AB. These would be the *given* objects of *Elem.* I.23 (see Fig. 8). The fact that Apollonios states these as "arbitrary" is probably an indication that he does not acknowledge a difference between the two different ways of being *given* that we find in Euclid's *Data*.[59] If so, his understanding of given accords more closely with ours, since there is no logical difference between the two ways in which objects can initially be taken as given. That is, at this point, G, E, A and B are unspecified parts of the letter-names of rays DG, DE, and line AB. Next, in [2], Apollonios draws a circle, **C**(GE), about center D, with an arbitrary distance, DG, intersecting the rays of the given angle at G and E, *Elem.* I.post.3(compass). Now points G and E are specified as the intersections of the rays and the circle. In the next step, [3], he draws a circle, **C**(BZ), around A as center with distance AB. The qualification of "in the same way" should probably be understood to mean with the same distance – that is, $AB = DG$. The use of this vague expression is likely an indication that Proklos took this construction from a summary, not from a text by Apollonios himself, an author not generally known for such imprecision. At any rate, this specifies A and B as both the endpoints of segment AB, and as the center of the circle and its intersection with line AB. If Apollonios allows the production of **C**(BZ) with distance $AB = DG$, this means that he understands the operation *Elem.* I.post.3(compass) to be an abstraction of the possible action of a real compass. At this stage of the construction, Z is an unspecified part of the letter-name of the circle. The next step, [4], involves another application of *Elem.* I.post.3(compass), cutting off $BZ = GE$. The text does not say whether we should understand this as the distance from Z to B, or as the **Arc**(BZ). Both pairs are equal, but through the use of *Elem.* I.post.3(compass) we are only in position to assert that the distances are equal without adducing further considerations. Hence, this operation specifies point Z as the point on the circumference of **C**(BZ) such that the distance from B to Z is equal to that from G to E. Finally, in [5], line AZ is joined, *Elem.* I.post.1, which completes the construction.

Proklos does not give the demonstration but he hints, in [6], that it relies on a claim that **Arc**(BZ) = **Arc**(GE). If this were the case, it would likely have to depend on something like *Elem.* III.28, which shows that in equal circles, equal arcs stand on equal chords, and *Elem.* III.27, which shows that in equal circles angles that stand on equal arcs are equal. The justifications for these theorems can be traced back through *Elem.* III.26 to stand on *Elem.* III.24,

59 For a discussion of these two different ways of being given, see Sidoli, "The concept of *given*", 2018.

Elem. I.8 and *Elem.* I.4, which are the three propositions of Euclid's *Elements* that are famously proved through superposition. Since we know that Apollonios accepted superposition and used it in his work,[60] it is possible that he was reworking the foundations of geometry to rely more directly on superposition and to be less centered on a theory of triangles. Another possibility is that Proklos was wrong about the demonstration, and that it relied instead on a claim that the distances *GE* and *BZ* are equal, which would follow directly from *Elem.* I.post.3(compass). If this were the case, Apollonios' argument could have depended only on something like *Elem.* I.8, which asserts side-side-side congruence, and could have been shown through superposition, or through more purely constructive assumptions.

Whatever may have been the case with the demonstration, we can make a few certain claims about the construction. It relies on something like *Elem.* I.post.3(compass) to transfer distances and something like *Elem.* I.post.1 to join lines. It uses these principles to build an isosceles version of the triangle used in *Elem.* I.23, which had been produced in *Elem.* I.22. Moreover, it does not rely on previously established *problems*, but builds everything from constructive axioms. In this regard, it is different from the problems of Euclid's *Elements*,[61] and establishing this constructive difference was probably one of Apollonios' objectives.

As we will see, the procedure employed by Menelaos in M.*Sph.* I.1 is directly analogous to that in Apollonios' version of *Elem.* I.23. Ibn 'Irāq's version of Menelaos' *Spherics* I.1 begins as follows:[62]

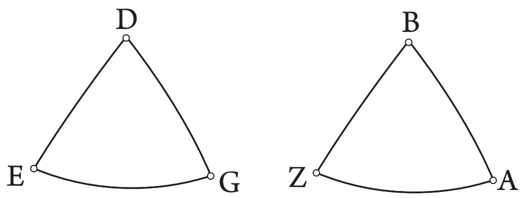

Fig. 9: Diagram for M.*Sph.* I.1. For the medieval manuscript figure, see Leid.or. 390, f. 1*b*.

60 Acerbi, "Two Approaches", 2010, p. 166-168.

61 Sidoli, "Uses of construction", 2018, p. 434-442.

62 It should be noted that al-Harawī's version of the *demonstration* for this proposition is somewhat different, but the construction is essentially the same.

> We want to show how to erect an angle equal to a known[63] angle that great circles contain on a known arc of a great circle at a known point on it.
> For, [1] let the known arc of the great circle be arc *AB*, and the known point point *B*, and the known angle that the great circles contain angle *GDE*. And, we want to erect an angle equal to angle *GDE* at point *B* of arc *AB*.
> Then, [2] we make point *D* a pole and with whatever distance we trace arc *GE*, and [3] we make point *B* a pole and with the same as that distance we trace arc *AZ*.[64] And [4] we cut off from that arc an arc equal to arc *GE*,[65] which is *AZ*. And [5] we trace an arc of a great circle passing through points *B* and *Z*,[66] which is *BZ*.[67]

Reading through this construction, we will see that it follows step-by-step, [1]-[5], the same procedure as that set out in Apollonios' method. In order to discuss Menelaos' construction it may be useful to introduce the same terminology and diagramming style used to discuss Theodosios' *Spherics* above (see Fig. 10).

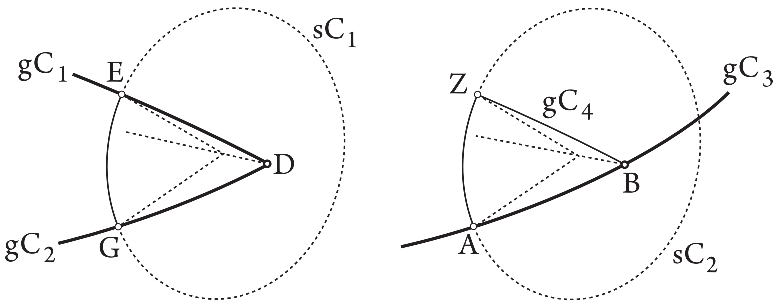

Fig. 10: Perspective diagram of all of the objects mentioned in M.*Sph*. I.1. Not in our sources. Dotted objects are discussed in the text, but neither constructed, named, nor drawn into the diagram.

