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Chapter 3

Axiomatic spherical geometry

9 Basic axioms

In this section we set up a system of axioms for spherical geometry. We assume the reader possesses some familiarity with the axiomatic method used for plane geometry. Recall that in the plane we have several undefined terms (e.g., "point," "line," and "plane"). Then we have three kinds of statements: definitions, axioms, and theorems. A definition is a statement that says exactly what a term means. An axiom (also known as a "postulate") is a statement which is assumed about the terms being discussed. That is, an axiom is a statement whose truth is taken for granted without proof. An axiom should set out some fundamental property of a system. In contrast, a theorem is a statement whose truth is established via a proof by making use of the axioms and other theorems which have already been proven. (The terms "lemma" and "corollary" refer to types of theorems; the term "proposition" can refer to either a theorem or an axiom.)

An axiom tends to be a statement that is self-evident, but this is not necessary. Sometimes one must assume statements whose truth is not obvious. Even if some axioms are not evident, exhibition of the axioms makes clear what the assumptions are in a discussion. On the other hand, a theorem tends to be a statement whose truth is not obvious, but this is also not necessary.

In axiomatic spherical geometry, we will also have undefined terms: "point," "great circle," and "sphere." Our understanding of what these terms mean will come solely from the axioms. Theorems shall be proven solely based on the assumptions from the axioms, and from theorems previously proven. In particular, a sphere will not be assumed to consist of points equidistant from a center in space. In fact, it will not be assumed to have any such thing as a center. A great circle will not be assumed to be the intersection of a plane with a sphere.

The reader may wonder why we would do this. Are not the results of the previous chapter an adequate enough beginning to our study of spherical geometry? There are several answers. First, all mathematics makes use of assumptions which are not proven. It is simply good manners to be explicit about what those assumptions are. In the previous sections we did not do this thoroughly. We assumed all sorts of facts about the nature of space without explicitly stating them. Second, the axiomatic method will base assumptions solely on intrinsic properties of the sphere, where the previous section made use of properties of space where the sphere lives.

Once we have axioms for spherical geometry, we should check that the sphere of radius r in space satisfies these axioms, in order to conclude that theorems that arise from the axioms apply to the sphere of radius r in space. For most axioms this verification will be simple (in fact, in most cases it will turn out that we have already checked them in §5) but in other cases it will require some work.

Our system of axioms attempts to minimize the number of axioms. Consequently most of the results in this section and the next are obvious enough that the reader may not find them very interesting. The reader may profit by skimming the next two sections for now and only returning later.

In plane geometry a line is found to be in one-one correspondence with the set of real numbers $\mathbf{R}$. This correspondence allows the creation of a notion of distance between points. The great circle is treated similarly on the sphere. For a circle of radius 1 we see that if we label some starting point with 0 we may proceed around that circle one way or the other: one direction (say counterclockwise) around may be considered a positive direction and the other may be considered negative. We may label each point with a real number which is the signed distance around the circle from the origin point — positive if traced around the circle counterclockwise and negative if traced clockwise. What makes the situation different from the line is that after tracing around the circle a multiple of $\pm 2\pi$ we return to the starting point. (See Figure 3.1.) Thus 0 is the same point as $\pm 2\pi$. Tracing around again, 0 is the same point as $\pm 4\pi$. In fact, with this labeling, a point with label x is the same as the point with label $x \pm 2\pi, x \pm 4\pi, x \pm 6\pi, \ldots$. We call the set $\{\ldots x - 6\pi, x - 4\pi, x - 2\pi, x, x + 2\pi, x + 4\pi, x + 6\pi, \ldots\}$ an *equivalence class* of real numbers modulo 2π. Any element of this set is said to be a *representative* of the equivalence class. We create the set called $\mathbf{R}/2\pi$ (read: "R modulo 2π" or "R mod 2π") which is the set of all equivalence classes of real numbers modulo 2π. If x and y are two real numbers, we will write $x = y (\bmod 2\pi)$ if $x - y$ is an integer multiple of 2π. Then x and y are both a representative of the same equivalence class of real numbers modulo 2π. Furthermore, it will make sense to add and subtract elements "modulo 2π" in $\mathbf{R}/2\pi$. Let X be a class in $\mathbf{R}/2\pi$ with a representative x and Y be a class with representative y. We write $X = Y (\bmod 2\pi)$ if $x = y (\bmod 2\pi)$. We define $X + Y$ to be the class with representative $x + y$ and $X - Y$ to be the class with representative $x - y$. One must check that the operations $'+'$ and $'-'$ do not depend on the choice of x and y (and we say that $'+'$ and $'-'$ are "well defined"). For this one must check that if $x_1 = x_2 (\mathrm{mod}\, 2\pi)$ and $y_1 = y_2 (\mathrm{mod}\, 2\pi)$, then $x_1 + y_1 = x_2 + y_2 (\mathrm{mod}\, 2\pi)$ and $x_1 - y_1 = x_2 - y_2 (\mathrm{mod}\, 2\pi)$.

Given any equivalence class X in $\mathbf{R}/2\pi$ there exists a unique x, $-\pi < x \leq \pi$ such that x is a representative of X. Given a real y which is a representative of X, the process of obtaining such an x is called "reduction" of a number modulo 2π, and we speak of "reducing" $y(\mathrm{mod}\, 2\pi)$ to obtain x.

Axiom 9.1 (A-1) [1] *A sphere and a great circle are sets of points. There are at least two points on the sphere.*

Axiom 9.2 (A-2) *There is a one-one correspondence between the points of a great circle and the points of* $\mathbf{R}/2\pi$.

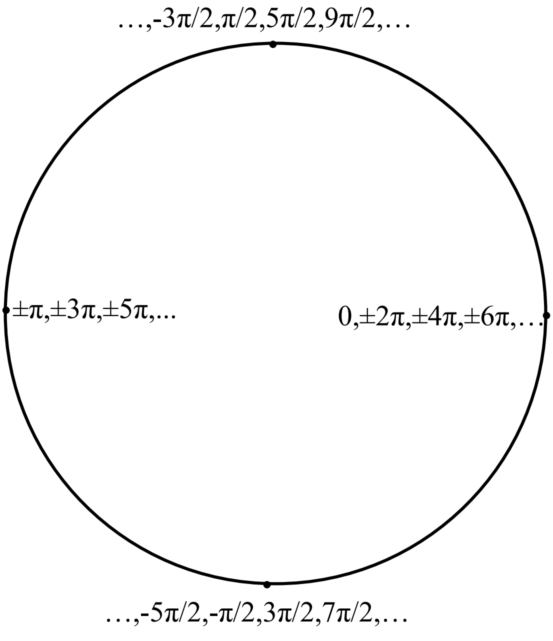

Figure 3.1: The labeling of a circle with $\mathbf{R}/2\pi$.

This axiom sets up a coordinate on the great circle. There will in general be more than one such one-one correspondence, but once we have one, we will not use another one-one correspondence for that great circle unless it is in some sense "consistent" with the first. (See Proposition 9.14.)

We next define what it means for two points on the sphere to be antipodal.

[1] The axioms for spherical geometry will be labeled A-1, A-2, A-3, etc.

Definition 9.3 *Two points on a sphere are said to be* antipodal *if there exists a great circle passing through them whose one-one correspondence takes these two points to elements of* $\mathbf{R}/2\pi$ *which differ by* $\pi \pmod{2\pi}$*. Each point is said to be an antipode of the other.*

In the standard three-dimensional model sphere, this will simply mean what it meant in Definition 5.4: that the line segment between the two points on the sphere passes through the center of the sphere. But Definition 9.3 succeeds in defining the notion of antipodal using only intrinsic properties of the sphere. This will not depend on what great circle contains the two points, via another axiomatic property for spherical geometry:

Axiom 9.4 (A-3) *A point on the sphere has no more than one antipode.*

We may then speak of <u>the</u> antipode of a point. If B is the antipode of A, then we write $B = A^a$ and $A = B^a$. We leave it to Exercise 1 to check the simple fact that the antipode of the antipode of a point is the given point.

Proposition 9.5 *If a great circle contains a point, it also contains the antipode of that point.*

(See Exercise 2.)

Proposition 9.6 *Suppose two points are antipodal. Given <u>any</u> great circle which contains the two points, its one-one correspondence takes the two points to elements in* $\mathbf{R}/2\pi$ *which differ by* π.

That is, if two points are a semicircle apart on some great circle, they are a semicircle apart on any great circle containing them.

Proof. Suppose the two points A and B lie on a great circle on which the one-one correspondences for them differs by a value other than $\pi (\mathrm{mod}\, 2\pi)$. Then there is a third point C different from A and B whose label under the one-one correspondence differs by π from the label for A. So C is an antipode for A by definition. But then A has two antipodes, a contradiction of Axiom 9.4. ◇

Axiom 9.7 (A-4) *If two distinct points on a sphere are not antipodal then there exists a unique great circle passing through them.*

Definition 9.8 *If two distinct points* A *and* B *are not antipodal, the unique great circle passing through* A *and* B *is denoted by* $\bigcirc AB$.

If two distinct points are antipodal, then by definition of antipodal there is at least one great circle containing them. But there could be more than one such great circle.

Proposition 9.9 *Through any point there exists at least one great circle.*

For the proof, see Exercise 3.

Proposition 9.10 *Every point has an antipode.*

For the proof, see Exercise 4. Our next axiom concerns what happens when two great circles meet.

Axiom 9.11 (A-5) *Two distinct great circles meet in at least one point.*

We can easily prove more than that:

Proposition 9.12 *Two distinct great circles meet in exactly two points which are antipodal.*

Proof. Since the two circles meet in a point, each must also pass through the antipode of that point by Proposition 9.5. If there were a pair of non-antipodal points in the intersection, there could only be one great circle passing through them, by Axiom 9.7. $\diamondsuit$

Definition 9.13 *Let A and B be two points, Γ a great circle containing them with one-one correspondence ℓ to $\mathbf{R}/2\pi$. Let δ be the representative of the equivalence class of $\ell(A) - \ell(B)$ which lies in $(-\pi, \pi]$. Then the distance $d(A, B)$ between A and B is the absolute value of δ.*

Note that $d(A, B) = d(B, A)$ because $(\ell(A) - \ell(B))$ is the negative of $(\ell(B) - \ell(A))$ (details are left to Exercise 5). The reader may note one problem with Definition 9.13: what happens if the two points have more than one great circle containing them? Could we get two values for the distance between them? The answer is no, because by Axiom 9.7, this could only occur if the two points are antipodal, and by Axiom 9.6, $\ell(A) - \ell(B)$ must be equal to π modulo 2π.

Note that since $0 \leq d(A, B) \leq \pi$ then $d(A, B) \pm 2\pi$ represents the same element in $\mathbf{R}/2\pi$. Note also that $\pm d(A, B) = \ell(A) - \ell(B) (\bmod 2\pi)$ so

$$\ell(A) = \ell(B) \pm d(A, B) (\bmod 2\pi). \tag{3.1}$$

We have assumed by axiom that every great circle has a coordinate ℓ which induces the distance function of Definition 9.13. But this coordinate is not always convenient: sometimes it is important to be able to choose which point is zero, and decide which way around the circle is positive:

Proposition 9.14 *Given one-one correspondence ℓ for a great circle, the functions $-\ell$ and $\ell + k$ (where k is a constant) are also one-one correspondences on the same great circle which induce the same distance $d(\cdot, \cdot)$.*

The proof is in Exercise 7. We speak of these one-one correspondences as being "equivalent" to ℓ because of the fact that they induce the same distance on the circle.

Proposition 9.15 *Given a point A on great circle Γ, there are exactly two points on Γ at distance d for $0 < d < \pi$ and exactly one such point if $d = 0$ (point A) or π (the antipode of A). The two points at distance d are antipodal if $d = \frac{\pi}{2}$.*

Proof. Let ℓ be the one-one correspondence for Γ. The points at distance d from A are those which under ℓ correspond to $\ell(A) \pm d$ modulo 2π. If $d = 0$ or $d = \pi$ there is exactly one such point (A and an antipode, respectively). If $0 < d < \pi$ $(\ell(A) \pm d)(\text{mod}\, 2\pi)$ are distinct elements of $\mathbf{R}/2\pi$, so correspond to distinct points on Γ since ℓ is a one-one correspondence. If $d = \frac{\pi}{2}$, $\ell(A) + d - (\ell(A) - d) = \pi$, so the points are antipodal. $\diamondsuit$

Definition 9.16 *Given three distinct points A, B, and C, we say that B is between A and C if A and C are not antipodal, B is on $\bigcirc AC$, and $d(A, C) = d(A, B) + d(B, C)$.*

Proposition 9.17 *Suppose A, B, and C are points. Then B is between A and C if and only if A, B, and C lie on a great circle with one-one correspondence ℓ such that $0 < d(A, B) < d(A, C) < \pi$ and*

(1) $\ell(C) = \ell(A) + d(A, C)$ and $\ell(B) = \ell(A) + d(A, B)(\text{mod}\, 2\pi)$

or

(2) $\ell(C) = \ell(A) - d(A, C)$ and $\ell(B) = \ell(A) - d(A, B)(\text{mod}\, 2\pi)$.

That is, B is between A and C if the labels $\ell(A), \ell(B), \ell(C)$ are "increasing" or "decreasing" modulo 2π without spanning more than π. Note that (1) (respectively, (2)) holds for ℓ if and only if (1) (respectively, (2)) holds for a one-one correspondence $\ell + k$. Also, (1) holds for ℓ if and only if (2) holds for $-\ell$. So in Proposition 9.17 the one-one correspondence ℓ can be replaced by any of those mentioned in Proposition 9.14.

Proof. Suppose B is between A and C. Then A and C are distinct and not antipodal, so determine the great circle $\bigcirc AC$ with one-one correspondence ℓ and B is on $\bigcirc AC$. As noted in (3.1), we must have $\ell(C) = \ell(A) \pm d(A, C)$ and $\ell(B) = \ell(A) \pm d(A, B)(\text{mod}\, 2\pi)$, in one of the four possible combinations of the $\pm$. Since A and C are distinct and not antipodal, $0 < d(A, C) < \pi$. Since $d(A, C) = d(A, B) + d(B, C)$, $d(A, B) < d(A, C) < \pi$. Since A and B are distinct, $0 < d(A, B)$.

Without loss of generality we may assume $\ell(C) = \ell(A) + d(A, C)(\text{mod}\, 2\pi)$; the case with "$-$" is similar (or we may replace ℓ by $-\ell$ by Proposition 9.14). If $\ell(B) = \ell(A) - d(A, B)(\text{mod}\, 2\pi)$, then $\ell(C) + \ell(B) = 2\ell(A) + d(A, C) - d(A, B)(\text{mod}\, 2\pi)$, and since B is between A and C, $d(A, C) - d(A, B) = d(B, C)$. Now $d(B, C) = \pm(\ell(B) - \ell(C))$ reduced modulo 2π. If $d(B, C) = \ell(B) - \ell(C)(\text{mod}\, 2\pi)$, then $\ell(C) + \ell(B) = 2\ell(A) + \ell(B) - \ell(C)(\text{mod}\, 2\pi)$, so $2\ell(C) = 2\ell(A)(\text{mod}\, 2\pi)$, so $2(\ell(C) - \ell(A)) = 0(\text{mod}\, 2\pi)$, so $\ell(C) - \ell(A) = 0$ or π modulo 2π. Thus A and C are identical or antipodal, a contradiction.

If $d(B, C) = \ell(C) - \ell(B)(\text{mod}\, 2\pi)$, then a similar argument shows that $2(\ell(B) - \ell(A)) = 0(\text{mod}\, 2\pi)$, so A and B are identical or antipodal, another contradiction since $0 < d(A, B) < \pi$.

Conversely, suppose A, B, and C lie on a great circle with one-one correspondence ℓ, $0 < d(A, B) < d(A, C) < \pi$ and (1) holds. Then A, B, and C are distinct and A and C are not antipodal. Subtracting these two equations, $\ell(C) - \ell(B) = d(A, C) - d(A, B)(\text{mod}\, 2\pi)$. Now $0 < d(A, C) - d(A, B)$

and $d(A,C) - d(A,B) < d(A,C) < \pi$. So $\ell(C) - \ell(B)$ reduced modulo 2π is $d(A,C) - d(A,B) > 0$, so $d(B,C) = d(A,C) - d(A,B)$, or $d(A,C) = d(A,B) + d(B,C)$, as desired. $\diamondsuit$

Proposition 9.17 can be used to prove properties of betweenness such as:

Proposition 9.18 *If C is between A and D and B is between A and C then B is between A and D and C is between B and D.*

Proof. If C is between A and D then we apply Proposition 9.17 to conclude either (1) or (2) there. By Proposition 9.14 we may assume (1) so that $\ell(D) = \ell(A) + d(A,D)$ and $\ell(C) = \ell(A) + d(A,C)$ modulo 2π with $0 < d(A,C) < d(A,D) < \pi$. We again apply Proposition 9.17 to the assumption that B is between A and C; then we are forced to conclude that $\ell(B) = \ell(A) + d(A,B) (\mathrm{mod}\, 2\pi)$ with $0 < d(A,B) < d(A,C) < \pi$. But then $\ell(D) = \ell(A) + d(A,D)$ and $\ell(B) = \ell(A) + d(A,B) (\mathrm{mod}\, 2\pi)$ for $0 < d(A,B) < d(A,C) < d(A,D) < \pi$, so by Proposition 9.17, B is between A and D. Thus $d(A,D) = d(A,B) + d(B,D)$ and we already had by assumption that B is between A and C so $d(A,C) = d(A,B) + d(B,C)$. Then $\ell(D) = \ell(A) + d(A,D) = \ell(B) - d(A,B) + d(A,D) = \ell(B) + d(B,D) (\mathrm{mod}\, 2\pi)$ and $\ell(C) = \ell(A) + d(A,C) = \ell(B) - d(A,B) + d(A,C) = \ell(B) + d(B,C) (\mathrm{mod}\, 2\pi)$. Furthermore $\pi > d(B,D) = d(A,D) - d(A,B) > d(A,C) - d(A,B) = d(B,C) > 0$. Then by Proposition 9.17, C is between B and D. $\diamondsuit$.

Definition 9.19 *Given two non-antipodal points A and B, the* (spherical) arc *$\overset{\frown}{AB}$ is the set of points consisting of A, B, and all points on $\bigcirc AB$ between A and B. The* measure *of $\overset{\frown}{AB}$ is $d(A,B)$.*

Definition 9.20 *Two spherical arcs are said to be* congruent *if they have the same measure. If the two arcs are $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ then we write $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.*

Our understanding of the notion of poles in the previous section of a great circle arose from the fact that a line perpendicular to the plane of a great circle meets the sphere in two points. We shall want to have an understanding of properties of the poles which depend solely on intrinsic properties of the sphere and not its placement relative to space:

Axiom 9.21 (A-6) *For any great circle there exists a point such that the great circle consists of points at spherical distance a quarter circle from the given point.*

Such a point is defined to be a *pole* of the great circle.

Proposition 9.22 *The antipode of a pole in Axiom 9.21 is the only other point which satisfies the same property.*

For a proof, see Exercise 9.

Axiom 9.21 guarantees that a great circle consists of points at a quarter circle from the pole. But could there be any points at a quarter circle from the pole which are not on the great circle? The answer is no:

Theorem 9.23 *A great circle is the set of all points on the sphere at spherical distance of a quarter circle from its pole.*

Proof. By the definition of pole, all points on the great circle (call it Γ) are at distance $\frac{\pi}{2}$ from a pole. Now conversely, suppose that a point Q is a quarter circle from the pole P. Then there exists a great circle passing through P and Q. Then $\bigcirc PQ$ is distinct from Γ since P is not on Γ. So $\bigcirc PQ$ meets Γ at two points A and A^a. Then on Γ, Q is at distance $\frac{\pi}{2}$ from P by definition of Q and A, A^a are both at distance $\frac{\pi}{2}$ by definition of P. By Proposition 9.15, Q must be the same as either A or A^a. This places Q on Γ. $\diamondsuit$

Theorem 9.24 *Every point is a pole of some great circle.*

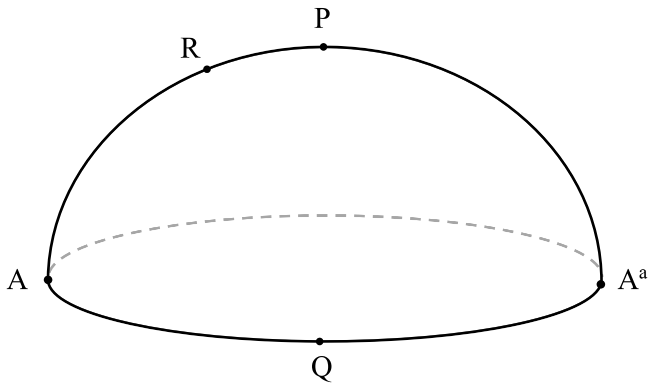

Figure 3.2: Diagram for proof of Theorem 9.24.

Proof. (See Figure 3.2.) Let the given point be P. There exists a great circle passing through P by Proposition 9.9. Let Q be one of its poles. There exist two (antipodal) points A and A^a on this great circle at distance $\frac{\pi}{2}$ from P (using the one-one correspondence between the great circle and $\mathbf{R}/2\pi$). Since Q is at distance $\frac{\pi}{2}$ from A, it is not antipodal, so there exists a unique great circle passing through Q and A. Note that all of A, A^a and Q are at distance $\frac{\pi}{2}$ from P. We claim that $\bigcirc AQ$ is the set of all points at spherical distance $\frac{\pi}{2}$ from P. Let R be a pole of $\bigcirc AQ$. We claim $R = P$ or $R = P^a$. Since R is a pole of $\bigcirc AQ$, it is a quarter circle from Q. Since Q is a pole of $\bigcirc PA$ and by Theorem 9.23, R lies on $\bigcirc PA$. On $\bigcirc PA$, R is also a quarter circle from A since it is a pole of $\bigcirc AQ$, so by Proposition 9.15, R must be P or P^a. $\diamondsuit$

Definition 9.25 *Given a point on the sphere, a great circle of which the point is a pole is called a* polar circle *or* polar *of the point.*

By Theorem 9.23, a point has a unique polar.

Proposition 9.26 *If A, B, C are points on a sphere, $\overset{\frown}{BA}$ and $\overset{\frown}{BC}$ are both quarter circles, and A and C are neither the same nor antipodal then B is a pole of great circle $\bigcirc AC$.*

Proof. Point B is the pole of some great circle (by Theorem 9.24) which contains A and C. But there is only one such circle $\bigcirc AC$ since A and C are neither the same nor antipodal. $\diamondsuit$

Proposition 5.8 gives us a way to define the term "small circle" which depends only on intrinsic properties of the sphere.

Definition 9.27 *A small circle on a sphere is the set of all points on the sphere at a fixed spherical distance $\rho < \frac{\pi}{2}$ from a point P. The point P is called the* (spherical) center *of the small circle, and the quantity ρ is called the* (spherical) radius *of the small circle. The set of all points at distance less (greater, respectively) than ρ from P is called the interior (exterior, respectively) of the small circle.*

It is not hard to see that the points on the small circle are also equidistant from the antipode of P; see Exercise 16.

Definition 9.28 *Let A and B be two points on a sphere which are not antipodal. Then the* spherical ray $\overset{\hookrightarrow}{AB}$ *is the set of all points C on the great circle $\bigcirc AB$ such that C is on $\overset{\frown}{AB}$ or B is between A and C. We say that A is the* vertex *or* endpoint *of the spherical ray. The* open ray *with vertex A through B consists of all points on $\overset{\hookrightarrow}{AB}$ except for A.*

See Figure 3.3. We leave several natural properties of rays to the exercises.