In the *exposition*, [1], Menelaos' sets out two great circles, gC_1 and gC_2, intersecting with a given angle at point *D*, and some other given great circle gC_3, on which is set a given point, *A*. Then, in [2], he draws a small circle, sC_1, about

63 The term *known* (*ma'lūm*) is a standard Arabic translation of the Greek word for *given*; see Rashed, Bellosta, *Apollonius de Perge*, 2010, p. 467-469; Sidoli, Isahaya, *Thābit ibn Qurra's* Restoration, 2018, p. 210.

64 Both by *Elem.* I.post.3(compass), transferring the span.

65 Again by *Elem*. I.post.3(compass), transferring the span.

66 Th.*Sph*. I.20.

67 Krause, *Die* Sphärik *von Menelaos*, 1936, p. 3 (Arabic).

D with any polar radius, say DG, using *Elem.* I.post.3(compass),[68] and [3] he draws another circle with the same polar radius, $AB = DG$, about B as pole, such that $sC_2 = sC_1$. The text does not say how this can be done but makes it explicit that such an operation is possible. Again, the simplest way to understand this is as an operation of a normal compass used to transfer a polar radius, or distance, *Elem.* I.post.3(compass). In [4], the text states that we cut off **Arc**(BZ) = **Arc**(GE) from sC_2. This can again be done with a normal compass, *Elem.* I.post.3(compass). Finally, [5], Z is joined to B by **gArc**(BZ), Th.*Sph.* I.20.

The *demonstration* need not concern us in detail here, but involves an argument in solid geometry, using Th.*Sph.* I.15, one of the most important solid theorems of Theodosios' *Spherics* I, propositions from Euclid's *Elements* dealing with the circle and solid configurations, and Menelaos' definition of a spherical angle as determined by the dihedral angle of the planes of the great circles that contain it. In fact, the argument is similar to the sorts of argument that we encounter in Theodosios' *Spherics* II and III, but Menelaos has neither constructed nor named any of the solid objects. Instead, he describes the objects, inviting the reader to imagine them and their properties, which have been established in propositions of the *Elements* and Theodosios' *Spherics*. This is a consistent feature of Menelaos' approach,[69] as will be described in more detail in the following section. In Section 4, I will argue that this feature of Menelaos' style is also adumbrated in Theodosios' *Spherics*.

With the establishment of this *problem*, which reproduces exactly the operations of Apollonios' construction of an angle equal to a given angle, *Elem.* I.23, Menelaos makes it clear that his text will work with the analogy between a great circle and a straight line, $gC_S \leftrightarrow l_E$, and a small circle and a circle, $sC_S \leftrightarrow C_E$, but now applied to producing analogs to the theorems of *Elements* I, Euclid's first theory of the plane triangle. The definitions and opening propositions of the Menelaos' *Spherics*, along with Menelaos' assumption that his readers will have already mastered the *Elements* and Theodosios' *Spherics* make it certain that he meant us to understand his use of this analogy as deliberate. In the following section, we will look at a couple of examples of theorems from Mene-

68 Ibn ʿIrāq's text assumes that this circle could also be a great circle and gives a *demonstration* in two cases; see Krause, *ibid.*, p. 3 (Arabic). Although mathematically this might be possible, it is more common for Greek mathematical texts to simply handle the more difficult case. Moreover, there is only one case in Al-Harawī's and Gerard's texts; see Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 507-509; Bjørnbo, *Studien über Menelaos*' Sphärik, 1902, p. 33. This first case was probably added by Ibn ʿIrāq.

69 The only exception is the so-called Sector Theorem, M.*Sph.* III.1.

laos' *Spherics* that are direct analogs to, or divergences from, propositions of Euclid's *Elements*.

3.3. *Examples of Analogs and Divergences between Meneaos'* Spherics *and Euclid's* Elements *I and VI*

It is well known that many of Menelaos' theorems are analogs to theorems of the Euclid's *Elements*, or demonstrate properties that do not hold for plane triangles, and any number of propositions could be adduced to show these relationships.[70] Indeed, the first ten propositions of Menelaos' *Spherics* I are each an analog to one or more propositions of Euclid's *Elements* I, and then M.*Sph.* I.11 demonstrates a difference from the plane situation by showing what happens when an exterior angle of a spherical triangle is greater than or equal to an opposite interior angle, situations that *Elem.* I.16 shows do not arise for plane triangles. It seems certain that Menelaos is deliberately inviting his readers to conceive of these new spherical figures as sometimes analogous to, and sometimes different from, plane rectilinear figures.

In order to see an example of how Menelaos develops analogs to propositions from the *Elements*, and how he differentiates his approach from that of Theodosios, we will first work through M.*Sph.* I.4. In Ibn ʿIrāq's version, this proposition reads as follows:

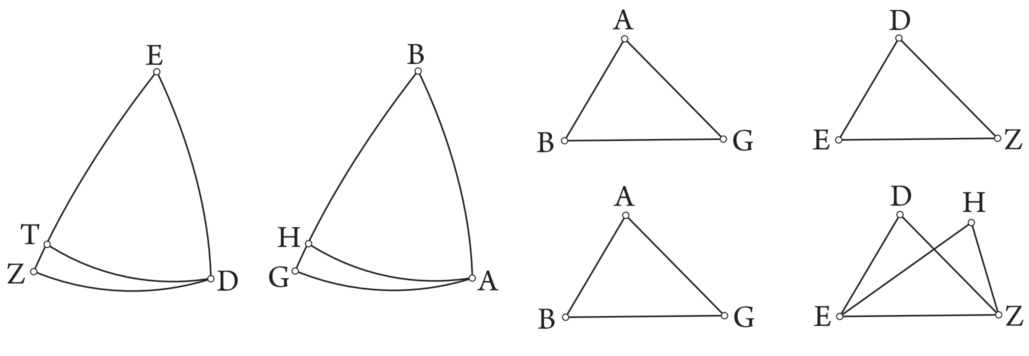

Fig. 11: (left) Diagram for M.*Sph.* I.4. (right, top) Diagram for *Elem.* I.4. (right, bottom) Diagram for *Elem.* I.8. For the medieval manuscript figures, see Leid.or. 390, f. 3*a*, and Vat.gr. 190, f. 18*v*, 21*r*.

> If two sides of a trilateral figure are equal to two sides of another trilateral figure, each to its correspondent, and the base is equal to the base,

70 Bjørnbo, *Studien über Menelaos'* Sphärik, 1902, p. 32-45; Heath, *A History*, 1921, vol. II, p. 262-264; Rashed, Papadopoulos, *Menelaus'* Spherics, 2017, p. 129-204.

> then the angles of those figures that the equal sides contain are equal; and if the angle is equal to the angle, then the base is equal to the base. For, let there be two trilateral figures, which are *ABG* and *DEZ*, and let side *AB* be equal to side *DE*, and side *BG* equal to side *EZ*. Then, I say that [1] if base *AG* is equal to *DZ*, then the angle that is at point *B* is equal to the angle that is at point *E*; and [2] if the angle that is at point *B* is equal to the angle that is at point *E*, then base *AG* is equal to base *DZ*. Indeed, [3] we make points *B* and *E* poles and we rotate two arcs of circles with distance of points *A* and *D*.[71] So, these two circles are equal to one another.[72] And, [4] because arc *BG* is equal to arc *EZ* and arc *BH* to arc *ET*, then arc *GH* must be equal to arc *ZT*.[73] So, [5] two equal segments of equal circles were erected upon their[74] diameters that extend from points *H* and T at right angles,[75] which are those from which equal arcs *GH* and *TZ* have been cut off, and they are not half of the segments, and arcs *AG* and *ZD*, which are equal, were produced, then arc *AH* is equal to arc *DT*.[76] So, [7] the angle that is at *B* is equal to the angle that is at *E*.[77]
> Again, [8] likewise we show that, of the angle that is at *B*, if it is equal to the angle that is at *E*, then base *AG* is equal to base *DE*, because arc *AH* is equal to arc *DT*.[78] And that is what we wanted to show.[79]

The proposition is in two parts. Part 1, stated in [1], is an analog to *Elem*. I.8, side-side-side congruence, and Part 2, stated in [2], is an analog to *Elem*. I.4, side-angle-side congruence. In fact, the *demonstration* for Part 2 is only sketched in Ibn ʿIrāq's version. In Euclid's *Elements*, both of these congruence theorems are demonstrated through superposition, but *Elem*. I.8 also uses an indirect argument, relying on the claim that two different triangles cannot be constructed on the same side of the same base with two pairs of equal sides, *Elem*. I.7. Since Menelaos does not use indirect arguments, he is free to introduce the analog to *Elem*. I.8 first, thus applying Th.*Sph*. I.11 before Th.*Sph*. I.12. His argument is constructive, and direct.

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71 *Elem*. I.post.3(compass).

72 They have equal polar radii.

73 *Elem.* I.c.n.3.

74 This must mean $\mathbf{sC}(AH)$ and $\mathbf{sC}(DT)$.

75 Th.*Sph.* I.15.

76 Th.*Sph.* II.11.

77 M.*Sph.* I.def.3.

78 This is a sketch of an argument that would apply Th.*Sph.* II.12.

79 Krause, *Die* Sphärik *von Menelaos*, 1936, p. 5 (Arabic).

With two spherical triangles **sphT**(ABG) and **sphT**(DEZ), such that the sides are equal pairwise, **gArc**(AB) = **gArc**(DE), **gArc**(BG) = **gArc**(EZ), and **gArc**(AG) = **gArc**(DZ), then if they are equilateral or isosceles, by the definition of a spherical angle, the angles between the equal sides will be equal, so Menelaos does not consider these cases, as the more trivial (see Fig. 12). Otherwise, in [3], he can consider one side to be greater than another, and so use *Elem.* I.post.3(compass), to draw circles C and C' about poles B and E passing through points A and D, so as to cut off the lesser arc from the greater – that is, **gArc**(BA) = **gArc**(BH) = **gArc**(ED) = **gArc**(ET). Hence, by subtraction, **gArc**(HG) = **gArc**(TZ), as stated in [4]. Then, in [5], since the B and E are the poles of C and C', this establishes all of the conditions for Th.*Sph.* II.11, because **gArc**(AG) = **gArc**(DZ) must stand on equal chords, so that Menelaos can assert that **Arc**(AH) = **Arc**(DT) in C and C'. Now, in [7], since these arcs are the same as the dihedral angles at B and E, which are, by definition, the angles of the spherical triangles, the spherical angles are equal. Finally, in [8], the proof of the converse is sketched by pointing out that if, instead of starting with **gArc**(AG) = **gArc**(DZ), it is assumed that the spherical angles at B and E, and the sides about them are equal, then **Arc**(AH) = **Arc**(DT), so that the conditions are satisfied to apply Th.*Sph.* II.12 to show that **gArc**(AG) = **gArc**(DZ), because the subtended chords are equal.

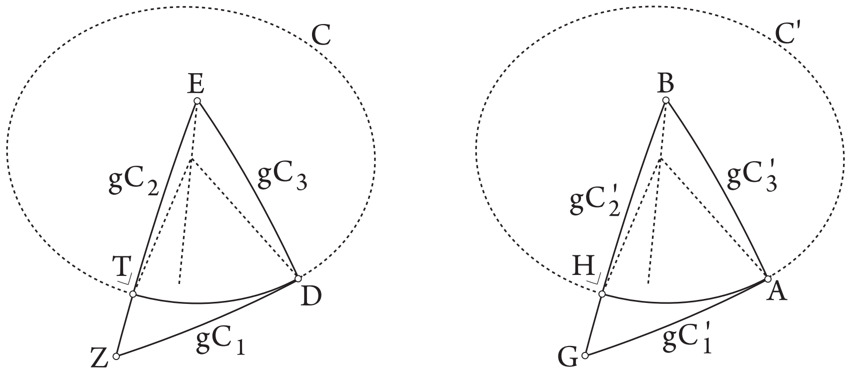

Fig. 12: Perspective diagram of all the objects mentioned in M.*Sph.* I.4. Not found in our sources.

Hence, although Menelaos produced a theorem whose mathematical content is analogous to that in *Elem.* I.8 and I.4, he takes a completely different approach. He does not use superposition, but rather construction, and directly applies two of Theodosios' congruence theorems, which are essentially propositions of solid geometry. Moreover, although Menelaos calls on the solid features of Th.*Sph.* II.11 and II.12, he neither constructs nor names any objects not on the surface of the sphere – in particular, the subtended chords, which are the subject of Theodosios' congruence theorems. This is essentially the same as his approach in his only *problem*, M.*Sph*. I.1. Although the solid geometry plays a crucial role in the argument, there seems to be a deliberate attempt to downplay its significance in the geometric configuration and its presence in the figure.