Proposition 9.29 *If A and B are not antipodal and*

$$d(A, B) = \ell(B) - \ell(A)(\bmod 2\pi)$$

or

$$d(A, B) = \ell(A) - \ell(B)(\bmod 2\pi),$$

respectively, then $\overset{\hookrightarrow}{AB}$ consists of all points C satisfying $\ell(C) = \ell(A) + x$ or $\ell(A) - x \pmod{2\pi}$, respectively, where $0 \le x < \pi$.

This follows directly from Proposition 9.17. See Exercise 10.

Corollary 9.30 *Given any spherical ray $\overset{\hookrightarrow}{AB}$ and a real number x, $0 \le x < \pi$, there exists a unique point C of $\overset{\hookrightarrow}{AB}$ such that $d(A, C) = x$.*

(See Exercise 11.)

Corollary 9.31 *If C is a point of $\overset{\hookrightarrow}{AB}$ and $C \neq A$, then $\overset{\hookrightarrow}{AB} = \overset{\hookrightarrow}{AC}$.*

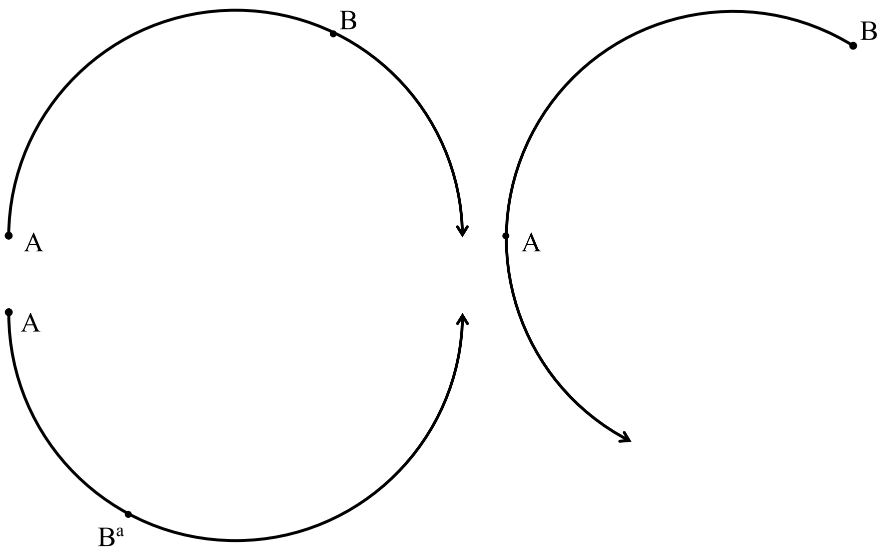

Figure 3.3: Three spherical rays from the same great circle: rays $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{AB^a}$ are opposite; rays $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{BA}$ intersect in $\overset{\frown}{AB}$.

(For the proof of Corollary 9.31, see Exercise 12.)

Proposition 9.32 *The intersection of the rays $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{BA}$ is $\overset{\frown}{AB}$.*

(See Figure 3.3.) The proof is Exercise 13.

Definition 9.33 *Spherical rays $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{AC}$ are said to be opposite if they are not the same ray but A, B, and C lie on a single great circle.*

(See Figure 3.3.)

Definition 9.34 *Let A be a point, and $\overset{\hookrightarrow}{AB}$ a spherical ray with vertex A. Then the* great semicircle $A\overset{\frown}{B}A^a$ *is the union of the ray $\overset{\hookrightarrow}{AB}$ and the point A^a.*

Proposition 9.35 *The great semicircle ABA^a is the union of the arcs $\overset{\frown}{AB}$ and $\overset{\frown}{A^aB}$.*

See Exercise 15.

Note that a spherical ray contains no pair of antipodal points. The set of points consisting of antipodes of the points on the spherical ray is another spherical ray on the same great circle whose vertex is the antipode of the vertex of the given ray. Every point on the great circle of that ray is either on the ray or its antipode is on the ray, but not both. (See Exercise 24.)

Definition 9.36 *The unique point of a spherical arc whose spherical distance to the end points is the same is called the* midpoint *of the arc.*

That a midpoint exists and is unique is a consequence of Proposition 9.30.

Proposition 9.37 *A spherical ray and a great circle intersect in exactly one point unless the ray is contained in the great circle.*

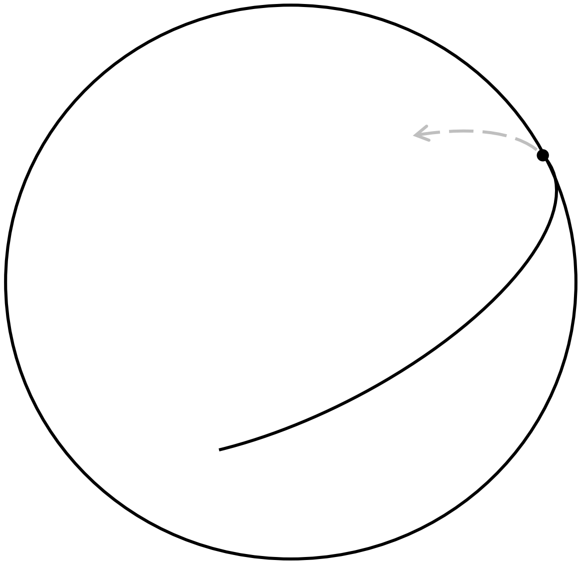

Figure 3.4: Intersection of a spherical ray and a great circle.

Proof. Suppose the great circle (Γ_1) does not contain the ray ($\overset{\hookrightarrow}{r}$). Then a second great circle (Γ_2) which does contain $\overset{\hookrightarrow}{r}$ meets Γ_1 in two points which divide Γ_2 into two great semicircles. Because those two points are antipodal, exactly one of them lies on $\overset{\hookrightarrow}{r}$. $\diamondsuit$

Definition 9.38 *The set of all points lying less than a quarter circle from a given point is called a* hemisphere*. The given point is said to be the* center *of the hemisphere.*

If A is a point, then we leave it to Exercise 19 to check that every point on the sphere is either in the hemisphere centered at A, the hemisphere centered at A^a, or on the polar of A. Also, a hemisphere contains no pair of antipodal points, and the antipode of a point in a hemisphere centered at A is in the hemisphere centered at A^a.

Definition 9.39 *Given a great circle with poles A, A^a, the hemispheres with centers at A and A^a are said to be the* sides *of the great circle. The polar circle* *of A is said to be the* edge *of each hemisphere. Two points lying in the same hemisphere are said to be on the* same side *of the great circle. If one lies in the hemisphere centered at A and the other lies in the hemisphere centered at A^a, the points are said to be on* opposite sides *of the great circle.*

Definition 9.40 *A given subset of the sphere is said to be spherically convex if for every pair of non-antipodal points, the unique arc between them is contained in the given subset.*

Proposition 9.41 *A great circle and a spherical ray are spherically convex.*

Proof. Given two non-antipodal points on a great circle, the spherical arc between them is a subset of the great circle by definition, so the great circle is spherically convex.

Suppose we are given two points C and D on a spherical ray $\overset{\hookrightarrow}{AB}$. By Proposition 9.29, if $\ell(B)-\ell(A)=d(A,B)(\bmod 2\pi)$, then the points C and D of $\overset{\hookrightarrow}{AB}$ satisfy $\ell(C)=\ell(A)+d(A,C)(\bmod 2\pi)$ and $\ell(D)=\ell(A)+d(A,D)(\bmod 2\pi)$ where we may assume without loss of generality that $d(A,C)<d(A,D)$ so that $0\leq d(A,C)<d(A,D)<\pi$. Suppose E is between C and D. Then since either $C=A$ or C is between A and D, E must also be between A and D (by Proposition 9.18). Since $\ell(D)=\ell(A)+d(A,D)(\bmod 2\pi)$ then by Proposition 9.17, $\ell(E)=\ell(A)+d(A,E)(\bmod 2\pi)$ for $0<d(A,E)<d(A,D)<\pi$. Then by Proposition 9.29, E is on $\overset{\hookrightarrow}{AD}=\overset{\hookrightarrow}{AB}$.

If $\ell(A)-\ell(B)=d(A,B)(\bmod 2\pi)$ then the argument is analogous. $\diamondsuit$

In Exercise 18, we will also see that open spherical rays and spherical arcs are spherically convex.

Axiom 9.42 (A-7) *A hemisphere is spherically convex.*

See Figure 2.6. In §13, Exercises 11 and 12, we will see that the interior of small circles are also spherically convex. We leave it to Exercise 17 to show that the intersection of spherically convex sets is also spherically convex, so that using hemispheres and interiors of small circles we may create many other spherically convex sets.

Proposition 9.43 *If two points are on opposite sides of a great circle and are not antipodal, the arc between them meets the great circle.*

Proof. Let A and B be the points and Γ the given great circle. Since A and B are on opposite sides of Γ, A and B^a are on the same side of Γ (see Exercise 19). Thus the arc $\overset{\frown}{AB^a}$ does not meet Γ. But $\overset{\hookrightarrow}{B^aA}$ meets Γ in a unique point, which thus must be between A and B, as desired. $\diamondsuit$

The following consequence of the convexity of a hemisphere is a useful tool in the proof of a number of theorems, including Propositions 10.8, 10.9, 10.11, 13.1, and Theorem 15.3.

Proposition 9.44 *If $\bigcirc AB$ is a great circle, and C is not on that great circle, then the points of $\overleftrightarrow{AC}$ other than A are on the same side of $\bigcirc AB$ as C.*

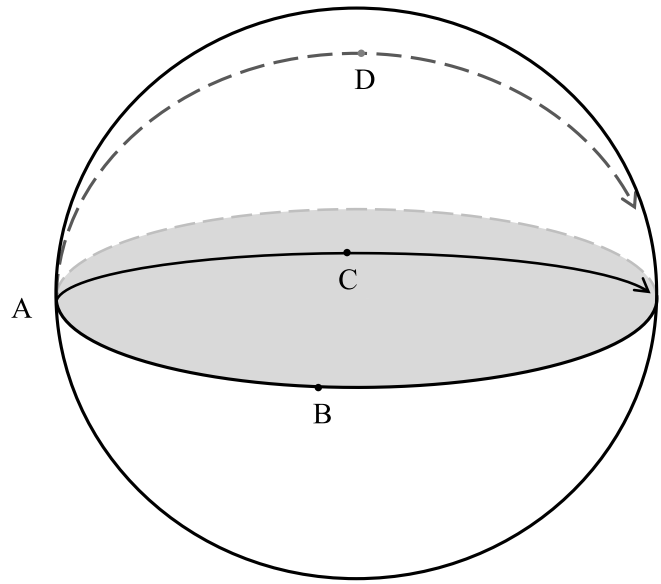

Figure 3.5: The points of $\overleftrightarrow{AC}$ and $\overleftrightarrow{AD}$ lie on the same side of $\bigcirc AB$, except for A.

Proof. (See Figure 3.5.) Suppose $X \neq A$ is on $\overleftrightarrow{AC}$ and is on $\bigcirc AB$. Then $\overleftrightarrow{AX}=\overleftrightarrow{AC}$ is contained in $\bigcirc AB$, so C is on $\bigcirc AB$, a contradiction.

Suppose X is on $\overleftrightarrow{AC}$ and belongs to the side of $\bigcirc AB$ opposite to C. By Proposition 9.43 above, $\overset{\frown}{CX}$ meets Γ in some point Y. Since $\overleftrightarrow{AC}$ is spherically convex, point Y must be on $\overleftrightarrow{AC}$. Since $\overleftrightarrow{AC}$ meets Γ in only one point, and A is such a point, $A = Y$. But an open ray is spherically convex (Exercise 18), and points $C \neq A$ and $X \neq A$ are on the ray $\overleftrightarrow{AC}$, so Y is also different from A, a contradiction. Thus the assumption was false: X must be on the same side of Γ as C. ♢

Definition 9.45 *We say that two (great or small) circles are* tangent *if they meet in a single point. This point is called the point of tangency. A* chord *of a small circle is a spherical arc between two points of the small circle.*

Since a pair of great circles must meet in two points, we only have tangent circles if one of the circles is a small circle. We will characterize tangency in later sections.

Exercises §9

1. Explain why the antipode of the antipode of a point is the given point.
2. Prove Proposition 9.5.
3. Prove Proposition 9.9.
4. Prove Proposition 9.10.
5. Justify the statement that $d(B, A) = d(A, B)$.
6. Prove that for any pair of points A, B on a sphere, $d(A^a, B^a) = d(A, B)$.
7. Prove Proposition 9.14.
8. Prove that there are infinitely many great circles.
9. Prove Proposition 9.22.
10. Prove Proposition 9.29.
11. Prove Corollary 9.30.
12. Prove Corollary 9.31.
13. Prove Proposition 9.32.
14. If A is a point, $B \neq A$, and $B \neq A^a$ then $m\overset{\frown}{AB} + m\overset{\frown}{A^aB} = \pi$.
15. Prove Proposition 9.35.
16. Given a small circle with center P, prove that the points of the circle are also equidistant from the antipode of P.
17. Prove that the intersection of spherically convex sets is also spherically convex.
18. Prove that open spherical rays and spherical arcs are spherically convex.
19. If A is a point, then every point on the sphere is either in the hemisphere centered at A, the hemisphere centered at A^a, or on the polar of A. Also, a hemisphere contains no pair of antipodal points, and the antipode of a point in a hemisphere centered at A is in the hemisphere centered at A^a.
20. Suppose A is between B and C. Prove that $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{AC}$ are well-defined opposite rays.
21. Suppose A, B, and C are three (distinct) points on a great circle no two of which is antipodal. Prove that one of them is between the other two or between the antipodes of the other two.

22. Prove that rays $\overset{\hookrightarrow}{AB}$ and $\overset{\hookrightarrow}{AC}$ are opposite if and only if one of them consists of the set of points X on $\bigcirc AB$ such that $\ell(X) = \ell(A) + x(\bmod 2\pi)$, $0 \leq x < \pi$ and the other consists of the set of points X on $\bigcirc AB$ such that $\ell(X) = \ell(A) - x(\bmod 2\pi)$, $0 \leq x < \pi$. (Here ℓ is the one-one correspondence for $\bigcirc AB$.)

23. If a great circle has a pole lying on a second great circle, the poles of the second great circle lie on the first great circle. Conclude that a set of great circles all pass through a particular point if and only if their poles all lie on a single great circle.

24. Verify the assertions made earlier: a spherical ray contains no pair of antipodal points. The set of points consisting of antipodes of the points on the spherical ray is another spherical ray on the same great circle whose vertex is the antipode of the vertex of the given ray. Every point on the great circle of that ray is either on the ray or its antipode is on the ray, but not both.

25. Suppose that we are given two spherical rays whose endpoints are neither the same nor antipodal. Suppose that the points of the rays (endpoints excepted) lie on the same side of the great circle passing through those endpoints. Then the rays meet in exactly one point.

10 Angles

We now discuss angles on the sphere and their measurement.

Definition 10.1 *Let A, B, and C be points of a sphere which do not lie on a single great circle. Then the* (spherical) angle $\prec ABC$ *with* vertex B *is the union of the spherical rays* $\overset{\hookrightarrow}{BA}$ *and* $\overset{\hookrightarrow}{BC}$. *These rays are known as the* sides *of the angle. The* interior *of the angle is the intersection of two hemispheres: the side of* $\bigcirc AB$ *containing* C *and the side of* $\bigcirc BC$ *containing* A.

See Figure 3.7. Note that the assumption that the points A, B, C are not on the same great circle means that $\overset{\hookrightarrow}{BA}$ and $\overset{\hookrightarrow}{BC}$ are not the same and not opposite rays.

In the plane, the measure of angles comes from the desire to measure how far apart the side rays are. The process of assigning measure comes about by placing a protractor centered at the vertex of an angle. A protractor is essentially a circle whose perimeter has been labeled, typically with degrees. The measure of an angle is determined by looking at the two points where the sides of the angle intersect the protractor circle. The measure of the angle is the measure of the arc on the protractor circle between those two points.

The motivation on the sphere is the same.

Definition 10.2 *Let* $\prec ABC$ *be a spherical angle. Choose points* D *and* E *on* $\overset{\hookrightarrow}{BA}$ *and* $\overset{\hookrightarrow}{BC}$*, respectively, which are both at a quarter circle from* B*. Then the* measure *of* $\prec ABC$ *is defined to be* $m\ \overset{\frown}{DE}$*. (See Figure 3.6.)*

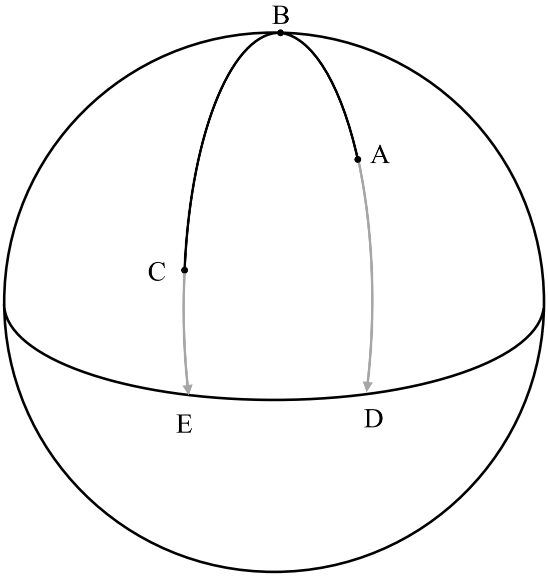

Figure 3.6: Definition 10.2: $m \prec ABC = m\ \overset{\frown}{DE}$.

Here we may regard the protractor circle as the polar circle of B ($\bigcirc DE$) and the vertex B as the center of the protractor. We say that $\prec ABC$ is *acute*, *right*, or *obtuse* if its measure is, respectively, less than, equal to, or greater than $\frac{\pi}{2}(90°)$.

Recall that in §5 (see Figure 2.4) we noted several different equivalent ways of understanding the measure of a spherical angle in space. Only one of those was intrinsic to the sphere, and this is the understanding of Definition 10.2.

Definition 10.3 *Two spherical angles are said to be* congruent *if their measures are the same. If the angles are* $\prec ABC$ *and* $\prec DEF$ *then we write* $\prec ABC \cong \prec DEF$*. The angles are said to be* supplementary *if their measures add up to* π *radians (*$180°$*). The angles are said to be* complementary *if their measures add up to* $\frac{\pi}{2}$ *radians (*$90°$*).*

Proposition 10.4 *Suppose* $\prec ABC$ *and* $\prec DBC$ *are such that* $\overset{\hookrightarrow}{BA}$ *is opposite to* $\overset{\hookrightarrow}{BD}$*. Then* $\prec ABC$ *and* $\prec DBC$ *are supplementary.*

Proof. Choose points E, F, and G on $\overset{\hookrightarrow}{BA}$, $\overset{\hookrightarrow}{BC}$, and $\overset{\hookrightarrow}{BD}$ so that $m\ \overset{\frown}{BE} = m\ \overset{\frown}{BF} = m\ \overset{\frown}{BG} = \pi/2$. Since $\overset{\hookrightarrow}{BA}$ and $\overset{\hookrightarrow}{BD}$ are opposite, E and G are a semicircle apart on the same great circle, so $E = G^a$. All of E, F, and G lie on the polar of B. F cannot be the same as either E or G since if either were the case, neither $\prec ABC$ nor $\prec DBC$ would be an angle. Then EFG forms a great semicircle such that (by §9 Exercise 14) $m\ \widehat{EF} + m\ \widehat{FG} = \pi$ so $m \prec ABC + m \prec DBC = \pi$, so $\prec ABC$ and $\prec DBC$ are supplementary by definition. ♢

Definition 10.5 *Suppose we are given* $\prec ABC$ *and* $\prec DBE$ *such that* $\overset{\hookrightarrow}{BA}$ *is opposite to* $\overset{\hookrightarrow}{BD}$ *and* $\overset{\hookrightarrow}{BC}$ *is opposite to* $\overset{\hookrightarrow}{BE}$. *Then* $\prec ABC$ *and* $\prec DBE$ *are said to be* vertical *angles.*

Proposition 10.6 *Vertical angles are congruent.*

Proof. If $\prec ABC$ and $\prec DBE$ are vertical as in Definition 10.5, then $\prec ABC$ and $\prec DBE$ are both supplementary to $\prec CBD$, so must be congruent. ♢

Definition 10.7 *A* lune *is the union of an angle with the antipode of its vertex. The* measure *of a lune is the measure of its associated angle. If the angle is* $\prec ABC$ *then we use the notation* BCB^aAB *to refer to this lune, where* B^a *is the antipode of* B. *The points* B *and* B^a *are the* vertices *of the lune.*

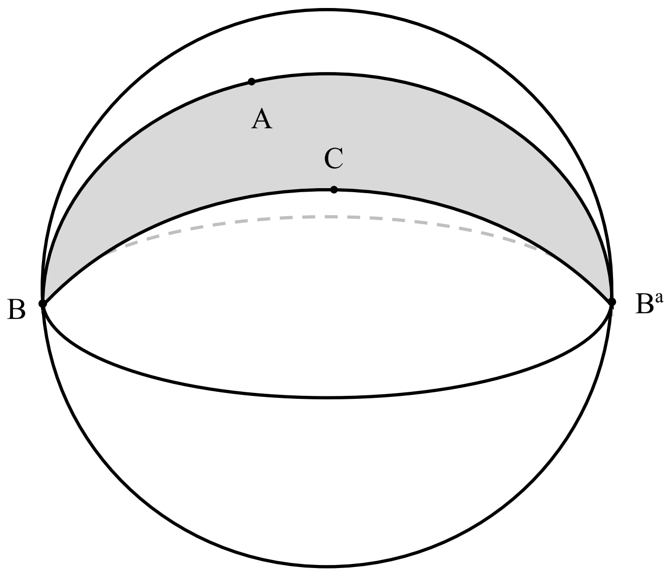

Figure 3.7: Angle $\prec ABC$, its interior, and the associated lune BAB^aCB.

Proposition 10.8 (Protractor theorem) *Suppose that we are given (1) a spherical ray* $r = \overset{\hookrightarrow}{PQ}$ *with* $\overset{\frown}{PQ}$ *a quarter circle and (2) a hemisphere* h *on one side of* r*. Then there exists a one to one to one correspondence between the set of real numbers* $-\pi < \theta \leq \pi$*, the points of the polar of* P *and the set of spherical rays which have the same endpoint as* r *such that (1)* $\theta = 0$ *is associated with* r *and* Q*, (2)* $\theta = \pi$ *is associated with the ray opposite to* r *and* Q^a *(3)* $0 < \theta < \pi$ *is associated with point* R *on the polar of* P *and ray* $\overset{\hookrightarrow}{PR}$ *such that* R *is in* h*,* $m\ \overset{\frown}{QR} = \theta$*, and* $\overset{\hookrightarrow}{QR}$ *is the same ray for all such* θ*, (4)* $-\pi < \theta < 0$ *is associated with point* S *on the polar of* P *and ray* $\overset{\hookrightarrow}{PS}$*, such that* S *is in the hemisphere opposite to* h*,* $m\ \overset{\frown}{QS} = -\theta$*, and* $\overset{\hookrightarrow}{QS}$ *is the same ray for all such* θ*, (5) if* r_1 *and* r_2 *are any two such rays forming an angle with associated real numbers* θ_1 *and* θ_2 *then the angle formed has measure found by reducing* $(\theta_1 - \theta_2)$ *modulo* 2π *to a value in* $(-\pi, \pi]$ *and taking the absolute value.*

Thus there exists a unique spherical ray r' *which (6) has the same endpoint as* r*, (7) passes through the hemisphere* h *and (8) forms an angle with* r *of measure* $\theta > 0$*. In fact, the spherical ray* r' *lies entirely in* h *except for its endpoint.*

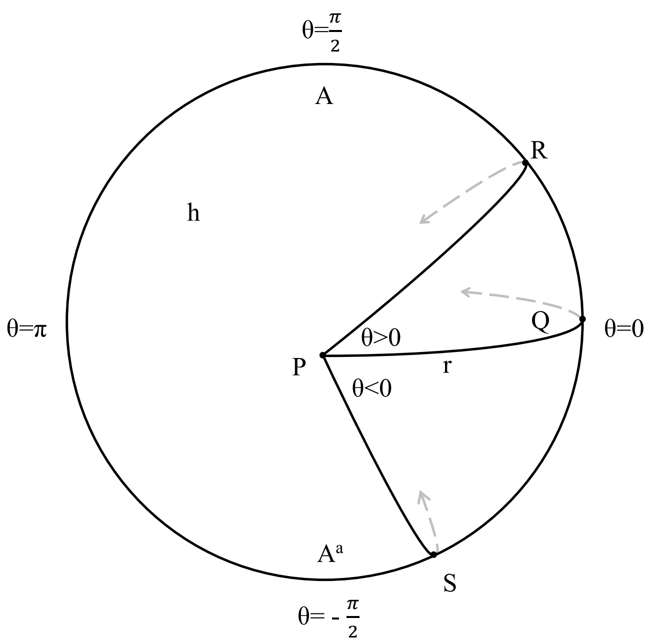

Figure 3.8: Proposition 10.8.