The opening run of propositions in Menelaos' *Spherics* I demonstrates analogies with, and differences from, Euclid's theory of plane triangles, particularly as concerns congruence and comparisons of the elements of triangles. This is, hence, directly analogous to the subject matter of *Elem*. I.1-I.26, prior to Euclid's development of his theory of parallelism. Among these early propositions of Menelaos' *Spherics* I is an important theorem that has implications for the concept of similarity – namely, Menelaos' *Spherics* I.18, which demonstrates angle-angle-angle congruence for spherical triangles. This implies that there will be no similarity, as distinct from congruence, for spherical triangles. Indeed, Menelaos explores various properties of spherical triangles that can be directly contrasted with properties of plane triangles that are developed in the first part of *Elements* VI, which is Euclid's theory of similarity in plane triangles.

In order to see an example of how Menelaos develops propositions that are different from those in Euclid's *Elements*, we will go through the details of M.*Sph*. I.26, which can be directly compared with *Elem*. VI.2, and which establishes a claim that has been used as an axiom for spaces of positive curvature in modern geometry.[80] Ibn 'Irāq's version of M.*Sph*. I.26 reads as follows:

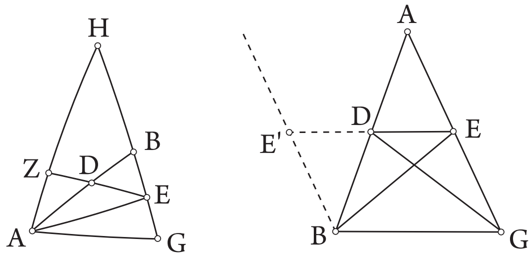

Fig. 13: (left) Diagram for M.*Sph*. I.26. (right) Diagram for *Elem*. VI.2, with additional objects in dotted lines. For the manuscript figures see Leid.or. 390, f. 12*a*, and Vat. gr. 190, f. 87*r*.

> If any two sides of a trilateral figure are partitioned into two halves, then the arc that is traced in it between the two midpoints is greater than half of the base.

80 Busemann, "Spaces with non-positive curvature", 1948, p. 1; Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 205.

For, [1] let there be a trilateral figure, on it *ABG*, and let two of its sides *AB* and *BG* be partitioned into two haves at points *D* and *E*. And we draw an arc in it between points *D* and *E*, which is arc *DE*. Then, I say that arc *DE* is greater than half of base *AG*.
Indeed, [2] we extend *DE* to *Z*,[81] and [3] we make *DZ* equal to *DE*,[82] and [4] we join arc *AZ* in it between *A* and *Z*,[83] and [5] we extend each of arcs *AZ* and *GB*.[84] And [6] let them meet at point *H*.[85]
Then, [7] sides *BD* and *DE* are equal to sides *AD* and *DZ*, each side to its correspondent,[86] and those sides contain equal angles,[87] so base *BE* is equal to base *AZ*.[88] And, again, [8] it is equal to arc *EG*, so arc *EG* is equal to arc *AZ*.[89] And [9] angle *ABE* is equal to angle *BAZ*,[90] so the two arcs *AH* and *HB*, joined, are equal to a semicircle – as is shown in the tenth proposition[91] – so, [10] arcs *AH* and *HE*, when joined, are greater than a semicircle. And [11] we produce arc *AE*,[92] so [12] angle *AEG* is less than angle *EAZ* – as is shown from the converse to the tenth proposition[93] – and [13] the two sides *ZA* and *AE* are equal to the two sides *GE* and *EA*, each side to its correspondent, and angle *ZAE* is greater than angle *AEG*, so base *ZE* is greater than base *AG*.[94] So, [14] arc *ED* is greater than half of *AG*.[95] And that is what we wanted to show.[96]

In order to discuss Menelaos' procedure, it may be helpful to work through the details of the argument. In the *exposition*, [1], a spherical triangle, **sphT**(ABG), composed of gC_1, gC_2, and gC_3, is assumed in which a great circle arc, gC_4 is drawn through it bisecting two of the sides at points *E* and *D* (see Fig.

81 Th.*Sph*. I.20.

82 *Elem*. I.post.3(compass).

83 Th.*Sph*. I.20.

84 Th.*Sph*. I.20 (twice).

85 They must meet in both directions by Th.*Sph*. I.6 and I.11 (see below).

86 That is, *BD* = *AD* and *DE* = *DZ*.

87 Vertical angles (see below).

88 M.*Sph*. I.4. Note that this is also justifiable by Th.*Sph*. III.3, as discussed above.

89 *Elem*. I.c.n.1.

90 Also from M.*Sph*. I.4.

91 M.*Sph*. I.10. The statement of this fact is probably an addition by Ibn ʿIrāq; it is not found in al-Harawī's version; see Rashed, Papadopoulos, *Menelaus*' Spherics, 2017, p. 527.

92 Th.*Sph*. I.20.

93 M.*Sph*. I.10(converse). The parenthetical statement must again be Ibn ʿIrāq's remark.

94 M.*Sph*. I.8.

95 Euclid's *Elements* V.15.

96 Krause, *Die* Sphärik *von Menelaos*, 1936, p. 20-21 (Arabic).

14). The *construction* proceeds, in [2], by extending **gArc**(ED) out towards Z, which is initially unspecified, using Th.*Sph.* I.20, and then, in [3], cutting off **gArc**(DZ) = **gArc**(ED) and specifying *Z*, with *Elem.* I.post.3(compass), next, in [4], joining *A* and *Z* with **gArc**(*AZ*), using Th.*Sph.* I.20, and, finally, in [5], extending **gArc**(*AZ*) and **gArc**(*GB*), with Th.*Sph.* I.20, such that they meet at *H*, in the direction of *B*. There is no discussion of the fact that these great circles, gC_5 and gC_1, must meet in both directions, but this fact follows from considering the implications of Th.*Sph.* I.6, which shows that great circles are concentric with the sphere, and Th.*Sph.* I.11, which proves that all great circles bisect each other. Together these two propositions imply that every two great circles bisect each other at the common section of their planes, *Elem.* XI.3. This is a crucial fact about spherical figures, because in a plane analog to this same configuration, the extension of such a line would be parallel to the far side, *AG*, of the original triangle (see Fig. 13 (right), dotted lines).

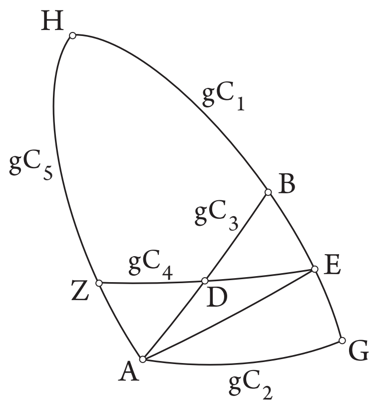

Fig. 14: Perspective diagram for M.*Sph.* I.26. Individual labels of the great-circle arcs are not found in our sources, but otherwise the diagram is close to the manuscript diagrams.