Proof. (See Figure 3.8.) Let A be the center of h, so A^a is the center of the opposite hemisphere. (So A, A^a are the poles of r.) Then both A and Q are a quarter circle from P, so $\bigcirc AQ$ is the polar of P (by Proposition 9.26). Points A and Q are neither the same nor antipodal because Q and Q^a are on the great circle of r but A is not). Let ℓ be the one-one correspondence coordinate for the polar of P such that $\ell(Q) = 0$ and $\ell(A) = \frac{\pi}{2}$. Then for $0 < \theta \leq \pi$, θ is associated with the point R on $\overset{\hookrightarrow}{QA}$ such that $m\,\overset{\frown}{QR} = \theta$. For $-\pi < \theta < 0$, θ is associated with the point S on $\overset{\hookrightarrow}{QA^a}$ such that $m\,\overset{\frown}{QS} = -\theta$. Next we can see that the points on the polar of P are in one-one correspondence with the rays emanating from P. Given a point X on the polar of P, we associate it with the ray $\overset{\hookrightarrow}{PX}$ emanating from P. If X_1 and X_2 are distinct, $\overset{\hookrightarrow}{PX_1}$ and $\overset{\hookrightarrow}{PX_2}$ are distinct because if $\overset{\hookrightarrow}{PX_1} = \overset{\hookrightarrow}{PX_2}$ then X_1 and X_2 are both the unique point on the ray at a quarter circle from P. Furthermore, given any ray s with endpoint P, it meets the polar of P at a point X other than P by Proposition 9.37: then $\overset{\hookrightarrow}{PX} = s$. Thus $X \leftrightarrow \overset{\hookrightarrow}{PX} = s$ is a one-one correspondence between points of the polar of P and the spherical rays emanating from P. This point X either lies on $\bigcirc PQ$, in h or in the hemisphere opposite to h. It then is associated with a value θ as in (1)-(4) above.

For $0 < \theta < \pi$ (corresponding to $\overset{\hookrightarrow}{PR}$), the ray $\overset{\hookrightarrow}{QR}$ is $\overset{\hookrightarrow}{QA}$ for all R, which is in h (except for Q) by Proposition 9.44. Similarly, for $-\pi < \theta < 0$, the ray $\overset{\hookrightarrow}{QS}$ is $\overset{\hookrightarrow}{QA^a}$ which lies in the hemisphere opposite to h (except for Q).

We see that property (5) holds as follows. Values θ_1, θ_2 correspond to rays $\overset{\hookrightarrow}{PX_1}$ and $\overset{\hookrightarrow}{PX_2}$. Then $m \prec X_1PX_2 = m\,\overset{\frown}{X_1X_2}$ whose measure is the absolute value of $(\theta_1 - \theta_2)$ reduced $\bmod 2\pi$ by definition of arc measure (Definition 9.13).

Further details are left to Exercise 3. ◇

Proposition 10.9 *Let $\prec ABC$ be a spherical angle and let P be a point on the sphere. Then the following conditions are equivalent:*

(1) P is in the interior of $\prec ABC$.

(2) P is on the same side of $\bigcirc BC$ as A and $m \prec PBC < m \prec ABC$.

(3) P is not on either of $\bigcirc AB$ or $\bigcirc BC$, and $m \prec ABC = m \prec ABP + m \prec PBC$.

(4) P does not lie on either side of the angle, but for any Q on $\overset{\hookrightarrow}{BA}$ other than B, there is a point R on $\overset{\hookrightarrow}{BC}$ different from B such that P is between Q and R.

Proof. (See Figures 3.9 and 3.10.) For P to satisfy any of these four properties, P cannot be on $\bigcirc AB$ or $\bigcirc BC$, so we assume this is the case throughout. Our line of argument rests substantially on obtaining statements equivalent to (1) through (3) on the polar circle of B. Let points D, E, and F be on $\overset{\hookrightarrow}{BA}$, $\overset{\hookrightarrow}{BP}$, and $\overset{\hookrightarrow}{BC}$, respectively, at a quarter circle from B. Since P is not on $\bigcirc AB$ or $\bigcirc BC$, E is different from D and F (Proposition 10.8).

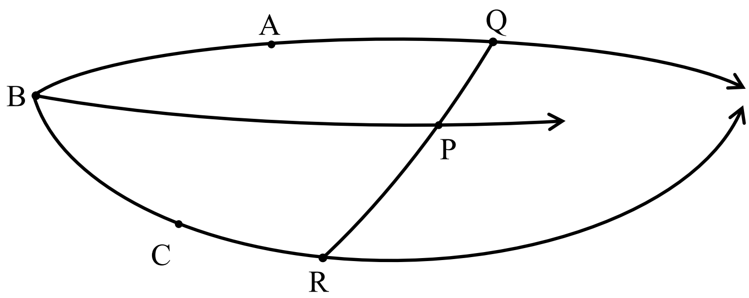

Figure 3.9: Proposition 10.9.

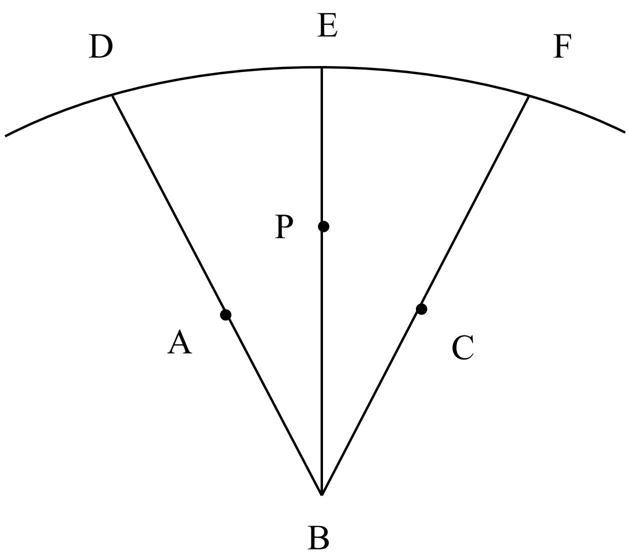

Figure 3.10: Proposition 10.9.

If (1) holds, then P and A are on the same side of $\bigcirc BC$, so (by Proposition 9.44) D and E are on the same side of $\bigcirc BC$. Thus $\overset{\hookrightarrow}{FE}$ and $\overset{\hookrightarrow}{FD}$ are the same ray (Proposition 10.8, where r is $\overset{\hookrightarrow}{BC}$). Similarly $\overset{\hookrightarrow}{DE}$ and $\overset{\hookrightarrow}{DF}$ are the same. Since the intersection of $\overset{\hookrightarrow}{DF}$ and $\overset{\hookrightarrow}{FD}$ is $\overset{\frown}{DF}$ (Proposition 9.32) E is on $\overset{\frown}{DF}$. Since E is different from D and F E is between D and F, so $m\ \overset{\frown}{DF}= m\ \overset{\frown}{DE}$ $+m\ \overset{\frown}{EF}$, so $m \prec ABC = m \prec ABP + m \prec PBC$, which gives (3).

If (3) holds, then E is different from D and F and $m\ \overset{\frown}{DF}= m\ \overset{\frown}{DE} +m\ \overset{\frown}{EF}$, so E is between D and F and $m\ \overset{\frown}{FE}< m\ \overset{\frown}{FD}$ (so $\overset{\hookrightarrow}{FE}=\overset{\hookrightarrow}{FD}$), so D and E are on the same side of $\bigcirc BC$ and ($m\ \overset{\frown}{FE}=$)$m \prec PBC < m \prec ABC(= m\ \overset{\frown}{DF})$, and finally P and A are on the same side of $\bigcirc BC$ (applying Proposition 9.44 to $\overset{\hookrightarrow}{BE}$ and $\overset{\hookrightarrow}{BD}$). Thus (2) holds.

If (2) holds, D and E are on the same side of $\bigcirc BC$ (applying Proposition 9.44 to $\overset{\hookrightarrow}{BE}$ and $\overset{\hookrightarrow}{BD}$). Applying Proposition 10.8 where r is $\overset{\hookrightarrow}{BC}$, $\overset{\hookrightarrow}{FE}=\overset{\hookrightarrow}{FD}$. Since $m \prec PBC < m \prec ABC$, $m\ \overset{\frown}{FE}< m\ \overset{\frown}{FD}$. Since $\overset{\hookrightarrow}{FE}=\overset{\hookrightarrow}{FD}$ and $m\ \overset{\frown}{FE}<$ $m\ \overset{\frown}{FD}$, E can only be between D and F. Thus $\overset{\hookrightarrow}{DE}$ is the same as $\overset{\hookrightarrow}{DF}$ so (by Proposition 10.8 applied where r is $\overset{\hookrightarrow}{BA}$) E and F are on the same side of $\bigcirc AB$, so P and C are on the same side of $\bigcirc AB$ (Proposition 9.44 applied to $\overset{\hookrightarrow}{BE}$ and $\overset{\hookrightarrow}{BF}$). Thus (1) holds.

Thus the first three conditions are equivalent. We now prove (1) is equivalent to (4).

If (4) holds, then P is on rays $\overset{\hookrightarrow}{RQ}$ and $\overset{\hookrightarrow}{QR}$. Since Q is on $\overset{\hookrightarrow}{BA}$, Q is on the same side of $\bigcirc BC$ as A. Since R is on $\bigcirc BC$, the points of $\overset{\hookrightarrow}{RQ}$ (except for R) are on the same side of $\bigcirc BC$ as Q, so on the same side of $\bigcirc BC$ as A. So P is on the same side of $\bigcirc BC$ as A. A similar argument shows that P is on the same side of $\bigcirc AB$ as C, so (1) holds.

Suppose that (1) holds, and Q is any point on $\overset{\hookrightarrow}{BA}$ other than B. Then Q is not on $\bigcirc BC$, so $\overset{\hookrightarrow}{QP}$ is not contained in $\bigcirc BC$. Thus $\overset{\hookrightarrow}{QP}$ meets $\bigcirc BC$ in a unique point R. R must be different from Q because Q is not on $\bigcirc BC$ but R is on $\bigcirc BC$. Every point of $\overset{\hookrightarrow}{QP}$ (except Q) is on the same side of $\bigcirc AB$ as P, so on the same side of $\bigcirc AB$ as C. Since R is different from Q, R is on the same side of $\bigcirc AB$ as C (and P). In particular R cannot be B or B^a. If R were on the ray opposite to $\overset{\hookrightarrow}{BC}$ ($\overset{\hookrightarrow}{BC}^a$), R and P would be on opposite sides of $\bigcirc AB$ (a contradiction). Thus R is a point on $\overset{\hookrightarrow}{BC}$ other than B, as desired. Then P is on $\overset{\hookrightarrow}{QR}$ and cannot be Q or R since P is not on $\bigcirc AB$ or $\bigcirc BC$. If R were between P and Q then P would be on the opposite side of $\bigcirc BC$ from Q, hence opposite from A, a contradiction of the assumption that P is in the interior of $\prec ABC$. So by definition of $\overset{\hookrightarrow}{QR}$, P is between Q and R. So (4) holds. This shows that (1) and (4) are equivalent, so all four conditions are equivalent. ♢

We need the above theorem in Proposition 10.13 and Proposition 11.12. Note that property (4) is too strong to be true in plane geometry, as in the plane it is not possible to let Q be any point on $\overset{\hookrightarrow}{BA}$. See Exercise 4.

Definition 10.10 *Let $\overset{\hookrightarrow}{BA}$ and $\overset{\hookrightarrow}{BC}$ form an angle $\prec ABC$ and let $\overset{\hookrightarrow}{BP}$ be a spherical ray. Then we say $\overset{\hookrightarrow}{BP}$ is* between *$\overset{\hookrightarrow}{BA}$ and $\overset{\hookrightarrow}{BC}$ if and only if P is in the interior of $\prec ABC$.*

Proposition 10.11 (Crossbar theorem) *Suppose that we are given angle $\prec ABC$ and D is a point in the interior of $\prec ABC$. Then the spherical ray $\overset{\hookrightarrow}{BD}$ intersects arc $\overset{\frown}{AC}$ in a point between A and C.*

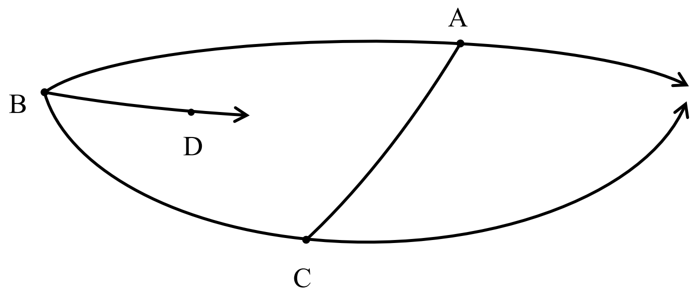

Figure 3.11: Proposition 10.11.

Proof. (See Figure 3.11.) Since D is in the interior of $\prec ABC$, it is on the same side of $\bigcirc BC$ as A. By Proposition 9.44, the points of spherical ray $\overset{\hookrightarrow}{BD}$ (except for B) are all on the same side of $\bigcirc BC$ as A. Similarly, we can argue that the points of spherical ray $\overset{\hookrightarrow}{BD}$ (except for B) are all on the same side of $\bigcirc AB$ as C. By definition, the points of $\overset{\hookrightarrow}{BD}$ (except for B) are all in the interior of $\prec ABC$. We know from Proposition 10.9 that the points of $\overset{\frown}{AC}$ are in the interior of $\prec ABC$, except for A and C. Now $\overset{\hookrightarrow}{BD}$ intersects $\bigcirc AC$ in a single point we call E (Proposition 9.37). Now E cannot be B since B is not on $\bigcirc AC$ so E must be in the interior of $\prec ABC$ since it is a point of $\overset{\hookrightarrow}{BD}$. So E is on the same side of $\bigcirc BC$ as A. Thus E must be on $\overset{\hookrightarrow}{CA}$. (If E were on the opposite ray $\overset{\hookrightarrow}{CA}^a$, it would be on the opposite side of $\bigcirc BC$ from A by Proposition 9.44.) Since E is on the same side of $\bigcirc AB$ as C, E must be on

$\overset{\hookrightarrow}{AC}$. Since E is on both $\overset{\hookrightarrow}{AC}$ and $\overset{\hookrightarrow}{CA}$, it must be on $\overset{\frown}{AC}$, and in fact must be between A and C since it is in the interior of $\prec ABC$. ♢

The notion of "measure of the angle between two lines" in the plane is not well-defined because when two lines meet, four angles are formed, and two (usually distinct) supplementary measures arise from these four angles. The same situation holds on the sphere when two great circles meet. Which of the two supplementary values one chooses depends on the situation. However, when the angles are all right angles, we may say that the great circles are perpendicular.

Proposition 10.12 *If two great circles are perpendicular then each of them passes through the poles of the other. Conversely, if one great circle passes through the poles of another, the two great circles are perpendicular.*

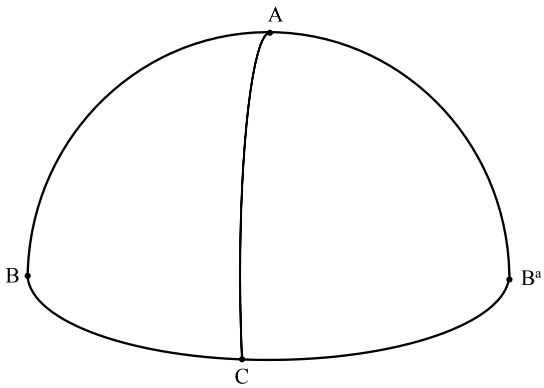

Figure 3.12: Proposition 10.12.

Proof. (See Figure 3.12.) If one of the circles passes through the pole A of the second circle, they are distinct circles, so meet in antipodal points B, B^a. Choose point C of the second circle such that C is a quarter circle from B. By definition of pole, $\overset{\frown}{AB}$ is a quarter circle. By definition, $m \prec ABC = m \overset{\frown}{AC}$. But $m \overset{\frown}{AC}= \frac{\pi}{2}$ since A is a pole of $\bigcirc BC$. Thus $\bigcirc AB$ is perpendicular to $\bigcirc BC$, as desired.

If the two circles are perpendicular then they are distinct, so meet at antipodal points B, B^a. Choose A on one circle and C on the other which are each a quarter circle from B. Then $m \overset{\frown}{AC}= m \prec ABC$ by definition; the latter is $\frac{\pi}{2}$ since the circles are perpendicular. Since $m \overset{\frown}{AB}= m \overset{\frown}{AC}= \frac{\pi}{2}$, A is a pole of $\bigcirc BC$, by Proposition 9.26. Similarly, C is a pole of $\bigcirc AB$. ♢

A useful fact in geometry is that if one desires to measure an angle between two lines (or planes) one may do so by measuring the angle between their perpendiculars at the point of intersection. The following result reflects this idea in spherical geometry.

Proposition 10.13 *Let A, B and C be points of a sphere which do not lie on a single great circle. Let A' be the pole of great circle $\bigcirc BC$ which is on the same side of $\bigcirc BC$ as A. Let C' be the pole of great circle $\bigcirc AB$ which is on the same side of $\bigcirc AB$ as C. Then $m\ \widehat{A'C'} = m \prec A'BC' = \pi - m \prec ABC$.*

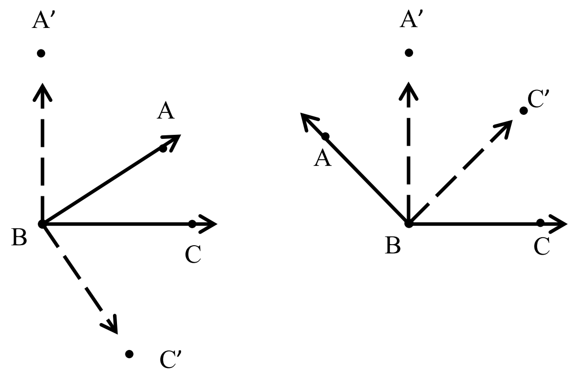

Figure 3.13: Proposition 10.13.

Proof. (See Figure 3.13.) By Proposition 10.12, $\bigcirc A'B$ is perpendicular to $\bigcirc BC$ and $\bigcirc C'B$ is perpendicular to $\bigcirc AB$. There are three cases to consider: $\prec ABC$ is acute, right, or obtuse. If $\prec ABC$ is acute, then (by Proposition 10.9) A is in the interior of $\prec A'BC$ and C is in the interior of $\prec ABC'$. Then[2] $m \prec A'BC' = m \prec A'BA + m \prec ABC + m \prec CBC' = \frac{\pi}{2} - m \prec ABC + m \prec ABC + \frac{\pi}{2} - m \prec ABC$, which is $\pi - m \prec ABC$, as desired. If $\prec ABC$ is obtuse, then A' and C' are both in the interior of $\prec ABC$. Then $m \prec ABC = m \prec ABA' + m \prec A'BC' + m \prec C'BC = \frac{\pi}{2} - m \prec A'BC' + m \prec A'BC' + \frac{\pi}{2} - m \prec A'BC'$, which is $\pi - m \prec A'BC'$. Thus $m \prec ABC = \pi - m \prec A'BC'$, as desired. If $\prec ABC$ is a right angle, then C' is on $\overrightarrow{BA}$ and A' is on $\overrightarrow{BC}$, so $\prec A'BC'$ is also a right angle and hence is supplementary to $\prec ABC$. $\diamondsuit$

Definition 10.14 *Suppose two small circles with centers A and B meet at point C. Then we define the measure of the angle between the circles to be $m \prec ACB$. (See Figure 3.14.)*

We occasionally will need to speak of the angle between a small circle and a great circle. We may follow the same trick used above: Let A be the center

[2] A detail is omitted in justifying the first equality here. It can be justified by using the one-one correspondence ℓ between rays emanating from B and the values $-\pi < \theta \leq \pi$. If ℓ is 0 on $\overset{\hookrightarrow}{BC}$ and $\frac{\pi}{2}$ on $\overset{\hookrightarrow}{BA'}$ then $\ell = m \prec ABC$ on $\overset{\hookrightarrow}{BA}$ since A, A' are on the same side of $\bigcirc BC$. Then since C, C' are on the same side of $\bigcirc AB$, the value of ℓ on $\overset{\hookrightarrow}{BC'}$ is $m \prec ABC - \frac{\pi}{2}$. The measures of the angles may be obtained from these protractor coordinates.

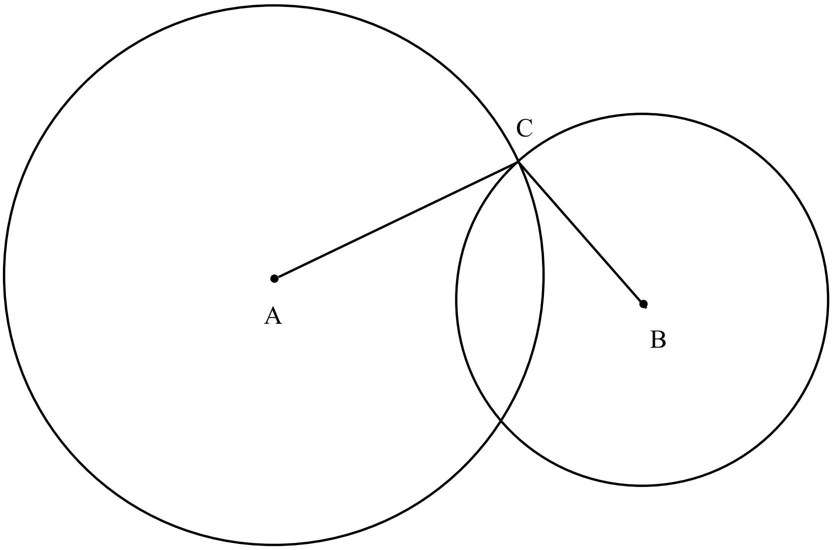

Figure 3.14: Definition 10.14.

of the small circle, B a pole of the great circle, and C a point of intersection. Then the measure of the angle between them is $m \prec ACB$. However, this cannot be made precise because a great circle has two poles, and we get two supplementary angle measures depending on which pole we use.[3] The ambiguity can be resolved by deciding a case by case basis which angle matters. As an example, in astronomy a star's daily path in the sky is a small circle but the horizon is a great circle, so the angle at which the star rises or sets is the angle between a small and great circle. One might choose the acute angle between the two circles as the desired angle measure. See §23, Exercise 10.

Proposition 10.15 *Let E be a great circle, let P be a pole of E, and let X be a point which is not a pole of E. Let M be the great circle passing through P and X, and let Y and Z be the (antipodal) points where E and M meet. Then the spherical distance from X to Y is different from the spherical distance from X to Z.*

Proof. (See Figure 3.15.) Point X lies on a great semicircle with endpoints Y and Z. If X is equidistant from Y and Z, then both distances are equal to $\frac{\pi}{2}$, which means $X = P$ or $X = P^a$, a contradiction. ♢

Definition 10.16 *Let E be a great circle, let P be a pole of E and let X be a point which is not a pole of E. Let M be the great circle passing through*

[3] In fact, since small circles also have two poles, the choice of poles A and B made in Definition 10.14 (instead of their antipodes) is somewhat artificial.

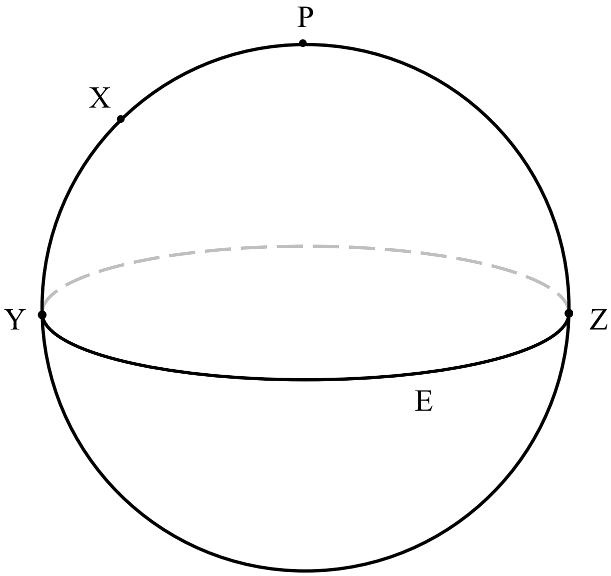

Figure 3.15: Proposition 10.15 and Definition 10.16.

P and X, and let Y and Z be the (antipodal) points where E and M meet. Suppose X is closer to Y than Z. Then the spherical distance from X to Y is said to be the distance *from X to E. If $X \neq Y$, then $\overset{\frown}{XY}$ is said to be the* shorter perpendicular *or* shorter altitude *from X to E, $\overset{\frown}{XZ}$ is the* longer perpendicular *or* longer altitude *from X to E, Y is the* foot *of the shorter perpendicular, Z is the* foot *of the longer perpendicular. The* distance *from P to E is defined to be $\frac{\pi}{2}$ radians (90°). (See Figure 3.15.)*

Note that, by Proposition 9.23, the distance to a great circle from one of its poles is the same as the distance between the pole and any point on the great circle. It is left to Exercise 8 to prove the following proposition.

Proposition 10.17 *Suppose that E is a great circle with a pole P and X is not on E and not one of the poles of E. If W is a point of E such that $\bigcirc XW$ is perpendicular to E, then W must be one of the two points where $\bigcirc PX$ meets E. (See Figure 3.15.)*

The last proposition of this section will be critical to relating spherical distance and angle measure, and be central to proving congruence of triangles. We shall accept it as an axiom. The reader can either accept it as self-evident for a sphere in space, or use solid geometry to justify it. (See Exercise 7.)

Axiom 10.18 (A-8) *Suppose that in spherical $\prec A_1B_1C_1$ and $\prec A_2B_2C_2$, we have $\overset{\frown}{A_1B_1} \cong \overset{\frown}{A_2B_2}$ and $\overset{\frown}{B_1C_1} \cong \overset{\frown}{B_2C_2}$. Then $\prec A_1B_1C_1 \cong \prec A_2B_2C_2$ if and only* *if* $\overset{\frown}{A_1C_1} \cong \overset{\frown}{A_2C_2}$.

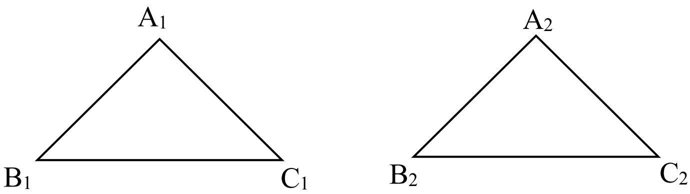

Figure 3.16: Axiom 10.18.

Exercises §10.

1. Given a spherical angle $\prec ABC$, prove that $\prec A^aB^aC^a$ is a well-defined spherical angle with the same measure.
2. In lune BCB^aAB, prove that $m \prec ABC = m \prec AB^aC$.
3. Fill in the details of the proof of Proposition 10.8.
4. Consider the analogue of property (4) of Proposition 10.9 in plane Euclidean geometry. For what Q does such an R exist?
5. Suppose that $\prec ABC$ is a right angle and $\overset{\frown}{AB}$ is a quarter circle. Prove that $\overset{\frown}{AC}$ is a quarter circle and $\prec ACB$ is a right angle.
6. Suppose that $\overset{\hookrightarrow}{BA}$, $\overset{\hookrightarrow}{BP}$, and $\overset{\hookrightarrow}{BC}$ are spherical rays. Using the one-one correspondence of Proposition 10.8, let D, E, and F be the corresponding points on the polar of B. Prove that $\overset{\hookrightarrow}{BP}$ is between $\overset{\hookrightarrow}{BA}$ and $\overset{\hookrightarrow}{BC}$ if and only if E is between D and F.
7. Use solid geometry to justify Axiom 10.18.
8. Prove Proposition 10.17.
9. Prove that two distinct great circles have a unique third great circle perpendicular to both.
10. Let X, Y, Z, and E be as in Definition 10.16. Let F be the great circle whose poles are Y and Z. Prove that F meets E in two points which are at distance $\frac{\pi}{2}$ from X, Y and Z.
11. Suppose that E is a great circle, P is one of the poles of E, and X is a point on E or on the same side of E as P. Prove that if h is the distance from X to E, $h = \frac{\pi}{2} - d(P, X)$. What happens if X is on the opposite side of E from P?

Historical notes. Let X-Y denote Book X, Proposition Y of Menelaus' *Sphaerica*. Then Axiom 10.18 is I-4.

11 Triangles

In plane geometry, a triangle arises by taking three points which do not lie on a line, and connecting them with the unique segments between pairs of these points. Thus three noncollinear points determine a unique triangle. We will do the same on a sphere (where instead the points will not lie on a single great circle, and we connect them with arcs of great circles). But before we do so, one problem will arise. If three points not on a single great circle are to determine a unique spherical triangle, is it possible that some pair of them are antipodal — which would mean that there is not a unique arc of a great circle between them? The answer, fortunately, is no:

Proposition 11.1 *If three points A, B, C on a sphere do not lie on a single great circle then no two of the points A, B, C are antipodal, so that the spherical arcs $\overset{\frown}{AB}, \overset{\frown}{BC}, \overset{\frown}{AC}$ are defined as in Definition 9.19.*

Proof. We proceed by contradiction. If one pair were antipodal (say A, B) then (the three points being distinct) C would not be antipodal to either A or B. Thus the great circles $\bigcirc AC$ and $\bigcirc BC$ are defined. Since B is antipodal to A, B lies on $\bigcirc AC$ (by Proposition 9.5), so A, B, and C lie on a single great circle, a contradiction. So no pair of A, B, or C is antipodal. $\diamondsuit$

Definition 11.2 *Let A, B, C be three points on a sphere which do not lie on a single great circle. Then the* spherical triangle *$\triangle^s ABC$ is the union of the three arcs $\overset{\frown}{AB}, \overset{\frown}{BC}, \overset{\frown}{AC}$. Each of the three points A, B, and C is called a* vertex *(plural:* vertices*) of $\triangle^s ABC$. The arcs $\overset{\frown}{AB}, \overset{\frown}{BC}, \overset{\frown}{AC}$ are called the* sides *of $\triangle^s ABC$. The spherical angles $\prec CAB$, $\prec ABC$ and $\prec BCA$ (also denoted by $\prec A$, $\prec B$, and $\prec C$) are the* angles *of $\triangle^s ABC$.*

By Proposition 11.1, no two of the points A, B, and C are antipodal, so the three spherical arcs in Definition 11.2 are well-defined. Also by Definition 9.19, each of the sides of the triangle is the shorter of the two great circle arcs between the vertices, so the measure of each of the three sides is less than π. We may also conclude from Proposition 10.9 that every side of a spherical triangle is in the interior of the opposite angle (except for the endpoints).

We define several types of special triangles.

Definition 11.3 *Two triangles are said to be colunar if they have two vertices in common and one pair of vertices which are antipodal.*

That is, the triangles have the form $\triangle^s ABC$ and $\triangle^s A^a BC$ where A and A^a are antipodal.

Proposition 11.4 *Given $\triangle^s ABC$ and the antipode A^a of A, the three points A^a, B, and C form a spherical triangle such that $\overset{\frown}{A^aB}$, $\overset{\frown}{A^aC}$, $\prec A^aBC$, and $\prec A^aCB$ are supplementary to $\overset{\frown}{AB}$, $\overset{\frown}{AC}$, $\prec ABC$, and $\prec ACB$, respectively.*

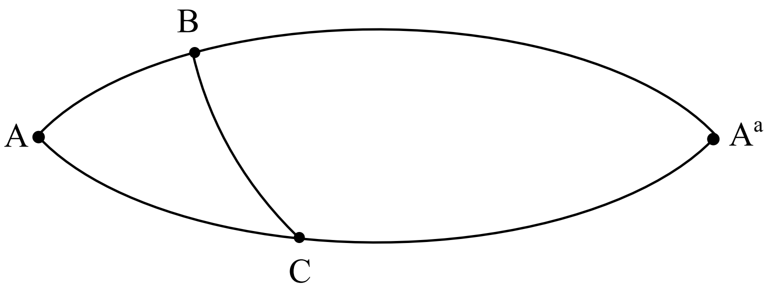

Figure 3.17: Colunar triangles $\triangle^s ABC$ and $\triangle^s A^a BC$.

Proof. See Exercise 1. ◇

Definition 11.5 *Two triangles are said to be antipodal if they can be expressed as $\triangle^s ABC$ and $\triangle^s A^a B^a C^a$, where A^a, B^a, and C^a are antipodal to A, B, and C, respectively.*

Proposition 11.6 *Given $\triangle^s ABC$, the three antipodes A^a, B^a, and C^a form a spherical triangle whose sides and angles are congruent to the corresponding sides and angles in $\triangle^s ABC$.*

Proof. See Exercise 2. ◇

Definition 11.7 *A spherical triangle is said to be a* right triangle *if at least one of its angles is a right angle. A side of a triangle which is opposite a right angle is said to be a* hypotenuse *of the right triangle. A side of the triangle which is* not *a hypotenuse is said to be a* leg *of the triangle.*

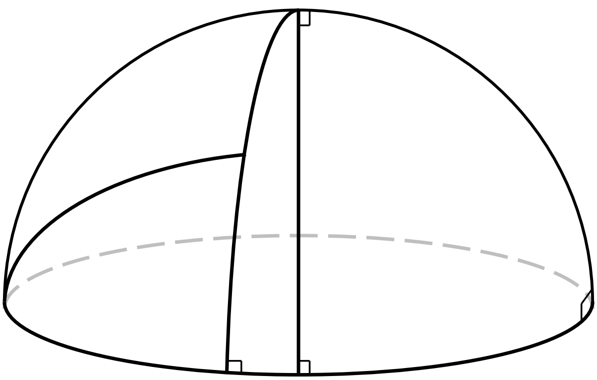

Figure 3.18: Triangles with one, two, and three right angles.

Definition 11.8 *If* $\triangle^s ABC$ *satisifes the property that one of its sides is a quarter circle, then we say that* $\triangle^s ABC$ *is* quadrantal. *A side which is a quarter circle is said to be a* right side.

Definition 11.9 *A triangle* $\triangle^s ABC$ *is said to be* isosceles *with* vertex A *if* $\overset{\frown}{AB} \cong \overset{\frown}{AC}$. *The angles at* B *and* C *are the* base angles *of the triangle. The triangle is said to be* equilateral *if all of its sides are congruent.*

Theorem 11.10 *In an isosceles spherical triangle, the angles opposite the congruent sides are also congruent.*

Proof. Suppose that in $\triangle^s ABC$, $\overset{\frown}{AB} \cong \overset{\frown}{AC}$. Then compare $\prec ABC$ and $\prec ACB$: $\overset{\frown}{AB} \cong \overset{\frown}{AC}$, $\overset{\frown}{BC} \cong \overset{\frown}{CB}$, and $\overset{\frown}{AC} \cong \overset{\frown}{AB}$. By Axiom 10.18, $\prec ABC \cong \prec ACB$, as desired. $\diamondsuit$

The converse of this is also true; this is §12, Exercise 1.

Let us now prove a few propositions in spherical geometry which demonstrate how the world of the sphere will be different from that of the plane. The reader should be used to the fact that in plane geometry, triangles may have only one right angle. This is not the case on the sphere, and the next proposition explains how this happens.

Proposition 11.11 *A pair of angles in a triangle are right angles if and only if the opposite sides are right sides.*

Proof. Suppose that in $\triangle^s ABC$, the angles $\prec B$ and $\prec C$ are both right. Then great circles $\bigcirc AB$ and $\bigcirc AC$ are both perpendicular to $\bigcirc BC$, so by Proposition 10.12, each passes through the poles of $\bigcirc BC$. Since $\bigcirc AB$ and $\bigcirc AC$ are distinct, they meet in only two points (which thus must be the poles of $\bigcirc BC$). Since A is one of these points, it is a pole of $\bigcirc BC$, so $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ are both right sides, as every point of a great circle lies a quarter circle from each pole.

Conversely, suppose that both $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ are right sides. By Proposition 9.26, A is a pole of $\bigcirc BC$. By Proposition 10.12, $\bigcirc AB$ and $\bigcirc AC$ are both perpendicular to $\bigcirc BC$, so $\triangle^s ABC$ has right angles at B and C. $\diamondsuit$

In plane geometry, if a triangle has one right angle, the other angles are acute. On a sphere, this also turns out differently.

Proposition 11.12 *Suppose* $\triangle^s ABC$ *has a right angle at* B. *Then one of the other angles of* $\triangle^s ABC$ *is acute, right, or obtuse, if and only if its opposite side is acute, right, or obtuse, respectively.*

Proof. By Proposition 10.12, $\bigcirc AB$ passes through the poles of $\bigcirc BC$. Let A' be the pole of $\bigcirc BC$ on the same side of $\bigcirc BC$ as A. Then $\overleftrightarrow{BA}$ and

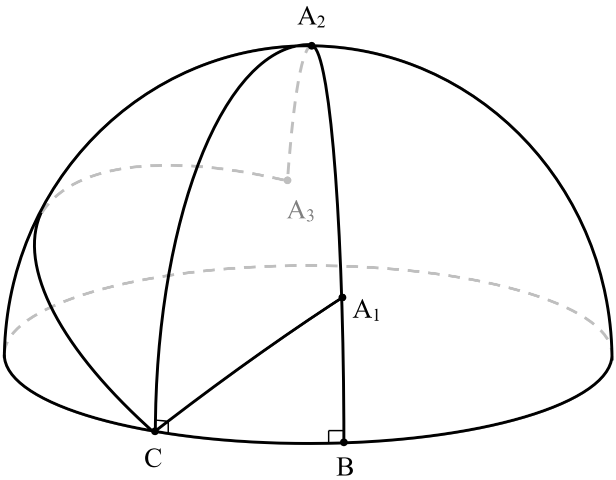

Figure 3.19: Proposition 11.12, cases $A = A_1, A_2, A_3$.

$\overset{\hookrightarrow}{BA'}$ are both perpendicular to $\bigcirc BC$ at B, and A, A' are on the same side of $\bigcirc BC$. By Proposition 10.8 $\overset{\hookrightarrow}{BA}=\overset{\hookrightarrow}{BA'}$.

If side $\overset{\frown}{BA}$ is right, then $A = A'$ (Corollary 9.30). Then by Proposition 10.12, $\bigcirc AC$ is perpendicular to $\bigcirc BC$, so $\prec C$ is a right angle, as desired. If side $\overset{\frown}{BA}$ is acute, then A is between B and A' by Proposition 9.17, since $m\ \overset{\frown}{BA}< m\ \overset{\frown}{BA'}= \frac{\pi}{2}$. From Proposition 10.9 we conclude that A is in the interior of $\prec A'CB$. By Proposition 10.9, $m \prec ACB < m \prec A'CB = \frac{\pi}{2}$, so $\prec ACB$ is acute, as desired.

If side $\overset{\frown}{BA}$ is obtuse, then by Proposition 9.17, A' is between B and A(since $m\ \overset{\frown}{BA}> m\ \overset{\frown}{BA'}= \frac{\pi}{2}$.) From Proposition 10.9 we conclude that A' is in the interior of $\prec ACB$. By Proposition 10.9, $m \prec ACB > m \prec A'CB = \frac{\pi}{2}$, so $\prec ACB$ is obtuse, as desired. $\diamondsuit$

If the assumption that $\prec B$ is right is relaxed to the assumption that $\prec B$ is the angle in the triangle closest to being right, then the same conclusion holds. See §16, Exercise 14 and §19, Exercise 17.

An arc between a vertex of a triangle and a point of the opposite side is called a *cevian* of the triangle from the vertex. If the point on the opposite side is the midpoint of that side, the cevian is called the *median* of the triangle from the vertex. The cevian which bisects the angle at the vertex is called an *angle bisector* from the vertex. An arc between a vertex of a triangle and the great circle containing the opposite side which is perpendicular to the great circle is called an *altitude* of the triangle from that vertex.

Recall that in a plane triangle, an altitude from one vertex of the triangle meets the interior of the opposite side provided that the other two angles are acute. A variation of this holds on the sphere.

Proposition 11.13 *Suppose that in $\triangle^s ABC$, $\sphericalangle A$ and $\sphericalangle B$ are not right angles. Then C is not a pole of $\bigcirc AB$. Let D and E be the feet of the shorter and longer altitudes from C to $\bigcirc AB$, respectively. Then we may determine the placement of D and E as follows:*

(1) D is between A and B if and only if $\sphericalangle A$ and $\sphericalangle B$ are both acute.

(2) E is between A and B if and only if $\sphericalangle A$ and $\sphericalangle B$ are both obtuse.

(3) neither D nor E is between A and B if and only if one of $\sphericalangle A$ and $\sphericalangle B$ is acute and the other is obtuse.

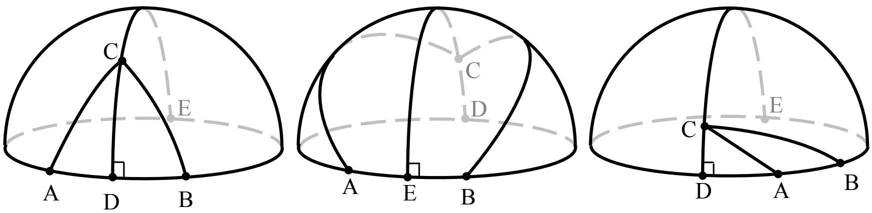

Figure 3.20: Proposition 11.13.

Proof. If C were a pole of $\bigcirc AB$, then by Proposition 9.23, $\overset{\frown}{CA}$ and $\overset{\frown}{CB}$ are right sides, so by Proposition 11.11, $\sphericalangle A$ and $\sphericalangle B$ would be right angles, a contradiction of our assumptions. So C is not a pole of $\bigcirc AB$. If neither $\sphericalangle A$ nor $\sphericalangle B$ are right angles then both D and E are distinct from A and B. (If D were equal to A, for example, then $\sphericalangle A$ would be right.) Note that D and E cannot be simultaneously between A and B because D and E are antipodal. So either (1) D is between A and B and E is not; (2) E is between A and B and D is not; or (3) neither D nor E is between A and B.

(1) Suppose D is between A and B. Then $\sphericalangle CAB$ is the same as $\sphericalangle CAD$ and $\sphericalangle CBA$ is the same as $\sphericalangle CBD$. Applying Proposition 11.12 to $\triangle^s CAD$ and $\triangle^s CBD$, we find that since $\overset{\frown}{CD}$ is acute, so $\sphericalangle A$ and $\sphericalangle B$ are also.

(2) Suppose E is between A and B. Then we apply the proof of (1), replacing D by E, and "acute" by "obtuse" to conclude that $\sphericalangle A$ and $\sphericalangle B$ are obtuse.

(3) Suppose neither D nor E is between A and B. We apply Proposition 11.12 again to $\triangle^s CAD$ and $\triangle^s CBD$ to conclude that $\sphericalangle CAD$ and $\sphericalangle CBD$ are acute. Then A and B are on the same side of the great circle $\bigcirc CD = \bigcirc CE$ because were they on opposite sides, the arc $\overset{\frown}{AB}$ would cross this great circle somewhere (by Proposition 9.43) and this could only occur at the intersection of $\bigcirc AB$ and $\bigcirc CD$ — that is, at either D or E — and this was ruled out by assumption. By Proposition 10.8, $\overset{\hookrightarrow}{DA}=\overset{\hookrightarrow}{DB}$. If A is between D and B then $\overset{\hookrightarrow}{AD}$ and $\overset{\hookrightarrow}{AB}$ are opposite (§9, Exercise 20) so $\sphericalangle CAB$ is supplementary to $\sphericalangle CAD$.

Furthermore (again assuming A is between D and B), $\overset{\hookrightarrow}{BA} = \overset{\hookrightarrow}{BD}$, so $\prec CBA$ is equal to $\prec CBD$. Then $\prec CAB$ is obtuse and $\prec CBA$ is acute. If B is between D and A then $\prec CAB$ is equal to $\prec CAD$ and $\prec CBA$ is supplementary to $\prec CBD$. Then $\prec CBA$ is obtuse and $\prec CAB$ is acute.

As to the opposite direction, suppose that $\prec A$ and $\prec B$ are both acute. If E is between A and B then from what we just proved, $\prec A$ and $\prec B$ are both obtuse, a contradiction. If neither D nor E is between A and B then from what we just proved, one of $\prec A$ or $\prec B$ is obtuse, a contradiction. So the only possibility remaining is that D is between A and B.

A similar argument works if $\prec A$ and $\prec B$ are both obtuse, or if one is obtuse and the other acute. $\diamondsuit$

Proposition 11.14 *If a triangle has two sides of measure a and the two opposite angles have measure A, then a and A are both acute, both right or both obtuse.*

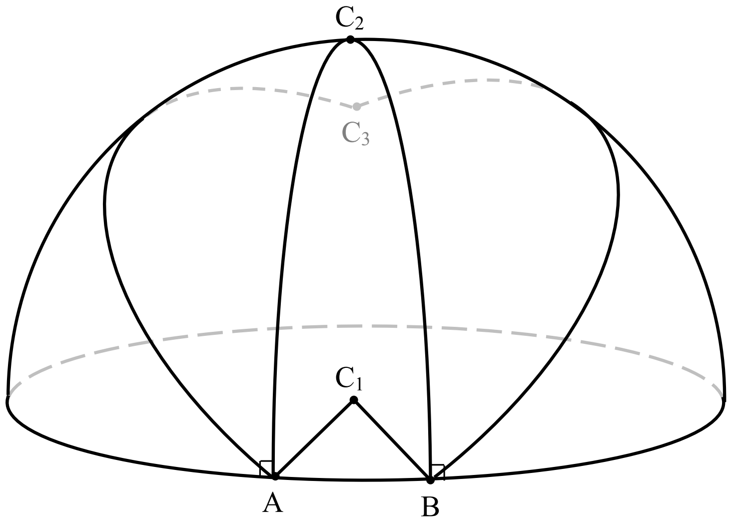

Figure 3.21: Proposition 11.14, cases $C = C_1, C_2, C_3$.