The *demonstration* begins, in [7], by setting up the conditions of M.*Sph.* I.4 so as to infer that **sphT**(*DBE*) ≅ **sphT**(*DAZ*), so that **gArc**(*BE*) = **gArc**(*AZ*) (see Fig. 13 (left)). This step depends on the claim that vertical spherical angles are equal, **Ang**(*BDE*) = **Ang**(*ZDA*). Although this fact is not demonstrated in the extant Greco-Roman works on the geometry of the sphere, something mathematically equivalent to this claim is assumed in Autolykos' *Moving Sphere* 10 and Th.*Sph.* II.22. It is relatively straightforward to reconstruct an argument for this using ancient techniques of the geometry of the sphere, as was done in a scholium to Theodosios' *Spherics*.[97] In [8], a substi-

97 Czinczenheim, Sphériques *de Théodose*, 2000, p. 415 (Scholium 294).

tution leads to the claim that **gArc**(GE) = **gArc**(AZ). In [9], it is claimed that since **Ang**(DBE) = **Ang**(DAZ), again by the congruence of the spherical triangles, Th.*Sph.* I.4, of which **Ang**(DBE) is an exterior angle, while **Ang**(DAZ) is an interior angle of **sphT**(AHB). This implies that **gArc**(AH) + **gArc**(HB) is a semicircle, as is shown in M.*Sph.* I.10 – which proves that (1) if an exterior angle of a spherical triangle is equal to an interior angle then the two arcs of the spherical triangle that meet the third point will together be equal to a semicircle; (2) if the exterior angle is less, then the two arcs are together greater than a semicircle; and (3) if the exterior angle is greater, then the two arcs are together less than a semicircle. Hence, in [10], **gArc**(AH) + **gArc**(HE) is greater than a semicircle, since by construction **gArc**(HE) > **gArc**(HB). In [11], a constructive step produces **gArc**(AE), joining those points, with Th.*Sph.* I.20, and, in [12], it is asserted that **Ang**(AEG) < **Ang**(EAZ). As noted by Ibn ʿIrāq, this follows as a converse to M.*Sph.* I.10 by assuming that this is not the case, and noting that it would immediately contradict the second part of M.*Sph.* I.10, as just stated. Hence, in [13], since there are two sides equal to two sides, **gArc**(ZA) = **gArc**(GE) and **gArc**(AE) = **gArc**(AE), and the angles contained by them are unequal, **Ang**(EAZ) > **Ang**(AEG), the greater base will subtend the greater base. This is established in M.*Sph.* I.8, which shows that if two spherical triangles have two sides equal and the angle contained by them unequal, then the greater side will subtend the greater angle – a direct analog to the plane theorem *Elem.* I.24. This means that **gArc**(ZE) > **gArc**(AG). Finally, this implies that **gArc**(ED) = **gArc**(ZE)/2 > **gArc**(AG)/2. This is an obvious mathematical fact, but it also follows directly from *Elem.* V.15, which shows that parts have the same ratio as their equimultiples.

By using the analogy $gC_S \leftrightarrow l_E$, this proposition can be directly compared with *Elem.* VI.2, with which it exhibits a number of conspicuous differences. *Elem.* VI.2 proves that a straight line is drawn parallel to a base of a triangle if and only if it cuts the sides of the triangle proportionally. That is, in Fig. 13 (right), above,

$$\mathrm{DE} \parallel \mathrm{BG} \iff \mathrm{BD:DA} = \mathrm{GE:AE}.$$

This proposition, together with *Elem.* VI.4, which shows that triangles that have equal angles are similar, implies that, in the plane situation, if E were taken so as to bisect AG, then $DE = BG/2$, and if DE were extended and cut off such that $DE' = DE$, analogously to what is done in the course of M.*Sph.* I.26, then $BE' \parallel GE$. Hence, the claim of M.*Sph.* I.26 shows a fundamental difference from *Elem.* VI.2 in terms of the absence of similarity, and the proof of M.*Sph.* I.26 reveals a key difference from the analogous situation in the plane with regard to the absence of parallelism. In this way, M.*Sph.* I.26 can be interpreted as making claims about the nature of spherical figures, by showing specifically how they crucially differ from plane figures as developed in theorems of Euclid's *Elements*.

A number of further comments can be made about this proposition. As usual for a proposition from such a text, each step of the proof can be justified by previously established propositions from the same treatise, or by claims from a well-defined toolbox,[98] which in the case of the Menelaos' *Spherics* is made up of Euclid's *Elements* and Theodosios' *Spherics*. In M.*Sph.* I.26, in contrast to M.*Sph.* I.1 and I.4, all of the objects introduced are formed by great-circle arcs, so that they model lines in the plain, with the analogy $gC_S \leftrightarrow l_E$. Furthermore, the argument itself also takes place entirely within the spherical surface. That is, it does not employ any solid objects, such as were used in the arguments for M.*Sph.* I.1 and I.4, above. Indeed, when we look at his *Spherics* as a whole, it seems that Menelaos had a preference for such purely intrinsic proofs, and attempted to dispense with solid arguments as soon as possible. Finally, as in Apollonios' revision of *Elem.* I.23, discussed above, Menelaos does not proceed by using previously established *problems* to introduce new objects, but rather depends directly on repeated uses of *Elem.* I.post.3(compass) and Th.*Sph.* I.20, which can be understood both as operations of the compass, from a practical perspective, and as analogous to the Euclidean postulates, *Elem.* I.post.3 and *Elem.* I.post.1, from a theoretical perspective. This seems, again, to indicate that Menelaos is following the constructive approach taken by Apollonios, as opposed to that employed by Euclid.

4. Implicit Solid Geometry in both *Spherics*

In the *demonstrations* of M.*Sph.* I.1 and I.4 that were discussed above, we saw that Menelaos introduced solid considerations, particularly when applying Th.*Sph.* II.11 and II.12, in such a way as to discuss solid objects without actually constructing or naming those objects. In fact, this is a standard feature of Menelaos' style, and there is only one proposition of the text that constructs and names solid objects – namely, the famous Sector Theorem, M.*Sph.* III.1.