Proof. Suppose that the triangle is $\triangle^s ABC$, the sides mentioned are $\overset{\frown}{BC}$ and $\overset{\frown}{AC}$, and the angles are $\prec A$ and $\prec B$. We have dealt with the case of right angles/right sides in Theorem 11.11. So we may assume here that a, A are each either acute or obtuse (not necessarily the same). Thus C is not a pole of $\bigcirc AB$. Let D and E be the feet of the shorter and longer perpendiculars from C to $\bigcirc AB$, respectively. By Proposition 11.13, because the angles at A and B are congruent, they must be both acute or both obtuse, so either D or E is between A and B (but not both, since D and E are antipodal). (Note that D and E are both different from A and B because $\prec A$ and $\prec B$ are not right angles.)

If a is acute then A and B belong to the hemisphere centered at C, so by Axiom 9.42, so do all points on $\widehat{AB}$. This would include either D or E, but it cannot be E because $\widehat{CE}$ is obtuse. Thus $\prec CAB = \prec CAD$. Applying Proposition 11.12 to $\triangle^s CDA$ we conclude that since $\widehat{CD}$ is acute by definition the opposite angle $\prec CAD$ (with measure A) is acute. (So $\prec CBD$ is also acute, having the same measure as $\prec CAD$.)

If a is obtuse, then A and B belong to the hemisphere centered at the antipode of C, so the points of $\widehat{AB}$ are also, by Axiom 9.42. This must include either D or E, and since $\widehat{CD}$ is acute, it could only be E. Thus $\prec A = \prec CAE$. Applying Proposition 11.12 to $\triangle^s CAE$ we conclude that the opposite angle $\prec A$ (with measure A) is obtuse since $\widehat{CE}$ is obtuse by definition. (So $\prec B$ is also obtuse, having the same measure A.) $\diamondsuit$

Definition 11.15 *The* interior *of a triangle is the intersection of the interiors of its angles. The* exterior *of a triangle is the set of points which are neither on a triangle nor in its interior.*

We now introduce a notion that has no equivalent in plane Euclidean geometry, and which will be helpful in understanding many relationships in spherical triangles which do not occur in plane triangles. We note that if $\triangle^s ABC$ is a spherical triangle, then by Definition 11.2 the point A is not on the great circle $\bigcirc BC$. Thus A must lie in one of the two hemispheres into which the sphere is divided by $\bigcirc BC$, and hence lies on a particular side of $\bigcirc BC$. Similarly, B lies on a particular side of $\bigcirc AC$ and C lies on a particular side of $\bigcirc AB$.

Definition 11.16 *Let $\triangle^s ABC$ be a spherical triangle. We define the* polar triangle *$\triangle^s A'B'C'$ of $\triangle^s ABC$ (also sometimes called the* supplemental triangle *or the* dual triangle*) as follows. (See Figure 3.22.) We let A' be the pole of $\bigcirc BC$ which lies on the same side of $\bigcirc BC$ as A. We define B' and C' analogously: B' is the pole of $\bigcirc AC$ on the same side of $\bigcirc AC$ as B, and C' is the pole of $\bigcirc AB$ on the same side of $\bigcirc AB$ as C.*

We need to check that $\triangle^s A'B'C'$ is a well-defined triangle.

Theorem 11.17 *If $\triangle^s ABC$ is a well-defined spherical triangle, $\triangle^s A'B'C'$ is also; i.e., the points A', B', C' do not lie on a single great circle.*

Proof. Suppose A', B', C' all lie on a single great circle Γ with poles N, S. By Proposition 9.23 the points on $\bigcirc BC$ are the set of all points on Γ at a quarter circle from A'. By the same proposition, since N and S are a quarter circle from A', N and S lie on $\bigcirc BC$. Similarly we conclude that N and S belong to $\bigcirc AB$ and $\bigcirc AC$. Since A, B, C do not all lie on the same great circle, $\bigcirc AB$

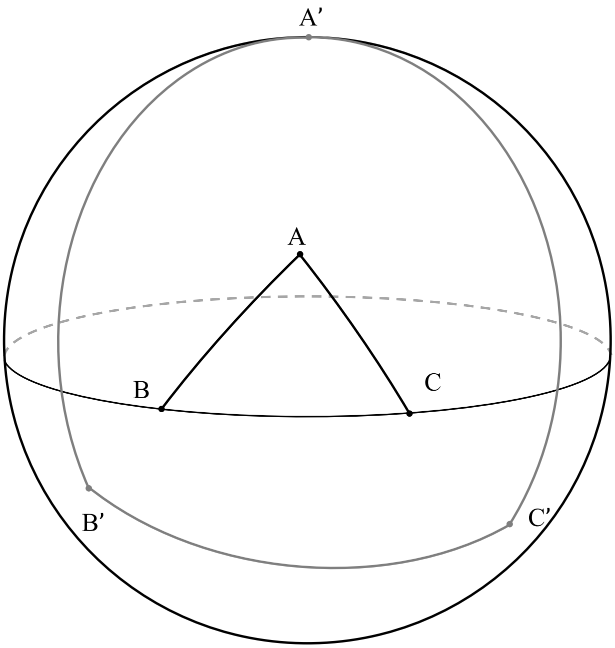

Figure 3.22: $\triangle^s ABC$ and its polar triangle $\triangle^s A'B'C'$.

and $\bigcirc BC$ must be different. Thus these two great circles meet in only two points (by Proposition 9.12), which we already know are N and S. But $\bigcirc AB$ and $\bigcirc BC$ also have the point B in common, so B must be the same as N or S. A similar argument shows A is the same as N or S, and C is the same as N or S. But then A, B, C all lie on a single great circle (any great circle passing through N and S contains all of A, B, C). This is a contradiction, so the assumption is false, and A', B', C' cannot all lie on a single great circle.$\diamondsuit$

The following theorem is one of the reasons the polar triangle is sometimes referred to as the dual triangle. (See [Je1994].[4])

Theorem 11.18 *The polar triangle of* $\triangle^s A'B'C'$ *is* $\triangle^s ABC$.

Proof. Since C and C' are on the same side of $\bigcirc AB$, and C' is a pole of $\bigcirc AB$, the distance between C and C' is strictly less than $\pi/2$. Since A' is a pole of $\bigcirc BC$, A' is at distance $\pi/2$ from C. Since B' is a pole of $\bigcirc AC$, B' is at distance $\pi/2$ from C. Since A' and B' are vertices of a spherical triangle, they are neither identical nor antipodal. Thus we conclude (from Proposition 11.1) that $\bigcirc A'B'$ is well-defined. Since A' and B' are both at distance $\frac{\pi}{2}$ from C, by Propositions 9.26 and 9.23 every point on $\bigcirc A'B'$ is at distance $\pi/2$

[4] Jennings defines the dual triangle to behave well with respect to the vector (cross) product, so that C' and $A \times B$ point in the same direction.

from C and C is one of the poles of $\bigcirc A'B'$. Since C and C' are at distance less than $\pi/2$ from each other, and C is a pole of $\bigcirc A'B'$, C' must be on the same side of $\bigcirc A'B'$ as C. Thus C is the pole of $\bigcirc A'B'$ on the same side of $A'B'$ as C'. In a similar manner, we can show B is the pole of $\bigcirc A'C'$ on the same side of $\bigcirc A'C'$ as B', and A is the pole of $\bigcirc B'C'$ on the same side of $\bigcirc B'C'$ as A'. By definition, this proves that $\triangle^s ABC$ is the polar triangle of $\triangle^s A'B'C'$.$\diamondsuit$

The following theorem is the justification for sometimes referring to the polar triangle as the supplemental triangle.

Theorem 11.19 *If $\triangle^s ABC$ is a spherical triangle and $\triangle^s A'B'C'$ is the polar triangle of $\triangle^s ABC$, then*

$$m \prec A' = \pi - m(\overset{\frown}{BC}) \tag{3.2}$$

$$m \prec B' = \pi - m(\overset{\frown}{AC}) \tag{3.3}$$

$$m \prec C' = \pi - m(\overset{\frown}{AB}) \tag{3.4}$$

$$m(\overset{\frown}{A'B'}) = \pi - m \prec C \tag{3.5}$$

$$m(\overset{\frown}{A'C'}) = \pi - m \prec B \tag{3.6}$$

$$m(\overset{\frown}{B'C'}) = \pi - m \prec A \tag{3.7}$$

Proof. The equation (3.6) follows immediately from Proposition 10.13. Equations (3.5) and (3.7) follow by permuting the vertices of the triangle.

To obtain the first three equations, we apply what we have just proven to $\triangle^s A'B'C'$, using the fact now known from Theorem 11.18 that the polar triangle of $\triangle^s A'B'C'$ is $\triangle^s ABC$. This gives us (3.2), (3.3), and (3.4). $\diamondsuit$

The foregoing theorems are valuable in spherical geometry because they show how statements involving sides and angles in one triangle may immediately be changed into statements involving corresponding angles and sides in another triangle (i.e., the polar triangle).

As in the plane, we sometimes consider objects with more than three sides.

Definition 11.20 *A spherical* quadrilateral *is the union of four spherical arcs of the form $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$, where A, B, C, and D are chosen so that no three of the points lie on a great circle, and the four arcs meet only at their endpoints. The quadrilateral is said to be spherically convex if for each of the four arcs of the quadrilateral, the points of the quadrilateral not on that arc lie on only one side of the great circle containing that arc. Each of the points A, B, C, and D is said to be a* vertex *of the quadrilateral.*

That is, the points of the quadrilateral not on $\overset{\frown}{AB}$ lie on the same side of $\bigcirc AB$, the points of the quadrilateral not on $\overset{\frown}{BC}$ lie on the same side of $\bigcirc BC$, and so on. We will refer to $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ as the "sides" of the quadrilateral, but this use of the word "side" must not be confused with the use of the word "side" used in the definition. No pair of the points A, B, C, and D is antipodal, so the arcs are well-defined (see Exercise 23). We use the notation $\diamond ABCD$ for the quadrilateral. A similar definition may be made for spherical *pentagon*, *hexagon*, or *n-gon*.

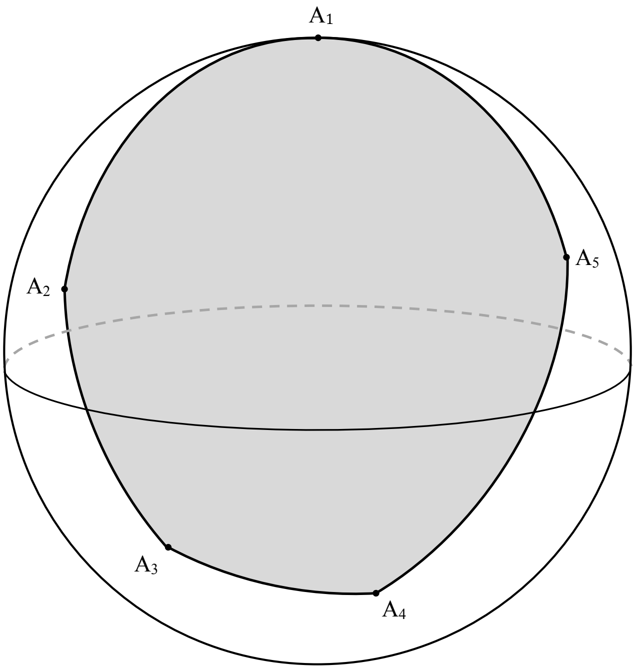

Figure 3.23: A spherically convex polygon $A_1A_2A_3A_4A_5$ and its interior.

Definition 11.21 *A spherical polygon of n sides, or n-gon, is the union of n spherical arcs $\overset{\frown}{A_1A_2}, \overset{\frown}{A_2A_3}, \ldots, \overset{\frown}{A_{n-1}A_n}, \overset{\frown}{A_nA_1}$ which meet only at their endpoints and such that of the n points $A_1, A_2, \ldots, A_n$, no three lie on a great circle. The n-gon is said to be* spherically convex *if for each of the n arcs of the n-gon, the points of the n-gon not on that arc lie on only one side of the great circle containing that arc. The n points are said to be the* vertices *of the n-gon. (See Figure 3.23.)*

A spherically convex n-gon is closely related to the notion of spherically convex set defined earlier. Given a great circle which contains a side of a spherically convex n-gon, one of the sides of that great circle contains the rest of the n-gon. Thus we obtain n hemispheres, one for each side of the n-gon. A hemisphere is a spherically convex set. Thus the intersection of the n hemispheres we obtain this way (one from each side of the n-gon) is a spherically convex set called the *interior* of the n-gon. The union of a spherically convex n-gon and its interior is also spherically convex; this is left to Exercise 24.

Exercises §11

1. Prove Proposition 11.4.

2. Prove Proposition 11.6.

3. Prove the "obtuse" case of Proposition 11.12 from the "acute" case using colunar triangles.

4. Prove the "obtuse" case of Proposition 11.14 from the "acute" case using colunar triangles.

5. Suppose that $\prec ABC$ is acute (respectively, obtuse). Prove that the foot of the shorter (longer) perpendicular from A to $\bigcirc BC$ is a point on $\overset{\hookrightarrow}{BC}$ other than B.

6. Suppose that $\triangle^s ABC$ has a right angle at C. Suppose that the side opposite angle C has measure $\frac{\pi}{2}$. Use Exercise 3 of §2 to prove that $\triangle^s ABC$ must have a right angle at either A or B.

7. Suppose that $\triangle^s ABC$ has a right angle at C. Suppose that the side opposite angle C has measure $\frac{\pi}{2}$. Using only propositions from spherical geometry (as opposed to using the solid geometry of Exercise 6), prove that $\triangle^s ABC$ must have a right angle at either A or B.

8. Suppose that $\triangle^s ABC$ has a right angle at C. Suppose that the side opposite angle C has measure less than $\frac{\pi}{2}$. Using only propositions from spherical geometry (as opposed to using solid geometry), prove that in $\triangle^s ABC$, sides $\frown{AC}$ and $\frown{BC}$ are both acute or both obtuse.

9. Suppose that $\triangle^s ABC$ has a right angle at C. Suppose that the side opposite angle C has measure greater than $\frac{\pi}{2}$. Using only propositions from spherical geometry (as opposed to using solid geometry), prove that of $\frown{AC}$ and $\frown{BC}$ one is acute and the other is obtuse.

10. Prove that in any spherical triangle, a cevian is always a well-defined spherical arc (that is, the vertex and a point of the opposite side are not antipodal). Then show that the points of the cevian are in the interior of the triangle, except for the endpoints.

11. Prove that in any spherical triangle, there is a well-defined angle bisector from each vertex which (except for the endpoints) lies in the interior of the triangle.

12. Show that the intersection of the interiors of only two of the angles of a spherical triangle must be the interior of the triangle.

13. Suppose that two points are in the interior of a spherical triangle. Prove that the spherical arc between them lies in the interior of the triangle.

14. Suppose that given a spherical arc, one endpoint is in the interior of a triangle and the other is in the exterior. Prove that the arc contains at least one point of the triangle.

15. In a spherical $\triangle^s ABC$, suppose D is a point of $\overset{\frown}{BC}$ different from B and C. Suppose that $m\ \overset{\frown}{AB} < \frac{\pi}{2}$ and $m\ \overset{\frown}{AC} < \frac{\pi}{2}$. Prove that $m\ \overset{\frown}{AD} < \frac{\pi}{2}$. What happens if the lengths of the two sides given are not less than $\frac{\pi}{2}$?

16. Suppose that a great circle meets a side of a spherical triangle but does not contain any of the vertices. Prove that the great circle contains at least one other point of the triangle not on that side. (This is the spherical analogue of a theorem in the plane known as Pasch's theorem.)

17. Suppose that the sides of a spherical triangle all have measure less than $\frac{\pi}{2}$. Prove that the sides of the polar triangle do not meet any sides of the original triangle.

18. Suppose that the angles of a spherical triangle all are obtuse. Prove that the sides of the polar triangle do not meet any sides of the original triangle.

19. Suppose that the sides of a spherical triangle all have measure greater than $\frac{\pi}{2}$. Prove that the sides of the polar triangle do not meet any sides of the original triangle.

20. Suppose that the angles of a spherical triangle all are acute. Prove that the sides of the polar triangle do not meet any sides of the original triangle.

21. Suppose $\triangle^s ABC$ has a right side $\overset{\frown}{AC}$. Then one of the other sides of $\triangle^s ABC$ is acute, right or obtuse, if and only if its opposite angle is acute, right or obtuse, respectively. (Hint: look at the polar triangle.)

22. Suppose an altitude of a spherical triangle is acute (respectively, obtuse). Prove that it is contained in (respectively, contains) an altitude of the polar triangle and these altitudes are supplementary. If an altitude is right, then it coincides with an altitude of the polar triangle.

23. Prove that in a spherical $\diamond ABCD$, the "sides" $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, and $\overset{\frown}{DA}$ and "diagonals" $\overset{\frown}{AC}, \overset{\frown}{BD}$ are well-defined in the sense that no pair of the points is antipodal. Then prove that the diagonals meet in a single point if and only if the quadrilateral is spherically convex.

24. Prove that the union of a spherically convex n-gon and its interior is also spherically convex.

25. Suppose that $A_1A_2\dots A_n$ is a spherically convex n-gon. Let $A_{n+1} = A_1$. For $i = 1, 2, \dots n$ let A'_i be the pole of $\bigcirc A_iA_{i+1}$ on the same side of $\bigcirc A_iA_{i+1}$ as the other vertices of the n-gon (which are all on the same side by definition of spherically convex n-gon). Prove that $A'_1A'_2\dots A'_n$ is also a spherically convex n-gon. (We call $A'_1A'_2\dots A'_n$ the polar n-gon of $A_1A_2\dots A_n$.)

26. Prove that if $A'_1A'_2\dots A'_n$ is the polar n-gon of the spherically convex n-gon $A_1A_2\dots A_n$, then $A_1A_2\dots A_n$ is the polar n-gon of $A'_1A'_2\dots A'_n$.

27. Let $A_1A_2\dots A_n$ be a spherically convex n-gon and $A'_1A'_2\dots A'_n$ its polar n-gon. Let $A_{n+1} = A_1$ and $A'_{n+1} = A'_1$, $a_i = m\ \overset{\frown}{A_iA_{i+1}}$, and $a'_i = m\ \overset{\frown}{A'_{i-1}A'_i}$. Prove that for all i, $a_i + m \prec A'_i = \pi$ and $a'_i + m \prec A_i = \pi$.

Historical notes. We let X-Y denote Book X, Proposition Y of Menelaus' *Sphaerica*. Then Theorem 11.10 is I-2.

12 Congruence

In this section we discuss what conditions are needed on sides and angles of two triangles for the two triangles to be congruent. We first state the definition, which is the same as for plane geometry.

Definition 12.1 *Two spherical triangles $\triangle^s ABC$ and $\triangle^s DEF$ are said to be* congruent *if $\overset{\frown}{AB} \cong \overset{\frown}{DE}$, $\overset{\frown}{AC} \cong \overset{\frown}{DF}$, $\overset{\frown}{BC} \cong \overset{\frown}{EF}$, $\prec A \cong \prec D$, $\prec B \cong \prec E$, and $\prec C \cong \prec F$. We say that* corresponding sides and angles *are all congruent.*

The corresponding sides and angles in Definition 12.1 are determined by the specific one-one correspondence of the vertices in each triangle: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$. Thus to say that $\triangle^s ABC \cong \triangle^s DEF$ is different from saying that $\triangle^s ACB \cong \triangle^s DEF$.

The reader should be familiar with the triangle congruence propositions of plane geometry, often abbreviated by SAS, SSS, and ASA. The first of these indicates that if two triangles have two pairs of congruent corresponding sides and the corresponding angles included between those sides are congruent, then the triangles are congruent, i.e., all of the other corresponding sides and angles are congruent. The SSS and ASA propositions are similar. These three propositions will turn out to hold for spherical triangles as well. In order to avoid confusion, we shall sometimes write "planar SSS" to refer to the SSS congruence proposition for planar triangles, and "spherical SSS" to refer to the corresponding proposition for spherical triangles. We will use similar language with SAS and ASA.

In high school plane geometry courses, these congruence propositions are usually assumed as axioms. Logically this is not necessary: one need only assume one of them (typically SAS) and the others can be proven.

Here all our congruence properties for spherical triangles will be established as theorems. The main vehicle for proving them will be Axiom 10.18.

Theorem 12.2 (SSS Congruence) *Suppose that in the spherical triangles* $\triangle^s A_1B_1C_1$ *and* $\triangle^s A_2B_2C_2$*, we have:*

$$\overset{\frown}{A_1B_1} \cong \overset{\frown}{A_2B_2}, \overset{\frown}{B_1C_1} \cong \overset{\frown}{B_2C_2}, \overset{\frown}{A_1C_1} \cong \overset{\frown}{A_2C_2} .$$

Then $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$*.*

Proof. We apply Proposition 10.18 to conclude that $\prec A_1B_1C_1 \cong \prec A_2B_2C_2$. By permuting the letters A, B, and C, we can conclude the same for the other corresponding angles. Thus $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$. ♢

Theorem 12.3 (SAS Congruence) *Suppose that in the spherical triangles* $\triangle^s A_1B_1C_1$ *and* $\triangle^s A_2B_2C_2$*, we have:*

$$\overset{\frown}{A_1B_1} \cong \overset{\frown}{A_2B_2}, \overset{\frown}{B_1C_1} \cong \overset{\frown}{B_2C_2}, \prec B_1 \cong \prec B_2.$$

Then $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$*.*

Proof. We apply Proposition 10.18 to conclude that the sides opposite the congruent pair of angles are congruent. But then we have an SSS correspondence between the triangles, and Theorem 12.2 implies that the triangles are congruent. ♢

As with plane geometry, we also have an ASA congruence theorem. However, we present a proof of it which makes use of the magic of polar triangles.

Theorem 12.4 (ASA Congruence) *Suppose that in the spherical triangles* $\triangle^s A_1B_1C_1$ *and* $\triangle^s A_2B_2C_2$*, we have:*

$$\overset{\frown}{B_1C_1} \cong \overset{\frown}{B_2C_2}, \prec B_1 \cong \prec B_2, \prec C_1 \cong \prec C_2.$$

Then $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$*.*

Proof. We let $\triangle^s A_i'B_i'C_i'$ be the polar triangle of $\triangle^s A_iB_iC_i$ for $i = 1, 2$. By Theorem 11.19, $m \prec A_1' = \pi - m\ \overset{\frown}{B_1C_1} = \pi - m\ \overset{\frown}{B_2C_2} = m \prec A_2'$ (so $\prec A_1' \cong \prec A_2'$,) $m\ \overset{\frown}{A_1'B_1'} = \pi - m \prec C_1 = \pi - m \prec C_2 = m\ \overset{\frown}{A_2'B_2'}$ (so $\overset{\frown}{A_1'B_1'} \cong \overset{\frown}{A_2'B_2'}$), and $m\ \overset{\frown}{A_1'C_1'} = \pi - m \prec B_1 = \pi - m \prec B_2 = m\ \overset{\frown}{A_2'C_2'}$ (so $\overset{\frown}{A_1'C_1'} \cong \overset{\frown}{A_2'C_2'}$). Thus we obtain an SAS correspondence between $\triangle^s A_1'B_1'C_1'$ and $\triangle^s A_2'B_2'C_2'$. By the spherical SAS congruence theorem, $\triangle^s A_1'B_1'C_1' \cong \triangle^s A_2'B_2'C_2'$. Thus all corresponding sides and angles of $\triangle^s A_1'B_1'C_1'$ and $\triangle^s A_2'B_2'C_2'$ have the same measure. By using Theorem 11.19 again in the same way in the other direction, we can show that corresponding sides and angles of spherical $\triangle^s A_1B_1C_1$ and $\triangle^s A_2B_2C_2$ are congruent, so $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$, as desired. ♢

In another moment's reflection we realize that by applying the same polar triangle trick to SSS congruence, we obtain a much more surprising theorem, one that does not hold in plane geometry.