98 Saito, "Index," 1997, *Girishia Sugaku*, 1998; Netz, *Shaping*, 1999, p. 216-235.

We can call this style an *implicitly solid* approach, to contrast it with the fully *surface* style of M.*Sph*. I.26 and the *explicitly solid* style of M.*Sph*. III.1.[99] With the exception of M.*Sph*. III.1, Menelaos always uses the implicitly solid or fully surface styles, with a systematic tendency to introduce the implicitly solid approach in the early part of a theory, where necessary, and then to dispense with it as soon as possible, so that he can work entirely on the surface of the sphere.

In fact, all three styles are also found in Theodosios' *Spherics*. For example, we have already seen the fully surface style in Th.*Sph*. II.14 and the explicitly solid style in Th.*Sph*. II.11, II.12, and III.3. Theodosios also employs the implicitly solid manner of handling solid objects, but it is not used systematically, or preferentially. Indeed, in Theodosios' *Spherics* this approach to introducing solid objects may have simply been used so as to avoid cluttering the diagrams, without any deliberate attempt to develop a more intrinsic geometry.

In order to see how this was done, we may look at two examples from Th.*Sph.* II.13, which is, in fact, the first theorem in the text in which either of Th.*Sph*. II.11 or II.12 is applied. Th.*Sph.* II.13 shows that if a pair of great circles are tangent to a pair of small circles and intersect other small circles parallel to the original pair, then they cut off certain pairs of similar arcs from all the small circles between the tangent pair, and the arcs of the great circles between any two small circles are equal. Because the theorem is long and somewhat involved, we will not read through the whole thing. Instead, we will simply look at couple of passages from the *demonstration*.

Initially, in the *exposition*, it is asserted that there are three parallel circles, **pC**(*ABGD*), **pC**(*EZHQ*), and **pC**(*KL*), with great circles, **gC**(*AKG*) and **gC**(*BLD*), tangent to **pC**(*KL*) at points *K* and *L*, respectively (see Fig. 15). Then, in the construction, point *M* is taken as the pole of the parallels, Th.*Sph*. I.21, and **gArc**(*MK*) and **gArc**(*ML*) are drawn, Th.*Sph*. I.20. The demonstration begins by showing that **gArc**(*MK*) and **gArc**(*ML*) will pass through the poles of **gC**(*AKG*) and **gC**(*BLD*), respectively, and then continues as follows:

99 Notice that propositions in the fully surface style may also rely on propositions that themselves explicitly introduce solid objects – such as Th.*Sph*. II.11 in M.*Sph*. I.26 – but they do not themselves make any mention of solid objects.

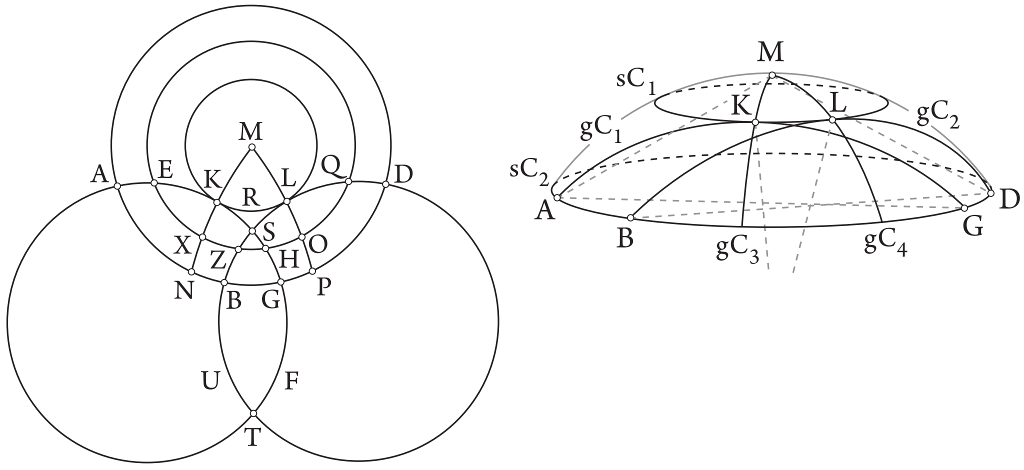

Fig. 15: (left) Diagram for Th.*Sph.* II.13. For the manuscript diagram see Vat.gr. 204, f. 15*v*. (right) Perspective diagram of a selection of the objects in Th.*Sph.* II.13. Not found in our sources.

> And since equal and upright segments of circles, *KM* and what is continuous with them, are set up on diameters in equal circles, *AEKHGT* and *BZLQDT*, from points *K* and *L*, while *KM* and *LM* are equal circumferences [that is, arcs] cut off from them,[100] being less than half of the whole, and the straight line joining from *M* to *A* is equal to the straight line joining from *M* to *D*,[101] therefore, the cut-off circumferences are equal.[102] Therefore, circumference *AK* is equal to circumference *LD*.[103]

This is an application of Th.*Sph.* II.11 in which both the segments and the circles upon which they are perpendicular are great circles, and the lines that join the endpoints of the arcs are polar radii of **pC**(*ABGD*). In this case, the diameters upon which the segments stand, as well as the polar radii, are simply described, but they are neither constructed nor named. In the accompanying perspective diagram, these are drawn as dotted lines, in gray (see Fig. 15 (right)). A few steps later in the *demonstration*, it is established that **gArc**(*AKG*) = **gArc**(*DLB*), after which the text reads, "Therefore the straight line joining from *A* to *G* is equal to the straight line joining from *D* to *B*", which is an application of *Elem*. III.28.[104]

100 Th.*Sph.* I.def.5, *Elem.* III.28.

101 Th.*Sph.* I.def.5.

102 Th.*Sph.* II.11.

103 Czinczenheim, Sphériques *de Théodose*, 2000, p. 99.

104 *Ibid.*

In both of these cases, we see that Theodosios applies solid considerations without actually introducing any objects that are not on the surface of the sphere. The usual idiom for this style of argument is that a line is said to be that "joining from *A* to *B*". This expression is the primary indication of the implicitly solid style, because although such lines, and the figures they produce, are needed for the argument, in terms of the usual idioms of Greek mathematical texts, they are merely implied – the lines are never explicitly introduced in the *exposition* or *construction*, they are never assigned a specific letter-name, and they are not depicted in the diagram, in contrast to the approach in explicitly solid propositions.

In fact, the implicitly solid style is used in 10 propositions of the Theodosios' *Spherics*: II.13, II.19, II.22, II.23, III.5, III.6, III.7, III.8, III.12, and III.13. The fully surface style, however, is used in 12 propositions: I.21, II.4-II.8, II.14, II.16, II.18, II.20, III.9, III.10, and III.14. Theodosios' overwhelming preference, however, is for explicitly solid methods, which are used in 34 propositions: I.1-I.20, II.1-II.3, II.9-II.12, II.15, II.17, II.21, III.1-III.3, and III.11.[105] As we see, Theodosios uses all three approaches and shows no reluctance to use any of them. Although for individual groups of theorems or theories, he often starts with an explicitly solid approach, then moves on to the implicitly solid style and finally ends with fully surface methods, this probably simply comes about because later results are based on those that come before; and, hence, he does not need to further consider the solid objects that were involved in establishing the earlier theorems. This can be contrasted with Menelaos' style which shows a strong preference for avoiding explicitly solid constructions. While the use of such stylistic features in the argument may strike us as mathematically superficial, they seem to have been a key feature of Menelaos' project of developing an intrinsic geometry of the sphere.

5. Conclusion

In comparing the methods of Theodosios and Menelaos, we have seen that although Menelaos sought to develop an intrinsic geometry of the sphere, on analogy with the geometry of the plane, he used the constructive and foundational methods of the Theodosios' *Spherics*. That is, he based his constructions on those in the earlier *Spherics*, and he used the congruence theorems of that text, particularly Th.*Sph*. II.11 and II.12. Furthermore, the core analogy that

105 I do not include Th.*Sph*. I.22 and I.23 in these considerations (see note 21, above).

Menelaos systematically applies in developing his theory of spherical figures – namely, $gC_S \leftrightarrow l_E$ and $sC_S \leftrightarrow C_E$ – was already at work, albeit faintly, in Theodosios' *problems*. Finally, in his approach to construction, Menelaos also appears to have been influenced by the work of Apollonios in rewriting the foundations of plane geometry.

When we speak of Menelaos developing an intrinsic geometry of spherical figures, what we mean is that he demonstrated propositions that are analogous to, and different from, specific propositions of Euclid's *Elements*, or of later reworkings of this material, such as those by Apollonios and, perhaps, himself. That is, he does not start by assuming certain intrinsic properties of spherical figures, such as that which he shows in M.*Sph*. I.26, discussed above, but rather with the methods and propositions that were handed down to him from Euclid, Apollonios, and Theodosios. Just as Theodosios produced analogies between various properties of the sphere and spherical figures with those established in *Elements* III, the theory of the circle, Menelaos explored analogies and differences with the properties of the plane triangle, as set out in *Elements* I and VI. In following this technique of investigating similarities and differences with the known properties of well-established objects, Menelaos was pursuing a common strategy of Greco-Roman mathematicians.[106]

As we saw, the most important theorems of the Menelaos' *Spherics*, and in particular the early congruence theorems, rely on operations that are abstractions of the motions of a rigid compass in three dimensional space, as well as Theodosios' congruence theorems, which are themselves explicitly cast in terms of solid configurations. Nevertheless, when Menelaos employs these, he does so in such a way as to downplay the underlying solid nature of these methods. From our perspective, this boarders on a superficial, essentially stylistic, avoidance, because, despite the fact that he does not construct or name the solid objects, they continue to play an essential role his proofs. Indeed, the intrinsic approach that Menelaos takes could be said to be more stylistic than essential. He does not construct solid objects or name them, but he still uses them. He does not postulate any properties of spherical figures that set them apart from plane objects. Instead, he shows individual similarities and differences with plane objects, often by showing proposition by proposition analogs and divergences with Euclid's *Elements*. In this sense, Menelaos' spherical geometry is still embedded in his Euclidean sources, in both methodology and conception.

106 Fried And Unguru, *Apollonius*, 2001, p. 332-363; Fried, "The Use of Analogy", 2003; Acerbi, "Homeomeric lines", 2010.

If we consider the importance of the role of construction, however, we can make a more sympathetic reading of Menelaos' practice. In such a reading we would distinguish the ontological status of objects that have been constructed, and objects that are merely considered in the argument. This distinction would be similar to that between objects that are introduced as part of a *problem-construction* and those that are introduced in a *proof-construction*, and may, in fact, be counterfactual.[107] In this interpretation, Menelaos shows an overwhelming preference for only constructing objects that are on the surface of the sphere – the only exceptions are certain solid lines in M.*Sph.* III.1. Objects on the surface of the sphere, having been constructed and named, may be thought of as actually there – having been drawn on an actual sphere, or produced by some mental act. The other objects that are discussed but never named, would, then, have a sort of hypothetical status. Menelaos considers the properties that such objects would have if they were produced, and makes arguments based on these properties, but he does not claim that they are actually there. In this sense, one could argue that while he considers and makes use of the properties of these hypothetical objects, he never brings them fully into the discourse, so that the only objects that he actually needs to *construct*, or draw, are those on the surface of the sphere.

One of the most unfortunate circumstances surrounding Menelaos' brilliant new approach to the geometry of the sphere is how little it was appreciated or further developed in antiquity. Although the concept of the spherical triangle and some of the early propositions about triangles were used by Ptolemy in his work on spherical astronomy, Ptolemy does not adopt any of the more advanced material of Menelaos' spherical trigonometry. Although, at least the early parts of Menelaos' *Spherics* were read by mathematicians such as Pappus and Theon and used in their commentaries on astronomical texts, and some later readers may have supplied some lemmas and cases to the text itself,[108] no one appears to have taken up the project of exploring the similarities and differences of figures on the sphere in relation to those in the plane, nor of further investigating the metrical properties of spherical figures. Further investigations of such matters would have to wait until the classical Islamic period, when Menelaos' text was studied in its entirety, and his ideas and methods were pushed in new directions.

107 Sidoli, "Uses of construction", p. 410-417.

108 Acerbi, "Traces of Menelaus' *Sphaerica*," 2015 (especially p. 112).

References

ACERBI F., "Homeomeric lines in Greek mathematics", *Science in Context*, 23, 2010, p. 1-37.

ACERBI F., *La sintassi logica della matematica greca*, Archives-ouvertes.fr, Sciences de l'Homme et de la Société, Histoire, Philosophie et Sociologie des Sciences. https://hal.archives-ouvertes.fr/hal-00727063, 2011.

ACERBI F., "Traces of Menelaus' *Sphaerica* in Greek scholia to the *Almagest*", *SCIAMVS*, 2015, 16, p. 91-124.

ACERBI F., "Two Approaches to Foundations in Greek Mathematics: Apollonius and Geminus", *Science in Context*, 23, 2010, p. 151-186.

ACERBI F., "Types, function, and organization of the collections of scholia to the Greek mathematical treatises", *Trends in Classics*, 6, 2014, p. 115-169.