Theorem 12.5 (AAA Congruence) *Suppose that in the spherical triangles* $\triangle^s A_1B_1C_1$ *and* $\triangle^s A_2B_2C_2$*, we have:*

$$\prec A_1 \cong \prec A_2, \prec B_1 \cong \prec B_2, \prec C_1 \cong \prec C_2.$$

Then $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$*.*

Proof. We let $\triangle^s A_i'B_i'C_i'$ be the polar triangle of $\triangle^s A_iB_iC_i$ for $i = 1, 2$. By Theorem 11.19, $m \stackrel{\frown}{B_1'C_1'} = \pi - m \prec A_1 = \pi - m \prec A_2 = m \stackrel{\frown}{B_2'C_2'}$, so $\stackrel{\frown}{B_1'C_1'} \cong \stackrel{\frown}{B_2'C_2'}$. A similar argument shows that $\stackrel{\frown}{A_1'B_1'} \cong \stackrel{\frown}{A_2'B_2'}$ and $\stackrel{\frown}{A_1'C_1'} \cong \stackrel{\frown}{A_2'C_2'}$. Thus $\triangle^s A_1'B_1'C_1' \cong \triangle^s A_2'B_2'C_2'$ by the spherical SSS congruence (Theorem 12.2). Thus corresponding angles and sides of triangles $\triangle^s A_i'B_i'C_i'$ are congruent. Using Theorem 11.19 again to go back to the original triangles $\triangle^s A_iB_iC_i$, we conclude similarly that their corresponding sides and angles all have the same measure (so are congruent). By definition, $\triangle^s A_1B_1C_1 \cong \triangle^s A_2B_2C_2$, as desired. ♢

There are two other cases to consider: can we conclude the congruence of spherical triangles in the event of an SSA or SAA congruence of corresponding sides and angles?

In plane geometry, let us recall that an SAA correspondence does guarantee congruence of triangles. One way to see this is that if two pairs of corresponding angles of a triangle are congruent, so is the third pair because in both triangles, the sum of the measures of the angles is π. Then we have an ASA correspondence between the triangles, from which congruence of the triangles follows. This argument will turn out to fail on the sphere because by Theorem 13.7 the sum of the measures of the angles is greater than π, and this sum is not generally the same from one triangle to another.

However, the angle sum theorem in the plane depends on the parallel postulate, and in fact the SAA congruence theorem in the plane does not depend on the parallel postulate. The reader will find a proof of SAA congruence in the plane in [Mo1963] (or [MD1982]) which makes use of the exterior angle theorem in the plane (which states that an exterior angle of a triangle has measure greater than the measures of either of the opposite interior angles). The spherical analogue of this theorem also turns out to be false.

In plane geometry the SSA correspondence does not guarantee congruence of triangles, although we might say that it almost does. Knowledge of the measures of two sides and an angle that is not included leads to the so-called "ambiguous case" in determining the other sides of the triangle. The other sides are uniquely determined if the angle is a right angle, but otherwise there are either two possibilities for the other sides and angles, or the triangle cannot be constructed at all.

We can make an immediate observation for spherical geometry: either SSA and SAA both guarantee congruence of triangles or neither does. The reason is that if one guarantees congruence, then we could prove that the other does as well by employing the same argument with polar triangles used in the proofs of ASA and AAA congruence. For example, suppose there were an SAA congruence theorem. To prove that an SSA correspondence guarantees congruence, we would consider two triangles which have an SSA correspondence. Their polar triangles would have an SAA correspondence by Theorem 11.19, and hence would be congruent. The congruence of the pair of polar triangles would in turn guarantee (by Theorem 11.19 again) that the original triangles be congruent. A similar line of reasoning would allow us to use an SSA congruence theorem to prove an SAA congruence theorem.

It turns out that on a sphere neither SSA nor SAA correspondences guarantee congruence of triangles in general. A single counterexample will dispense with both. Let $A_1 = A_2$ be the pole of any great circle. Choose points B_1, B_2, C_1, and C_2 on the great circle such that $m\ \overset{\frown}{B_1C_1} \neq m\ \overset{\frown}{B_2C_2}$. Then by definition of pole, $\overset{\frown}{A_1B_1}$, $\overset{\frown}{A_1C_1}$, $\overset{\frown}{A_2B_2}$, and $\overset{\frown}{A_2C_2}$ are all quarter circles. Furthermore, $\prec A_1B_1C_1$, $\prec A_1C_1B_1$, $\prec A_2B_2C_2$, and $\prec A_2C_2B_2$ are all right angles by Proposition 10.12. Then triangles $\triangle^s A_1B_1C_1$ and $\triangle^s A_2B_2C_2$ are in both SSA and SAA correspondences, but are not congruent because $m\ \overset{\frown}{B_1C_1} \neq m\ \overset{\frown}{B_2C_2}$.

The counterexample above is somewhat extreme in that if the triangles involved have no more than one right angle, the situation is more like that in the plane. If there is exactly one right angle, we do obtain congruence theorems. If there are no right angles, then there are at most two possibilities for a triangle where two sides and an angle opposite one of them (or two angles and a side opposite one of them) are known.

Theorem 12.6 (SSAA Congruence) *(1)Suppose we have that $\triangle^s ABC$ and $\triangle^s DEF$ satisfy $\overset{\frown}{AB} \cong \overset{\frown}{DE}$, $\overset{\frown}{AC} \cong \overset{\frown}{DF}$, $\prec B \cong \prec E$, and $\prec C$ is not supplementary to $\prec F$. Then $\triangle^s ABC \cong \triangle^s DEF$. (2)Suppose that $\triangle^s ABC$ and $\triangle^s DEF$ satisfy $\overset{\frown}{AB} \cong \overset{\frown}{DE}$, $\prec C \cong \prec F$, $\prec B \cong \prec E$, and $\overset{\frown}{AC}$ is not supplementary to $\overset{\frown}{DF}$. Then $\triangle^s ABC \cong \triangle^s DEF$.*

Proof. We prove (1) and leave (2) to Exercise 8. If $\overset{\frown}{BC} \cong \overset{\frown}{EF}$ then $\triangle^s ABC \cong$

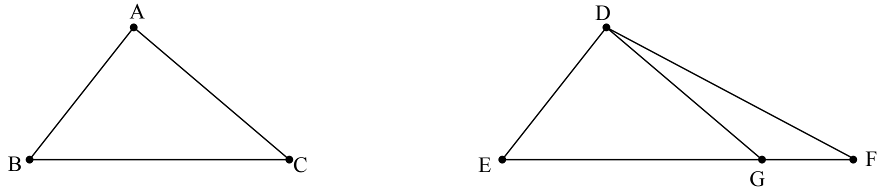

Figure 3.24: Theorem 12.6, part (1).

$\triangle^s DEF$ by SSS congruence. If this is not the case then one of the two arcs $\overset{\frown}{BC}$ and $\overset{\frown}{EF}$ is longer. Assume without loss of generality that $\overset{\frown}{EF}$ is the longer arc. Then there exists a point G on $\overset{\frown}{EF}$ not equal to E or F such that $\overset{\frown}{BC} \cong \overset{\frown}{EG}$.

Then $\triangle^s ABC \cong \triangle^s DEG$ by SAS congruence. Then $\prec DGE$ is congruent to $\prec ACB$ and $\overset{\frown}{AC} \cong \overset{\frown}{DG}$ (as corresponding parts). Since $\overset{\frown}{AC} \cong \overset{\frown}{DF}$ we also have $\overset{\frown}{DG} \cong \overset{\frown}{DF}$. Then in isosceles $\triangle^s DGF$ the angles opposite sides $\overset{\frown}{DG}$ and $\overset{\frown}{DF}$ are congruent, i.e., $\prec DGF \cong \prec DFE$. But $\prec DGF$ is supplementary to $\prec DGE \cong \prec ACB$, so $\prec DFE$ is supplementary to $\prec ACB$, a contradiction of the assumptions. Thus indeed we must have had $\overset{\frown}{BC} \cong \overset{\frown}{EF}$ as above. This concludes the proof of (1).$\diamondsuit$

This theorem suggests that if one is given two pair of sides in a spherical triangle and the opposite angles — then the other side and angle can generally be determined. This is simple to do in plane trigonometry (since the sum of the angle measures is known) but not so simple on the sphere. In fact, if the triangles are not isosceles it can be done with Napier's analogies (see §18, Exercise 6 and §19, Exercise 11). If the triangles are isosceles but the known angles are not right angles it can be done using right triangle trigonometry (see §17, Exercise 21).

Definition 12.7 *Two spherical triangles $\triangle^s ABC$ and $\triangle^s DEF$ are said to have a* hypotenuse-leg correspondence *if they have (corresponding) right angles at B and E, the opposite hypotenuses are congruent, and they have a pair of corresponding congruent (non-right) legs.*

Corollary 12.8 (Hypotenuse-leg Theorem) *Two triangles which have a hypotenuse-leg correspondence must be congruent.*

Proof. See Exercise 25. The reader should note that it is important that a leg is not a hypotenuse of the triangle. $\diamondsuit$

Definition 12.9 *Two spherical triangles $\triangle^s ABC$ and $\triangle^s DEF$ are said to have a* hypotenuse-angle correspondence *if they have (corresponding) right angles at B and E, the opposite hypotenuses are congruent, and they have a pair of corresponding congruent* non-right angles.

Corollary 12.10 (Hypotenuse-angle Theorem) *Two spherical triangles which have a hypotenuse-angle correspondence must be congruent.*

Proof. See Exercise 31. The assumption that the second angles are not right angles is important. $\diamondsuit$

Exercises §12

1. Prove the converse of Theorem 11.10: that if two angles in a spherical triangle are congruent, the opposite sides are congruent.

2. Suppose that $\triangle^s ABC$ is in an SAA correspondence with $\triangle^s DEF$ in the sense that $\prec A \cong \prec D$, $\prec B \cong \prec E$, and $\overset{\frown}{BC} \cong \overset{\frown}{EF}$. Suppose that $\triangle^s ABC$ is not congruent to $\triangle^s DEF$. Let A^a be the antipode of A, so that $\triangle^s A^a BC$ is colunar with $\triangle^s ABC$. Prove that there exists an SSA correspondence between $\triangle^s DEF$ and $\triangle^s A^a BC$.

3. Suppose that in spherical $\triangle^s ABC$, $m\ \overset{\frown}{AB} = m\ \overset{\frown}{AC}$ and D is the midpoint of $\overset{\frown}{BC}$. Prove that $\bigcirc AD \perp \bigcirc BC$.

4. Suppose that in spherical $\triangle^s ABC$, $m\ \overset{\frown}{AB} = m\ \overset{\frown}{AC}$ and D is the midpoint of $\overset{\frown}{BC}$. Prove that $\prec BAD \cong \prec CAD$.

5. Suppose that in spherical $\triangle^s ABC$, $m\ \overset{\frown}{AB} = m\ \overset{\frown}{AC} \neq \frac{\pi}{2}$ and D is between B and C such that $\overset{\frown}{AD} \perp \overset{\frown}{BC}$. Prove that D is the midpoint of $\overset{\frown}{BC}$. Is it important that $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$ be not right?

6. Suppose that a median of a triangle is perpendicular to the opposite side. Prove that the triangle is isosceles.

7. Suppose we are given two chords of a small circle. Show that the two chords have the same measure if and only if the perpendiculars from the center to each chord have the same measure. (Note that by Exercise 5, the perpendicular is the arc between the center and the midpoint of the chord.)

8. Prove part (2) of Theorem 12.6.

9. Prove that two sides of a spherical triangle are supplementary if and only if the opposite angles are supplementary.

10. Prove that the set of all points on a sphere which are at the same spherical distance from two given distinct points of the sphere must be a great circle which is perpendicular to a spherical arc between the two given points at its midpoint. (If the two given points are not antipodal, this great circle is called the *perpendicular bisector* of the spherical arc between the two given points.)

11. Prove that if three points do not lie on a single great circle, then there exists a unique small circle passing through them whose center is on each of the perpendicular bisectors of the arcs between the pairs of points. (This circle is called the *circumscribed circle* or *circumcircle* of the triangle of the three points and its center is the *circumcenter*.)

12. Let $\prec ABC$ be an angle. Let $\overset{\hookrightarrow}{r}$ be a spherical ray with vertex B on the same side of $\overset{\hookrightarrow}{BC}$ as A such that $\overset{\hookrightarrow}{r}$ forms an angle with $\overset{\hookrightarrow}{BC}$ of

measure half of $m \prec ABC$. Show that $\overset{\hookrightarrow}{r}$ also forms an angle of measure $\frac{1}{2}m \prec ABC$ with $\overset{\hookrightarrow}{BA}$. Then $\overset{\hookrightarrow}{r}$ is called the *angle bisector* of $\prec ABC$. Show that a point in the interior of a spherical angle is on the angle bisector of the angle if and only if its spherical distances to the sides of the angle are the same. Also, under these circumstances the distance to the sides is less than $\frac{\pi}{2}$.

13. Suppose that an angle bisector of an angle of a triangle is perpendicular to the opposite side. Prove that the triangle is isosceles.

14. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} < m \overset{\frown}{AC}$ and D is on $\overset{\frown}{BC}$ so that $\overset{\hookrightarrow}{AD}$ bisects $\prec BAC$. Prove that $m \prec ADB < m \prec ADC$ (so $\prec ADB$ is acute and $\prec ADC$ is obtuse).

15. Suppose that in $\triangle^s ABC$, D is on $\overset{\frown}{BC}$ so that $\overset{\hookrightarrow}{AD}$ bisects $\prec BAC$ or D bisects $\overset{\frown}{BC}$. Prove that if $m \overset{\frown}{AD} = \frac{\pi}{2}$ then $m \overset{\frown}{AB} + m \overset{\frown}{AC} = \pi$.

16. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} + m \overset{\frown}{AC} = \pi$. Suppose that D is on $\overset{\frown}{BC}$. Prove that $\overset{\hookrightarrow}{AD}$ bisects $\prec BAC$ if and only if D is the midpoint of $\overset{\frown}{BC}$ and in either case $m \overset{\frown}{AD} = \frac{\pi}{2}$.

17. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} \neq m \overset{\frown}{AC}$ and D is on $\overset{\frown}{BC}$ so that $\overset{\hookrightarrow}{AD}$ bisects $\prec BAC$ and D bisects $\overset{\frown}{BC}$. Prove that $m \overset{\frown}{AB} + m \overset{\frown}{AC} = \pi$. Can you remove the condition that $m \overset{\frown}{AB} \neq m \overset{\frown}{AC}$?

18. Suppose that base $\overset{\frown}{BC}$ of a spherical triangle $\triangle^s ABC$ is known, and the sum of the measures of the angles at B and C is known. Prove that regardless of the measures of the angles at B and C, the bisector of the angle at A passes through a fixed point.

19. Prove that in any $\triangle^s ABC$, the three spherical rays which bisect the three angles must intersect in a single point. (This point is called the *incenter* of the triangle.) Prove that the incenter of a triangle is at the same spherical distance $r < \frac{\pi}{2}$ from the great circles containing the sides of the triangle, and the feet of the three shorter perpendiculars from I to these three great circles lie on the sides of the triangle between the endpoints. Conclude that the small circle with spherical radius r and center I is tangent to each of $\bigcirc AB$, $\bigcirc BC$, and $\bigcirc AC$. (The circle is called the *inscribed circle* of $\triangle^s ABC$, its center is the *incenter*, and the number r is the *inradius* of $\triangle^s ABC$.

20. Suppose that $\triangle^s ABC$ has incenter I. Prove that I is the circumcenter of the polar triangle of $\triangle^s ABC$ and the inradius of $\triangle^s ABC$ is complementary to the circumradius of the polar triangle.

21. Suppose that in $\triangle^s ABC$, $m\ \widehat{AB} \neq m\ \widehat{AC}$ and $m\ \widehat{AB} + m\ \widehat{AC} = \pi$. Suppose D and E are on $\widehat{BC}$ so that $m\ \widehat{AD} + m\ \widehat{AE} = \pi$. Prove that $\widehat{BD} \cong \widehat{CE}$ and $\prec BAD \cong \prec CAE$. Can you remove the condition that $m\ \widehat{AB} \neq m\ \widehat{AC}$?

22. Following [Ca1889], we define a triangle to be *diametrical* if its vertices are at the same spherical distance from the midpoint of one of its sides. Prove that a triangle is diametrical if and only if the measure of one of its angles equals the sum of the measures of the other two.

23. Prove that in a diametrical triangle, two of its colunar triangles are also diametrical, and in the third the sum of the measures of the angles[5] must be 2π.

24. Let $\widehat{BC}$ be an arc and let k be a real number. Determine the set of all points A such that A, B, and C form a spherical triangle where $B + C - A = k$. What are the possible values of k? (Hint: see Exercise 22.)

25. Prove Theorem 12.8. Explain (by giving a counterexample) why it is important that the "leg" of the theorem is not a hypotenuse.

26. Suppose that a great circle is tangent to a small circle. Prove that the radius of the small circle to the point of tangency is perpendicular to the great circle.

27. Suppose that two great circles passing through a point A are tangent to a small circle at points B and C of the small circle (see Exercise 26). Prove that $\widehat{AB} \cong \widehat{AC}$.

28. Suppose that in a spherical right triangle, the two angles other than the right angle are acute. Prove that the hypotenuse is longer than either of the legs.

29. Suppose that in a spherical right triangle, the hypotenuse is acute and one of the non-right angles is acute. Prove that in this triangle, the measure of that angle is greater than or equal to the measure of the opposite side.

    What happens if the hypotenuse is obtuse? Right?

30. Suppose that an isosceles spherical triangle has legs which are acute. Prove that the measure of the base is less than the measure of the vertex angle. (Hint: see Exercise 29.)

31. Prove Theorem 12.10. Explain (by giving a counterexample) why it is important that the second angle not be a right angle.

[5] See [Ca1889].

32. Two spherical triangles $\triangle^s ABC$ and $\triangle^s DEF$ are said to have a *leg-angle correspondence* if they have (corresponding) right angles at B and E, a pair of corresponding legs is congruent, and a pair of corresponding non-right angles is congruent. Are two such triangles always congruent? If not, what conditions might be added to guarantee congruence?

33. Suppose that in a convex spherical quadrilateral the opposite sides are congruent. Prove that the opposite angles are congruent and the diagonals bisect each other.

34. Suppose that in a convex spherical quadrilateral the opposite angles are congruent. Prove that the opposite sides are congruent.

35. Suppose that in a convex spherical quadrilateral all the sides are congruent. Prove that the diagonals are perpendicular and bisect the angles. Do the diagonals have to be congruent?

36. Suppose that in a convex spherical quadrilateral all the angles are congruent. Prove that the diagonals are congruent. Do all the sides have to be congruent?

Historical notes. Most theorems in this section appear in Menelaus' *Sphaerica*, although most proofs are different since *Sphaerica* does not make use of the notion of polar triangle. We let X-Y denote Book X, Proposition Y of *Sphaerica*. Then Exercise 1 is I-3, Theorem 12.10 is I-12, Theorem 12.6 part (1) is I-13, Theorem 12.4 is I-14 and I-15, Theorem 12.6 part (2) is I-17, Theorem 12.5 is I-18, Exercise 16 is I-29 and I-30, and Exercise 21 is I-31.

13 Inequalities

Having considered many situations where objects in spherical geometry are equal or congruent, we now consider situations where inequalities occur. Recall that in the plane the sum of the measures of the angles in a triangle is π radians (180°). This is untrue for spherical triangles, and is probably the most important distinction between triangles in the plane and those on the sphere. We first examine this question in the case of right triangles.

Proposition 13.1 *In a right spherical triangle, the sum of the measures of the angles is larger than π (180°).*

Proof. Suppose without loss of generality that $\triangle^s ABC$ has a right angle at B. If either of the angles at A or C is right or obtuse, we are done. So we may assume that the angles at A and C are both acute. By Proposition 11.12, the opposite sides $\frown{AB}$ and $\frown{BC}$ are both acute also. Let A' be the pole of $\bigcirc BC$ on the same side of $\bigcirc BC$ as A. Then since $\frown{AB}$ is acute, A is between A' and B. Then $\prec A'AC$ is obtuse, since its measure is the supplement of the

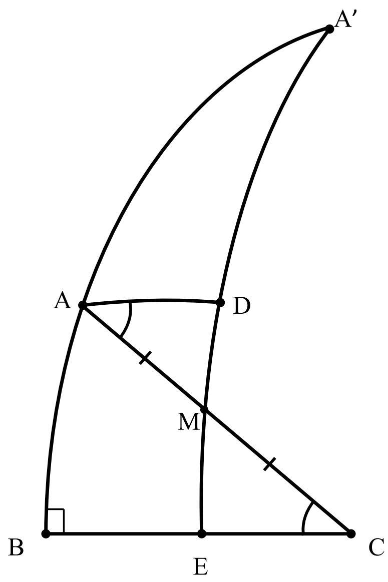

Figure 3.25: Figure for Proposition 13.1.

measure of $\prec A$. Since $\prec C$ is acute, $m \prec C < m \prec A'AC$. Then by Proposition 10.8, there exists a ray $\overset{\hookrightarrow}{r}$ emanating from A making an angle with $\overset{\hookrightarrow}{AC}$ whose measure is the same as that of $\prec C$, and such that $\overset{\hookrightarrow}{r}$ is on the same side of $\bigcirc AC$ as A'. The points of $\overset{\hookrightarrow}{r}$ are in the interior of $\prec A'AC$ (except for A). Let M be the midpoint of $\overset{\frown}{AC}$. By Proposition 10.11, $\overset{\hookrightarrow}{r}$ meets $\overset{\frown}{A'M}$ at a point we call D which lies between A' and M. Since A and A' are on the same side of $\bigcirc BC$, by Proposition 9.44, the points of $\overset{\hookrightarrow}{CA}$ (except for C) lie on the same side of $\bigcirc BC$ as A (so on the same side of $\bigcirc BC$ as A'); thus M is on the same side of $\bigcirc BC$ as A', so $\overset{\frown}{A'M}$ is acute. Since D is between M and A', $\overset{\frown}{A'D}$ is also acute. Since M is between A and C, M is in the interior of $\prec AA'C$. Thus $\overset{\hookrightarrow}{A'M}$ is also in the interior of $\prec AA'C$ (except for A'), and meets $\overset{\frown}{BC}$ at a point E between B and C. Since $\overset{\hookrightarrow}{A'M}$ passes through the pole A' of $\overset{\frown}{BC}$, it meets $\overset{\frown}{BC}$ at a right angle. Thus $\prec MEC$ is right. Since $\overset{\frown}{AM} \cong \overset{\frown}{MC}$ (definition of M), $\prec AMD \cong \prec CME$ (vertical angles) and $\prec MAD \cong \prec MCE$ (by definition of D), we obtain that $\triangle^s AMD \cong \triangle^s CME$ by spherical ASA congruence. Then since $\triangle^s CME$ has a right angle at E, $\triangle^s AMD$ has a right angle at D. Then $\triangle^s A'DA$ has a right angle at D also. Since $\overset{\frown}{A'D}$ is acute, by Proposition 11.12 applied to $\triangle^s A'AD$, $\prec A'AD$ is acute. But then $\prec BAD$ is obtuse. Since $\prec BAC$ is acute, $m \prec BAC < m \prec BAD$. Now all of ray

$\overset{\hookrightarrow}{A'M}$ is on the same side of $\bigcirc AB$ except for A', so D and M are on the same side of $\bigcirc AB$. By Proposition 10.9, M is in the interior of $\prec BAD$. Thus $m \prec BAD = m \prec BAC + m \prec CAD = m \prec A + m \prec C$. So the sum of the measures of $\prec A$ and $\prec C$ is greater than $\frac{\pi}{2}$ (90°). Since $\prec B$ is a right angle, the sum of the measures of $\prec A, \prec B$ and $\prec C$ is at least π (180°). ♢

Theorem 13.2 *The sum of the measures of the angles in a spherical triangle is greater than* π *radians (*180°*).*

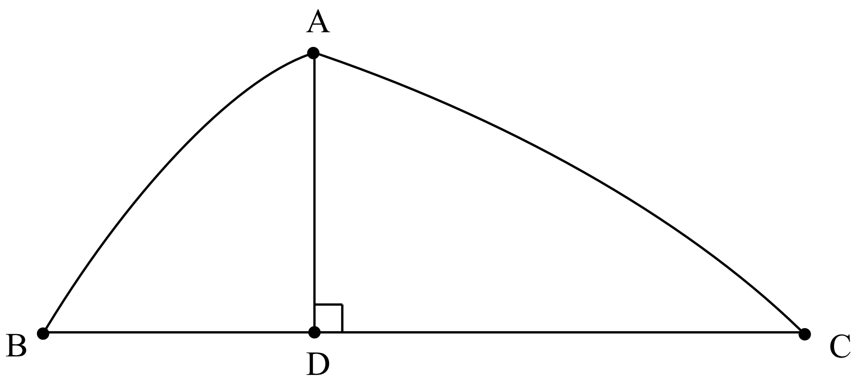

Figure 3.26: Figure for Proposition 13.2.