AUJAC G., "Le langage formulaire dans la géométrie grecque", *Revue d'histoire des sciences*, 37, 1984, p. 97-109.

BERGGREN J. L., "The relation of Greek spherics to early Greek astronomy", *Science & Philosophy in Classical Greece*, BOWEN A.C. (ed.), New York: Garland, 1991, p. 227-248.

BJØRNBO A. A., *Studien über Menelaos' Sphärik. Beiträge zur Geschichte der Sphärik und Trigonometrie der Griechen, Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft 14, Leipzig: Teubner, 1902.

BULMER-THOMAS I., "Theodosius of Bithynia", *Dictionary of Scientific Biography*, vol.13, GILLISPIE C. G. (ed.), New York: Charles Scribner's Sons, 1970, p. 319-321.

BUSEMANN H., "Spaces with non-positive curvature", *Acta Mathematica*, 80, 1948, p. 259-310.

CLAVIUS C., *Theodosii Tripolitae Sphaericorum libri III*, Roma: Domenico Basa, 1586.

CZINCZENHEIM C., *Edition, traduction et commentaire des* Sphériques *de Théodose*, Thèse de doctorat de l'université Paris IV, Lille: Atelier national de reproduction des thèses, 2000.

FRIED M., "The Use of Analogy in Book VII of Apollonius' *Conica*", *Science in Context*, 16, 2003, p. 349-365.

FRIED M., UNGURU S., *Apollonius of Perga's* Conica: *Text, Context, Subtex*t, Leiden: Brill, 2001.

FRIEDLEIN G., *Procli Diadochi in primum Euclidis Elementorum librum commentarii*, Leipzig: Teubner, 1873.

HEATH T. L., *A History of Greek Mathematics*, 2 vols., Oxford: Oxford University Press, 1921 (Reprint, New York: Dover, 1981.)

HEIBERG J. L., *Litterargeschichtliche Studien Über Euklid*, Leipzig: Teubner, 1882.

HEIBERG J. L., *Theodosius Sphaerica*, Abhandlungen der Akademie der Wissenschaften in Göttingen. Philologisch-Historische Klasse; n.F., Bd. 19, Nr. 3, Berlin: Weidmannsche Buchhandlung, 1927.

HEIBERG J. L., STAMATIS E. S., *Euclidis Elementa, Euclidis opera omnia*, vols. 1-5, Leipzig: Teubner, 1969-1977.

HOCHE R., *Nicomachi Geraseni Pythagorei introductionis arithmeticae libri ii*, Leipzig: Teubner, 1866.

HUFFMAN C. A., *Archytas of Terentum: Pythagorean, Philosopher and Mathematician King*, Cambridge: Cambridge University Press, 2005.

HULTSCH F., "Autolykos und Euklid", *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, Philologisch-Historische Klasse*, 38/39, 1886, p. 128-155.

KNORR W., *The Ancient Tradition of Geometric Problems*, Boston: Birkhäuser, 1986. (Reprinted, Dover: New York, 1993.)

KRAUSE M., *Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. Alī b. Irāq mit Untersuchungen zur Geschichte des Texte bei den islamischen Mathematikern*, Berlin: Weidmannsche Buchhandlung, 1936.

KUNITZSCH P., LORCH R., *Theodosius Spherica, Arabic and Medieval Latin*, Stuttgart: Franz Steiner, 2010.

NADAL R., TAHA A., PINEL P., "Le contenu astronomique des *Sphériques* de Ménélaos", *Archive for History of Exact Sciences*, 58, 2004, p. 381-436.

NETZ R., "Proclus' division of the mathematical proposition into parts: How and why was it formulated?", *Classical Quarterly* (N.S.) 49, 1999, p. 282-303.

NETZ R., *The Shaping of Deduction in Greek Mathematics*, Cambridge: Cambridge University Press, 1999.

NEUGEBAUER O., *A History of Ancient Mathematical Astronomy*, New York: Springer, 1975.

NIZZE E., *Theodosii Tripolitae, Sphaericorum libros tres*, Berlin: George Reimer, 1852.

NIZZE E., *Theodosius Von Tripolis, Drei Bücher Kugelschnitte*, Stralsund: Carl Löffler, 1826.

RASHED R., BELLOSTA H., *Apollonius de Perge, La section des droites selon des rapports*, Berlin: Walter de Gruyter, 2010.

RASHED R., PAPADOPOULOS A., *Menelaus'* Spherics*: Early Translation and al-Māhānī/al-Harawī's Version*, Berlin: Walter de Gruyter, 2017.

SAITO K., *Girishia Sugaku no Tool Box no Fukugen*, Research Report, Sakai: Osaka Prefecture University, 1998.

SAITO K., "Index of the propositions used in Book 7 of Pappus' *Collection*", *Jinbunn Kenkyu: The Journal of the Humanities, Chiba University*, 26, 1997, p. 155-188.

SCHMIDT O. H., *On the Relation between Ancient Mathematics and Spherical Astronomy*, PhD Thesis, Brown University, Department of Mathematics, 1943.

SIDOLI N., "The concept of *given* in Greek mathematics", *Archive for History of Exact Sciences* 72, 2018, p. 353-402.

SIDOLI N., "Review of Menelaus' Spherics: Early Translation and al-Māhānī/al-Harawī's Version by Roshdi Rashed and Athanase Papadopoulos", *Aestimatio*, 14, 2020, p. 14-21.

SIDOLI N., "The sector theorem attributed to Menelaus", *SCIAMVS*, 7, 2006, p. 43-79.

SIDOLI N., "Uses of construction in problems and theorems in Euclid's *Elements* I-VI", *Archive for History of Exact Sciences*, 72, 2018, p. 403-452.

SIDOLI N., ISAHAYA Y., *Thābit ibn Qurra's Restoration of Euclid's Data (Kitāb Uqlīdis fī al-Muṭaīyāt): Text, Translation, Commentaries*, New York: Springer, 2018.

SIDOLI N., KUSUBA T., "Al-Harawī's version of Menelaus' *Spherics*", *Suhayl*, 13, 2014, p. 149-212.

Sidoli N., Saito K., "The role of geometrical construction in Theodosius's *Spherics*", *Archive for History of Exact Sciences*, 63, 2009, p. 581-609.

Stone E., *Clavius's Commentary on the Spheriks of Theodosius Tripolitae*, London: J. Senex, 1721.

Ver Eecke P., *Les* Sphériques *de Théodose de Tripoli*, Bruges : Desclée De Brouwer, 1927 (Réimprimé, Paris : Albert Blanchard, 1959).