Proof. If the given triangle is right, we apply Proposition 13.1. If the given triangle has only one acute angle, then the sum of the measures of the other two (both obtuse) must be larger than π, so we are done.

Now suppose the triangle ($\triangle^s ABC$) has two acute angles (say at B and C). Then the foot of the shorter perpendicular from A to $\bigcirc BC$ is a point D between B and C. The sum of the measures of the non-right angles in $\triangle^s ADB$ is greater than $\frac{\pi}{2}$. The same is true in $\triangle^s ADC$. But the sum of the measures of the angles in $\triangle^s ABC$ is $m \prec A + m \prec B + m \prec C = m \prec BAD + m \prec CAD + m \prec B + m \prec C$ since D is between B and C. Then this sum is

$$(m \prec B + m \prec BAD) + (m \prec C + m \prec CAD),$$

which (by Proposition 13.1) is greater than $\frac{\pi}{2} + \frac{\pi}{2} = \pi$, as desired. ♢

Theorem 13.2 has a whole series of consequences that we obtain simply by making use of colunar and polar triangles on the sphere.

Theorem 13.3 *In any spherical triangle, the sum of the measures of the sides is less than* 2π*.*

Proof. Let the given triangle be $\triangle^s ABC$, and let its polar triangle be $\triangle^s A'B'C'$. By Theorem 13.2, $m \prec A' + m \prec B' + m \prec C' > \pi$. But by Theorem

11.19, $m \prec A' = \pi - m \overset{\frown}{BC}$, $m \prec B' = \pi - m \overset{\frown}{AC}$, and $m \prec C' = \pi - m \overset{\frown}{AB}$. Substituting these into the above inequality,

$$(\pi - m \overset{\frown}{BC}) + (\pi - m \overset{\frown}{AB}) + (\pi - m \overset{\frown}{AC}) > \pi,$$

or $m \overset{\frown}{AB} + m \overset{\frown}{AC} + m \overset{\frown}{BC} < 2\pi$, as desired. ♢

Theorem 13.4 *The sum of the measures of any two sides of a spherical triangle is greater than the measure of the third.*

No doubt this theorem, commonly known as the *triangle inequality* on the sphere, is intuitively obvious to most readers.

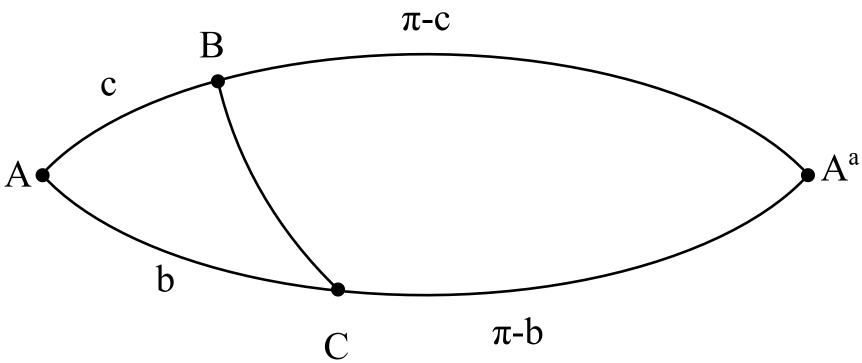

Figure 3.27: Figure for Proposition 13.4.

Proof. Suppose the triangle is $\triangle^s ABC$. Let $\triangle^s A^a BC$ be a triangle colunar with $\triangle^s ABC$. Then $m \overset{\frown}{A^a B} = \pi - m \overset{\frown}{AB}$ and $m \overset{\frown}{A^a C} = \pi - m \overset{\frown}{AC}$. Applying Theorem 13.3 to $\triangle^s A^a BC$, we obtain $(\pi - m \overset{\frown}{AB}) + (\pi - m \overset{\frown}{AC}) + m \overset{\frown}{BC} < 2\pi$, or $m \overset{\frown}{BC} < m \overset{\frown}{AB} + m \overset{\frown}{AC}$, as desired. The other two inequalities are obtained by permuting the vertices. ♢

Theorem 13.5 *If A, B, and C are three (distinct) points on a sphere, then $d(A, C) \leq d(A, B) + d(B, C)$, where equality occurs if and only if B is between A and C or $A = C^a$.*

Proof. If the points A, B, and C do not lie on a single great circle, they form a spherical triangle and the conclusion is immediate from Theorem 13.4 (where B cannot be between A and C because the three points are not on a great circle, and A cannot be antipodal to C by Proposition 11.1). If the points do lie on a great circle, then we leave the details to Exercise 20. ♢

Corollary 13.6 *Let $A_1, A_2, A_3, \ldots, A_n$ be distinct points on a sphere. Then*

$$d(A_1, A_n) \leq d(A_1, A_2) + d(A_2, A_3) + d(A_3, A_4) + \ldots + d(A_{n-1}, A_n). \quad (3.8)$$

Proof. See Exercise 21. ◊

Corollary 13.6 shows that if a sequence of points is chosen on a sphere, then the sum of the spherical distances traveled from each point to the next is always greater than or equal to the distance from the first point of the sequence to the last point in the sequence. This is a version of the principle stated earlier that the length of the shortest spherical path between two points on a sphere travels the route of a great circle arc of shortest length between the points. Note that such a great circle arc is not necessarily unique: if two points are antipodal, then any of the infinitely many great semicircles between them is a great circle arc of shortest length.

Of course, Corollary 13.6 is not the final word on the subject, since it only considers paths between two points on the sphere which consist of pieces which are great circle arcs. What about other paths?

In order to consider the last question, we need to use the integral and differential calculus — even to define what is meant by the length of a curve. The problem is unusually difficult because a minimum value is being sought for a function (length) which is defined on a set (the set of reasonably smooth curves) which cannot be parametrized with finitely many variables. Thus this minimum value problem does not belong in a typical course in calculus; instead it belongs to the field of *calculus of variations*. The interested reader will find a discussion of this problem in Exercises 54a and 54b.

We now use the notion of polar triangle to prove two more theorems which allow us to find relations among the measures of the angles of a spherical triangle.

Theorem 13.7 *In any spherical $\triangle^s ABC$, we have*

$$m \prec A + m \prec B < \pi + m \prec C \tag{3.9}$$

$$m \prec B + m \prec C < \pi + m \prec A \tag{3.10}$$

$$m \prec A + m \prec C < \pi + m \prec B \tag{3.11}$$

$$\pi < m \prec A + m \prec B + m \prec C < 3\pi \tag{3.12}$$

Proof. Let $\triangle^s A'B'C'$ be the polar triangle of $\triangle^s ABC$. By Theorem 13.4, we have $m \overset{\frown}{B'C'} + m \overset{\frown}{A'C'} > m \overset{\frown}{A'B'}$. By Theorem 11.19, we have (3.5), (3.6), and (3.7). Substituting these into the above inequality, we find that $(\pi - m \prec A) + (\pi - m \prec B) > \pi - m \prec C$, which gives us (3.9). We can permute the variables to obtain (3.10) and (3.11).

The first inequality in (3.12) is merely Theorem 13.2. The second inequality in (3.12) results from the fact that no angle can have measure greater than π, so the sum of the measures of three angles is less than 3π. ◊

To summarize Theorem 13.3 and 13.7: *in any spherical triangle, the sum of the measures of the sides is between* 0 *and* 2π *and the sum of the measures of the angles is between* π *and* 3π.

One might wonder whether we can improve on these statements, or whether there exist triangles whose side sums and angle sums approach these extremes. In fact, there are triangles which approach these extremes as closely as we would like, but being precise about their existence is a little tricky. We indicate informally how such extreme triangles might be found. A triangle with all three sides extremely small will clearly have the sum of the measures of its sides close to zero (but still slightly positive). Since such a triangle is close to being planar, the sum of the measures of its angles will be very close to π (but still slightly larger). For the other extremes, choose any great circle on the sphere with pole P, and then choose three points A, B, and C on the same side of that great circle but very close to it, all approximately equally spaced. Then the sides of the triangle will all be very close to the given great circle. As a result, the sum of the measures of the sides will be very close to 2π (the measure of the great circle). Since the great circles determined by the sides are all very close to the given great circle, the angles between the sides will all have measures close to π, hence the sums of the measures of the angles will be very close to 3π. For more details, see Exercises 46-49.

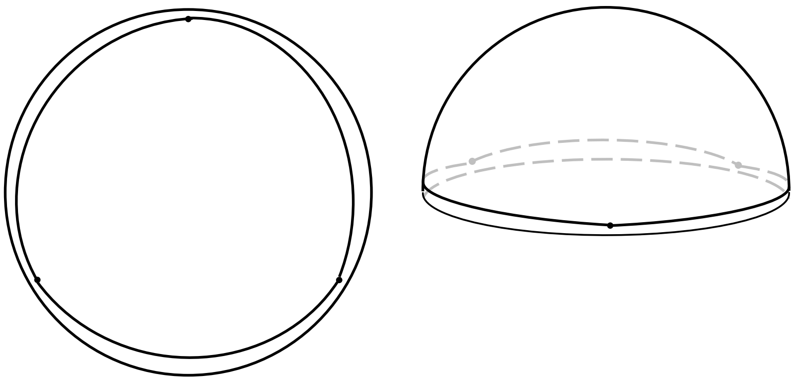

Figure 3.28: A triangle whose angle sum is near 3π and sums of whose sides is near 2π — top and side view.

One might further wonder if it is possible to have other combinations. For example, could one have a triangle, the sum of the measures of whose sides is very close to 2π, but the sum of the measures of whose angles is very close to π? This kind of question is explored in Exercises 50 and 51.

Definition 13.8 *Let $\triangle^s ABC$ be a spherical triangle. Let A^a, B^a, and C^a be the antipodes of A, B, and C, respectively. Then the* exterior angles *at A are the angles $\prec BAC^a$ and $\prec CAB^a$. The* opposite interior *angles (or* remote interior *angles) to $\prec BAC^a$ and $\prec CAB^a$ are the angles $\prec ABC$ and $\prec ACB$.*

Theorem 13.9 (Spherical Exterior Angle Theorem) *The measure of any exterior angle of a spherical triangle is less than the sum of the measures of the opposite interior angles and greater than the (absolute value of the) difference between those measures.*

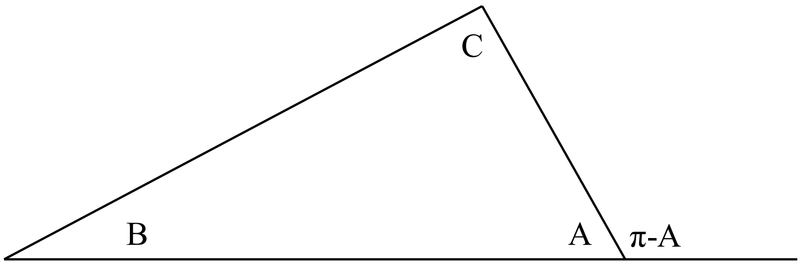

Figure 3.29: Figure for Theorem 13.9: $|B - C| < \pi - A < B + C$.

Proof. We let the triangle be $\triangle^s ABC$ and consider the exterior angle at A. By Theorem 13.2, $\pi < m \prec A + m \prec B + m \prec C$, so $\pi - m \prec A < m \prec B + m \prec C$. Since the exterior angle at A has measure $\pi - m \prec A$, this shows that the exterior angle at A has measure less than the sum of the measures of the angles $\prec B$ and $\prec C$. From equations (3.9) and (3.11) we find $\pi - m \prec A > m \prec B - m \prec C$ and $\pi - m \prec A > m \prec C - m \prec B$, respectively, so $\pi - m \prec A$ is greater than $|m \prec B - m \prec C|$, the absolute value of the difference between the measures of angles B and C. This is what we wanted. Inequalities for angles exterior at B and C are similar. $\diamondsuit$

Theorem 13.10 *Given two sides of a triangle whose measures are unequal and the angles opposite them, then the angles are unequal in measure and the larger angle is opposite the longer side. Similarly, given two angles of a triangle whose measures are unequal, the opposite sides have unequal measures and the longer side is opposite the larger angle.*

Proof. Suppose the triangle is $\triangle^s ABC$, and $m \overset{\frown}{AB} > m \overset{\frown}{BC}$. Choose point D on $\overset{\frown}{AB}$ such that $m \overset{\frown}{BD} = m \overset{\frown}{BC}$. By Theorem 11.10, $m \prec BDC = m \prec BCD$. Now $\prec BDC$ is exterior to $\triangle^s ADC$, so by Theorem 13.9,

$$m \prec BDC > |m \prec BAC - m \prec DCA| \geq m \prec BAC - m \prec DCA.$$

Then $m \prec BDC + m \prec DCA > m \prec BAC$. Since $m \prec BDC = m \prec BCD$, $m \prec BCD + m \prec DCA > m \prec BAC$. By Proposition 10.9, $m \prec BCD + m \prec DCA = m \prec BCA$, so $m \prec BCA > m \prec BAC$, as desired. The second sentence of the proof follows from the first and Theorem 11.10: if the opposite sides were equal in measure, the angles would also be (by Theorem 11.10).

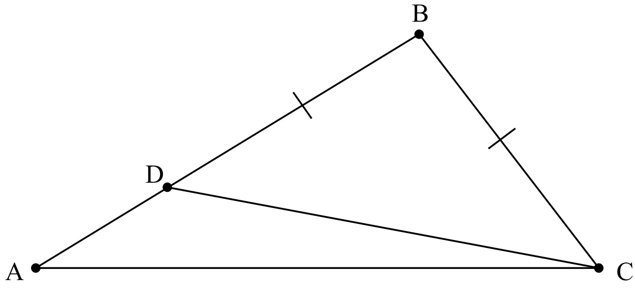

Figure 3.30: Figure for Theorem 13.10.

Thus the opposite sides are different in measure, and by the first sentence, the larger angle is opposite the longer side. ♢

Another approach to Theorem 13.10, using the triangle inequality, can be found in Exercise 3.

Proposition 13.11 *Suppose that $\triangle^s ABC$ has a right angle at C. Then side $\overset{\frown}{BC}$ is shorter than, congruent to, or longer than hypotenuse $\overset{\frown}{AB}$ if and only if $\overset{\frown}{BC}$ is acute, right, or obtuse, respectively. Furthermore, if $\overset{\frown}{BC}$ is not right, $m\ \overset{\frown}{AB}$ is between $m\ \overset{\frown}{BC}$ and $\pi - m\ \overset{\frown}{BC}$.*

Proof. We know from Proposition 11.12 (see Figure 3.19) that $\overset{\frown}{BC}$ is acute, right or obtuse if and only if $\prec BAC$ is acute, right or obtuse, respectively. By Theorem 13.10, Theorem 11.10 and §12, Exercise 1, this occurs if and only if the side opposite $\prec BAC$ is shorter than, congruent to, or longer than the side opposite $\prec ACB$, respectively — that is, if $\overset{\frown}{BC}$ is shorter than, congruent to, or longer than the hypotenuse $\overset{\frown}{AB}$.

Consider $\triangle^s A^a BC$. If $\overset{\frown}{BC}$ is acute, $\prec BAC$ is acute, so $\prec BA^aC$ is acute. So $m\ \overset{\frown}{A^aB} > m\ \overset{\frown}{BC}$, so $\pi - m\ \overset{\frown}{AB} > m\ \overset{\frown}{BC}$ and $m\ \overset{\frown}{AB} < \pi - m\ \overset{\frown}{BC}$. Since we already have $m\ \overset{\frown}{AB} > m\ \overset{\frown}{BC}$ from above, we conclude $m\ \overset{\frown}{BC} < m\ \overset{\frown}{AB} < \pi - m\ \overset{\frown}{BC}$. The case where $\overset{\frown}{BC}$ is obtuse is similar. ♢

It is worth making some observations about the order of logic in the proofs of the theorems in this section. The proof of Theorem 13.2 took considerable effort. Once this was done, the proofs of Theorems 13.3, 13.4, 13.7, 13.9, and 13.10 (as well as Exercise 9) follow rather quickly through the careful use of colunar and polar triangles. In fact, once we prove any of these, the others all follow fairly quickly in a similar manner. The real difficulty seems to be in proving one of them. In Exercise 27 the reader is encouraged to find a way to prove Theorem 13.3 as the first in the group (from which all the others would follow).

We conclude with the Hinge theorem on the sphere.

Proposition 13.12 (Spherical Hinge Theorem) *Suppose that* $\triangle^s ABC$ *and* $\triangle^s DEF$ *are spherical triangles with* $\overset{\frown}{AB} \cong \overset{\frown}{DE}$ *and* $\overset{\frown}{BC} \cong \overset{\frown}{EF}$*. Then* $m \prec ABC < m \prec DEF$ *if and only if* $m\ \overset{\frown}{AC} < m\ \overset{\frown}{DF}$*.*

Theorem 13.12 may be proven using the same method as used to prove the plane version, and we refer the reader to Exercise 28 for the argument.

Exercises §13

1. Prove that if spherical $\triangle^s ABC$ has a right angle at C, then $m \prec A$ and $m \prec B$ differ by no more than $\frac{\pi}{2}$ and their sum lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

2. Suppose that in $\triangle^s ABC$, $m\ \overset{\frown}{AB} < m\ \overset{\frown}{AC}$ and let D be the midpoint of $\overset{\frown}{BC}$. Prove that $m \prec ADB < m \prec ADC$.

3. Suppose that in a spherical triangle one angle is larger than another. By constructing inside the given triangle another triangle with two angles equal, prove that, of the two opposite sides, the larger angle is opposite the larger side.

4. Suppose that in $\triangle^s ABC$, D is chosen in the interior of the triangle. Prove that $m\ \overset{\frown}{DB} + m\ \overset{\frown}{DC} < m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC}$.

5. Suppose that D is chosen in the interior of $\triangle^s ABC$ so that $m \prec BAC + m \prec BDC < \pi$. Prove that $m\ \overset{\frown}{AC} > m\ \overset{\frown}{DC}$ and $m\ \overset{\frown}{AB} > m\ \overset{\frown}{DB}$.

6. Suppose that Γ is a great circle and X is not one of its poles. Let Y and Z be the feet of the shorter and longer perpendiculars, respectively, from X to Γ (see Definition 10.16). Using Proposition 13.11, prove that if W is any point of Γ other than Y and Z, $m\ \overset{\frown}{XY} < m\ \overset{\frown}{XW} < m\ \overset{\frown}{XZ}$. (Another approach to this problem is found in §17, Exercise 22.)

7. Let X, Y, Z and Γ be as in Exercise 6. If U and V are points of Γ other than Y and Z then: U is closer to Y than V if and only if $m\ \overset{\frown}{XU} < m\ \overset{\frown}{XV}$.

8. Suppose that a great circle meets a small circle at a point where the great circle is perpendicular to the radius of the small circle to the point of intersection. Prove that the great and small circles are tangent at the point of intersection. (That is, they meet at only one point.)

9. Prove that in any spherical triangle, the sum of the measures of two sides of a spherical triangle is (less than, equal to, greater than, respectively) π if and only if the sum of the measures of the opposite angles is (less than, equal to, greater than, respectively) π. Conclude that the measure of the exterior angle at one vertex of a triangle is greater than, equal to, or less than, the measure of the (interior) angle at a second vertex if the sum of the measure of the two sides not between the two given vertices

is, respectively, less than, equal to, or greater than π. (This is Book I, Proposition 10 of Menelaus' *Sphaerica*; it shows to what extent the planar version of the weak exterior angle theorem holds on the sphere.)

10. Let X, Y, Z, and Γ be as in Exercise 6. Let P be a pole of $\bigcirc XY$. If U and V are two points on Γ such that $m \overset{\frown}{YU} < m \overset{\frown}{YV} \leq m \overset{\frown}{YP} = \frac{\pi}{2}$, prove that $m \prec XUY > m \prec XVY \geq m \prec XPY = m \overset{\frown}{XY}$. Conclude that for W in Γ, $m \prec XWY$ is least when W is a pole of $\bigcirc XY$.

11. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} + m \overset{\frown}{AC} < \pi$ and $m \overset{\frown}{AB} \leq m \overset{\frown}{AC}$. Let D be a point between B and C. Prove that $m \overset{\frown}{AD} < m \overset{\frown}{AC}$.

12. Use Exercise 11 to conclude that the interior of a small circle is spherically convex. (Recall that a small circle by definition has radius less than $\frac{\pi}{2}$.) In fact, given two points on the small circle itself, or a point on the small circle and a point in the interior, then the points between them lie in the interior of the small circle.

13. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} + m \overset{\frown}{AC} < \pi$. Suppose D is between B and C and E is chosen on the same side of $\bigcirc BC$ as A so that $m \prec EDC = m \prec ABC$. If $m \overset{\frown}{AB} \leq \frac{\pi}{2}$, prove that $\overset{\hookrightarrow}{DE}$ meets $\overset{\frown}{AC}$ between A and C.

14. Suppose that in $\triangle^s ABC$, $a < \frac{\pi}{2}$, $b < \frac{\pi}{2}$, $a + c < \pi$ and $b + c < \pi$. Let D be a point in the interior of $\triangle^s ABC$. Prove that $m \prec ADB > m \prec ACB$. (See §16, Exercise 28 to see a partial converse of this statement.)

15. Suppose that in $\triangle^s ABC$, D lies on $\overset{\frown}{BC}$ so that $\overset{\hookrightarrow}{AD}$ bisects $\prec A$ (that is, $\prec BAD \cong \prec CAD$). Also assume $m \overset{\frown}{AB} + m \overset{\frown}{AC} < \pi$. If $m \overset{\frown}{AB} < m \overset{\frown}{AC}$, prove that $m \overset{\frown}{DB} < m \overset{\frown}{DC}$. What happens if $m \overset{\frown}{AB} + m \overset{\frown}{AC} \geq \pi$?

16. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} + m \overset{\frown}{AC} < \pi$. Let M be the midpoint of $\overset{\frown}{BC}$ and suppose $m \overset{\frown}{AB} < m \overset{\frown}{AC}$. Using Exercise 15, show that $m \prec BAM > m \prec CAM$.

17. Suppose that in $\triangle^s ABC$, $m \overset{\frown}{AB} < m \overset{\frown}{AC} < \frac{\pi}{2}$. Assume that the foot of the shorter perpendicular from A to $\bigcirc BC$ is not on $\overset{\frown}{BC}$. Let D be between B and C. Assume $m \prec BAD / m \prec CAD$ is a rational number. Prove that $m \prec BAD / m \prec CAD > m \overset{\frown}{BD} / m \overset{\frown}{CD}$.

18. Suppose that in $\triangle^s ABC$, D lies on $\overset{\frown}{BC}$ so that $\overset{\hookrightarrow}{AD}$ bisects $\prec A$ (that is, $\prec BAD \cong \prec CAD$). Also assume $m \overset{\frown}{AB} + m \overset{\frown}{AC} < \pi$. Let F be a point between A and D. If $m \overset{\frown}{AB} < m \overset{\frown}{AC}$, prove that $m \overset{\frown}{FB} < m \overset{\frown}{FC}$, $m \prec FCA < m \prec FBA$ and $m \prec FCB < m \prec FBC$.

19. Suppose that in $\triangle^s ABC$, G is the midpoint of $\overset{\frown}{BC}$. Also assume $m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC} < \pi$. Let H be a point between A and G. If $m\ \overset{\frown}{AB} < m\ \overset{\frown}{AC}$, prove that $m\ \overset{\frown}{HB} < m\ \overset{\frown}{HC}$, $m \prec HCA < m \prec HBA$ and $m \prec HCB < m \prec HBC$.

20. Prove Theorem 13.5.

21. Prove Corollary 13.6.

22. Prove that in any spherical right triangle, if two sides are acute then so is the third.

23. Let $\triangle^s A_iB_iC_i$ be two right triangles with right angles at C_i for $i = 1, 2$. Let the lengths of the sides be a_i, b_i and c_i. If $m \prec A_1B_1C_1 = m \prec A_2B_2C_2 < \frac{\pi}{2}$, and $c_1 < c_2$ prove that $a_1 < a_2$. If also $c_2 < \frac{\pi}{2}$, prove that $b_1 < b_2$.

24. Let $\triangle^s A_iB_iC_i$ be two right triangles with right angles at C_i for $i = 1, 2$. Let the lengths of the sides be a_i, b_i and c_i. Suppose that $c_1 = c_2$ and $m \prec A_1B_1C_1 < m \prec A_2B_2C_2 \leq \frac{\pi}{2}$. Prove that $b_1 < b_2$. If also $c_1 < \frac{\pi}{2}$ prove that $a_1 > a_2$.

25. Suppose that two sides of a spherical triangle are acute and the angle between them is not acute. Prove that the other two angles are acute.

26. In $\triangle^s ABC$, suppose that $C \geq \frac{\pi}{2}$, $a < \frac{\pi}{2}$ and $c < \frac{\pi}{2}$. Prove that $A < \frac{\pi}{2}$, $B < \frac{\pi}{2}$ and $b < \frac{\pi}{2}$.

27. Prove Theorem 13.3 as follows. Given an arbitrary spherical triangle $\triangle^s ABC$, let P be the center of its circumcircle. (See §12, Exercise 11.) Consider the three arcs $\overset{\frown}{PA}$, $\overset{\frown}{PB}$, and $\overset{\frown}{PC}$ and apply §12, Exercise 30.

28. Prove the Spherical Hinge Theorem as follows. First, we may arrange without loss of generality that $B = E$, $C = F$ and A is in the interior of $\prec DEF$. Construct the angle bisector of $\prec DEA$; it must meet $\overset{\frown}{DF}$ at some point G. Show that $\triangle^s DEG \cong \triangle^s AEG$, and then $m\ \overset{\frown}{AF} < m\ \overset{\frown}{FD}$.

29. Consider the following proposition: In $\triangle^s ABC$ and $\triangle^s DEF$, suppose that $\prec B \cong \prec E$ and $\prec C \cong \prec F$. Then $m\ \overset{\frown}{BC} < m\ \overset{\frown}{EF}$ if and only if $m \prec A < m \prec D$. Explain how this proposition is equivalent to the Hinge theorem on the sphere.

30. In $\triangle^s ABC$ and $\triangle^s DEF$, suppose that $\prec B \cong \prec E$, $\prec C \cong \prec F$, and $m \prec B < m \prec C < \pi - m \prec B$. If $m\ \overset{\frown}{BC} < m\ \overset{\frown}{EF}$ prove that $m\ \overset{\frown}{AB} < m\ \overset{\frown}{DE}$.

31. In $\triangle^s ABC$ and $\triangle^s DEF$, suppose that $\prec B \cong \prec E$, $\prec C \cong \prec F$, and $m\ \overset{\frown}{BC} < m\ \overset{\frown}{EF}$. Then prove that $m\ \overset{\frown}{AC} + m\ \overset{\frown}{DF}$ is equal to, less than, or greater than π, respectively, if $m\ \overset{\frown}{AB}$ is equal to, less than, or greater than $m\ \overset{\frown}{DE}$, respectively.

32. Given $\triangle^s ABC$ and $\triangle^s DEF$ so that $\overset{\frown}{AC} \cong \overset{\frown}{DF}$, $m \prec A > m \prec D$, $m \prec C < m \prec F$, and $m \prec B + m \prec E \geq \pi$. Prove that $m\ \overset{\frown}{BC} > m\ \overset{\frown}{EF}$ and $m\ \overset{\frown}{DE} > m\ \overset{\frown}{AB}$.

33. Given $\triangle^s ABC$ and $\triangle^s DEF$ so that $m\ \overset{\frown}{AB} < m\ \overset{\frown}{DE}$, $m\ \overset{\frown}{AC} > m\ \overset{\frown}{DF}$, $\prec A \cong \prec D$ and $m\ \overset{\frown}{BC} + m\ \overset{\frown}{EF} \leq \pi$. Prove that $m \prec B > m \prec E$ and $m \prec C < m \prec F$.

34. By §12 Exercise 11, a spherical triangle has a unique small circle passing through its three vertices called the *circumscribed circle* or *circumcircle.* Given a $\triangle^s ABC$ with circumcircle of center P, prove that (1) P is on the same (respectively, opposite) side of $\bigcirc BC$ as A if and only if (2) $m \prec A < m \prec B + m \prec C$ (respectively, $m \prec A > m \prec B + m \prec C$) if and only if (3) the median from A to $\overset{\frown}{BC}$ has measure greater (respectively, less) than $\frac{1}{2} m\ \overset{\frown}{BC}$. Conclude that the circumcenter is in the interior of the triangle if and only if the measure of any angle of the triangle is less than the sum of the measures of the other two angles. (Compare this problem to §12, Exercise 22.)

35. Prove that in a spherical triangle, the arc connecting the midpoints of two sides (a *midline*) has measure greater than half the measure of the third side of the triangle. (Hint. Suppose that in $\triangle^s ABC$, D, E are the midpoints of $\overset{\frown}{AB}, \overset{\frown}{AC}$, respectively. Extend $\overset{\frown}{DE}$ to F so $\overset{\frown}{DF}$ has twice the measure of $\overset{\frown}{DE}$. Then you must show $m\ \overset{\frown}{BC} < m\ \overset{\frown}{DF}$. Do this by applying the Spherical Hinge Theorem to $\triangle^s BDC$ and $\triangle^s FCD$.)

36. Suppose that in a spherical triangle the sum of the measures of two sides is less than π. Consider an arc from their common vertex to the opposite side. If this arc either bisects the angle or the opposite side, prove that the arc has measure less than $\frac{\pi}{2}$.

37. Extend Exercise 36 to prove that half the sum of the measures of the two given sides is greater than the measure of the arc constructed from the vertex to the opposite side.

38. Assume that Exercise 37 is true. Suppose then that in a triangle where the sum of the measures of two sides is less than π, the sides are unequal. We construct a segment from their common vertex to the opposite side whose measure is half the sum of the measures of those two sides. Prove

that both the vertex angle and base are divided into two unequal parts such that the larger part is adjacent to the shorter side.

39. Suppose that in $\triangle^s ABC$, $m\ \overset{\frown}{AB} < m\ \overset{\frown}{AC}$, $m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC} < \pi$, and M is the midpoint of $\overset{\frown}{BC}$. Suppose that D and E are chosen on $\overset{\frown}{BM}$ and $\overset{\frown}{MC}$, respectively, so that $\overset{\frown}{BD} \cong \overset{\frown}{EC}$. Prove that $m \prec BAD > m \prec EAC$ and $m\ \overset{\frown}{AD} + m\ \overset{\frown}{AE} < m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC}$. Hint. Exercise 36 shows that $m\ \overset{\frown}{AM} < \frac{\pi}{2}$. Choose H on $\overset{\hookrightarrow}{AM}$ so that $m\ \overset{\frown}{AM} = m\ \overset{\frown}{MH}$. Consider $\triangle^s ABH$ and $\triangle^s ADH$.

40. Suppose that in $\triangle^s ABC$, $m\ \overset{\frown}{AB} < m\ \overset{\frown}{AC}$, $m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC} < \pi$, and F is on $\overset{\frown}{BC}$ such that $\overset{\hookrightarrow}{AF}$ bisects $\prec BAC$. Suppose that D and E are chosen on $\overset{\frown}{BF}$ and $\overset{\frown}{FC}$, respectively, so that $\prec BAD \cong \prec EAC$. Prove that $m\ \overset{\frown}{BD} < m\ \overset{\frown}{EC}$ and $m\ \overset{\frown}{AD} + m\ \overset{\frown}{AE} < \pi$.

41. Suppose that a lune has vertices A and A^a, and that B is a point such that $\overset{\hookrightarrow}{AB}$ bisects the angle at A. Suppose that points C_i and D_i, $i = 1, 2$, are on the lune such that B is between C_i and D_i for $i = 1, 2$ and that $m \prec ABC_1 < m \prec ABC_2 \leq \frac{\pi}{2}$. Prove that $m\ \overset{\frown}{C_1D_1} > m\ \overset{\frown}{C_2D_2}$, and conclude that among all arcs with endpoints on the lune passing through B, the shortest is perpendicular to $\overset{\hookrightarrow}{AB}$.

42. Suppose that in $\triangle^s ABC$, $m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC} < \pi$, M is the midpoint of $\overset{\frown}{BC}$ and that D is between A and M. Prove that $m \prec BDC > m \prec BAC$.

43. Suppose that in $\triangle^s ABC$, $m\ \overset{\frown}{AB} + m\ \overset{\frown}{AC} < \pi$ and M is the midpoint of $\overset{\frown}{BC}$. Choose D and E on $\overset{\frown}{BM}$ and $\overset{\frown}{MC}$ so that $\overset{\frown}{BD} \cong \overset{\frown}{EC}$. Prove that $m \prec ADM + m \prec AEM > m \prec ABC + m \prec ACB$.

44. Suppose that in $\triangle^s ABC$, D and E are the midpoints of $\overset{\frown}{AB}$ and $\overset{\frown}{AC}$, respectively. Suppose that $\prec A$ is not acute. Prove that $m \prec ADE < m \prec ABC$ and $m \prec AED < m \prec ACB$.

45. Suppose that in $\triangle^s ABC$, D, E, and F are the midpoints of $\overset{\frown}{AB}, \overset{\frown}{BC}$, and $\overset{\frown}{AC}$, respectively. Suppose that $\prec A$ is not acute. Prove that both $m \prec BDE$ and $m \prec EFC$ are smaller than $m \prec A$.

46. Let δ be a real number between 0 and $\frac{\pi}{2}$. Suppose that a great circle has pole P and points A, B, and C are chosen at (the same) distance greater than $\frac{\pi}{2} - \delta$ from P. Suppose that the measures of angles $\prec APB$, $\prec BPC$, and $\prec APC$ are all $\frac{2\pi}{3}$. Prove that the perimeter of the spherical triangle $\triangle^s ABC$ is greater than or equal to $2\pi - 6\delta$.

47. Let δ, P, A, B, and C be as in Exercise 46. Show that if A, B, and C are far enough from P (i.e., close enough to the polar of P) then $m\prec A + m\prec B + m\prec C > 3\pi - 12\delta$. (Hint: let D and E be the points on $\overset{\hookrightarrow}{PA}$ and $\overset{\hookrightarrow}{PB}$ at distance a quarter circle from P. Let B be chosen so that $m\prec BDE < \delta$.)

48. Using Exercises 46 and 47 obtain a triangle $\triangle^s ABC$ such that both $m\,\overset{\frown}{AB} + m\,\overset{\frown}{AC} + m\,\overset{\frown}{BC} \geq 2\pi - 6\delta$ and $m\prec A + m\prec B + m\prec C > 3\pi - 12\delta$.

49. Using the triangle from Exercise 48, obtain a triangle $\triangle^s DEF$ whose spherical perimeter is less than 12δ and the sum of whose angles is less than or equal to $\pi + 6\delta$.

50. Is it possible for a spherical triangle to have the sum of its side measures close to 2π (360°) but have its angle sum close to π (180°)? If so, where could one place the vertices to produce such a triangle?

51. Is it possible for a spherical triangle to have the sum of its side measures close to 0 (0°) but have its angle sum close to 3π (540°)? If so, where could one place the vertices to produce such a triangle?

52. Let $A_1 A_2 \ldots A_n$ be a spherically convex n-gon. Prove that the sum of the measures of the angles of the n-gon is greater than $(n-2)\pi$.

53. Prove that the sum of the measures of the sides of a spherically convex n-gon is less than 2π. (Hint. Use the polar n-gon.)

54. (For readers familiar with calculus)

    (a) Assume that the length of a smooth curve $t \mapsto (x(t), y(t), z(t))$ in space is given by $\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$. For a curve on a sphere of radius r, show that in spherical coordinates (ϕ, θ) this formula is given by $\int_a^b r\sqrt{\phi'(t)^2 + \sin^2(\phi)\theta'(t)^2}\,dt$. (See Theorem 2.3.)

    (b) Assume that a curve $t \mapsto (r, \phi(t), \theta(t))$ for $a \leq t \leq b$ satisfies $\theta(b) = \theta(a)$ (i.e., the initial and terminal point of the curve lie on the same meridian in spherical coordinates). Show that the length of this curve is greater than or equal to the shortest great circle arc from the initial point to the terminal point.

55. (a) Prove the following theorem from solid geometry: (Euclid's Elements, Book XI, Proposition 20) Given four points O, A, B, C in space which do not all lie in a single plane, $m\angle AOB < m\angle AOC + m\angle BOC$.

    (b) Use part (a) to come up with another proof of Theorem 13.4.

56. (a) Use Exercise 55 and solid geometry to prove the following theorem: (Euclid's Elements, Book XI, Proposition 21) If O, A, B, C are four points in space which do not all lie in a single plane, then $m\angle AOB + m\angle AOC + m\angle BOC < 2\pi$.

    (b) Use part (a) to conclude another proof of Theorem 13.3.

Historical notes. The theorems and exercises of this section are featured extensively in the *Sphaerica* of Menelaus. Let X-Y denote Book X, Proposition Y. Then Theorem 13.4 is I-5, Exercise 4 is I-6, Theorem 13.10 is I-7 and I-9, Proposition 13.12 is I-8, Exercise 9 is I-10, Theorem 13.9 contains I-11, Exercises 29 and 31 are I-19, Exercise 32 is I-22, Exercise 34 contains I-23, Exercise 25 is I-24, Exercise 26 is I-25, Exercise 35 is I-26, Exercise 44 is I-27, Exercise 45 is I-28, Exercise 36 is I-32, Exercise 15 is I-33, Exercise 37 is I-34, Exercise 38 is I-35, Exercise 19 is I-36, Exercise 39 is I-37, Exercise 40 is I-38, Exercise 13 is II-3, and Exercise 11 is used in the proof of II-3 without justification.

14 Area

In §7 we justified the well-known classical formula for the surface area of a sphere and lune of radius r in space. Having done so we now accept these formulas as axioms of our axiomatic system. The main objective of this section is to obtain formulas for the areas enclosed by spherical triangles and polygons on the sphere.

Axiom 14.1 (A-9) *There exists an $r > 0$ such that the area of a sphere is $4\pi r^2$, the area of the region enclosed by a lune whose angle has radian measure θ is $2\theta r^2$, and the area of a spherical arc is 0.*

This may seem like a dramatic proposition to assume as an axiom, but because our system of axioms for spherical geometry does not presuppose a particular radius for the sphere, the exact value is not so important. By assuming that the area of a sphere is $4\pi r^2$ we are establishing the sphere's area as a unit by which all the others will be compared — in the same way that the area of a square is taken as a unit in the plane.

Axiom 14.2 (A-10) *If two triangles are antipodal, their interiors have the same area.*

Axiom 14.3 (A-11) *If two spherical regions have interiors which do not overlap then the area of the union of the regions is the sum of the areas of the regions.*

Because a spherical arc has zero area, the area of a triangle itself is zero, but the area of its interior is positive. Thus the area of the interior of a triangle is equal to the area of the union of the triangle with its interior. We can speak of the "area enclosed by a triangle" to refer to either of these, but it is also common to abuse language and refer to the "area of a triangle" when it is clear that we mean the area of the region enclosed by the triangle.

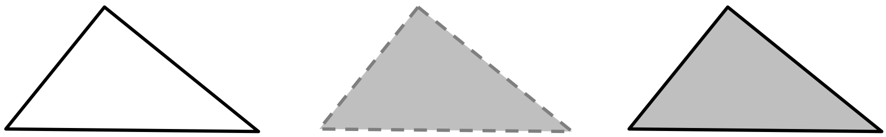

Figure 3.31: A triangle, its interior, and the union of the triangle and its interior.

Theorem 14.4 *The area enclosed by a spherical $\triangle^s ABC$ is given by*

$$area(\triangle^s ABC) = (m \prec A + m \prec B + m \prec C - \pi)r^2. \qquad (3.13)$$

This theorem might seem unusual, given that there is no theorem for the area of a plane triangle which depends only on the angles of the triangle. In the plane, two triangles can have the same angles without having the same area: they need only be similar. But after the results of the last section, the result should seem more natural: in spherical geometry, a triangle is uniquely determined (up to congruence) by its angles (by the AAA congruence theorem), so its area is also.

The history of the discovery of this theorem is complex. It is usually attributed to Albert Girard, who published a proof of it in 1629. However, his proof had logical weaknesses and is far more complex than the proof used today. Lagrange commented in 1800 that because of problems with the Girard proof, the theorem should instead be attributed to the Italian mathematician Bonaventura Cavalieri, who published a proof in 1632. The theorem was also apparently found but not published by Thomas Harriot in 1603. The proof given here is basically that given by Euler in 1781. (See [Ro1988], p. 31 and [Pa2014].)

Proof. Let A^a, B^a, and C^a be the antipodes of A, B, and C, respectively. The great circle $\bigcirc AB$ divides the sphere into two hemispheres; C is in one of these hemispheres and C^a is in the other. In this proof we refer informally to the hemisphere with C in it as the "top" hemisphere and the hemisphere with C^a in it as the "bottom" hemisphere. On $\bigcirc AB$, we have points A, B, A^a, B^a in that order. Then the top hemisphere can be written as the union of the regions bounded by four triangles: $\triangle^s ABC$, $\triangle^s BA^aC$, $\triangle^s A^aB^aC$, and $\triangle^s B^aAC$. (We neglect the contribution of the sides of the triangles themselves, which have area zero.) The union of the regions bounded by the first two of these ($\triangle^s ABC$ and $\triangle^s BA^aC$) is the lune ABA^aCA; its angle has measure $m \prec A$,

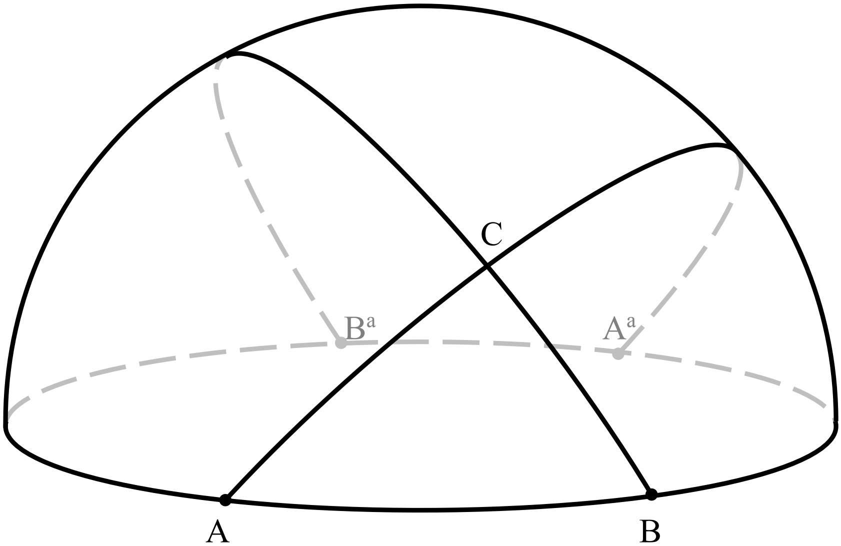

Figure 3.32: Theorem 14.4.

so it has area $2(m \prec A)r^2$. The union of the regions of the first and fourth triangles ($\triangle^s ABC$ and $\triangle^s B^a AC$) is the lune BAB^aCB; its angle has measure $m \prec B$, so it has area $2(m \prec B)r^2$. The lune CAC^aBC has angle whose measure is $m \prec C$ (so its area is $2(m \prec C)r^2$) and is the union of the regions of triangles $\triangle^s ABC$ and $\triangle^s ABC^a$. Having just written three lunes each as the union of two nonoverlapping triangular regions, we may write that the area of each lune is the sum of the areas of the two triangles of which it is the union:

$$2(m \prec A)r^2 = area(\triangle^s ABC) + area(\triangle^s BA^aC) \quad (3.14)$$
$$2(m \prec B)r^2 = area(\triangle^s ABC) + area(\triangle^s B^aAC) \quad (3.15)$$
$$2(m \prec C)r^2 = area(\triangle^s ABC) + area(\triangle^s ABC^a) \quad (3.16)$$

Note that in (3.16), $\triangle^s ABC^a$ is antipodal to $\triangle^s A^aB^aC$, hence has the same area, so we can replace (3.16) with

$$2(m \prec C)r^2 = area(\triangle^s ABC) + area(\triangle^s A^aB^aC). \quad (3.17)$$

Adding together (3.14),(3.15) and (3.17), we get

$$\begin{aligned} & (2m \prec A + 2m \prec B + 2m \prec C)r^2 \quad (3.18) \\ = \; & 3area(\triangle^s ABC) + area(\triangle^s BA^aC) \\ + \; & area(\triangle^s B^aAC) + area(\triangle^s A^aB^aC) \end{aligned}$$

The right side of (3.18) consists of the area of four triangular regions whose nonoverlapping union covers the top hemisphere. Since the area of the top hemisphere is $2\pi r^2$, we may write its area as the sum of the areas of the four triangles:

$$2\pi r^2 = area(\triangle^s ABC) + area(\triangle^s BA^aC) + area(\triangle^s B^aAC) + area(\triangle^s A^aB^aC).$$

Substituting this into (3.18), we obtain

$$(2m \prec A + 2m \prec B + 2m \prec C)r^2 = 2area(\triangle^s ABC) + 2\pi r^2$$

and solving for $area(\triangle^s ABC)$ we obtain (3.13) as desired. ♢

Definition 14.5 *In a spherical triangle $\triangle^s ABC$, the* spherical excess *is defined to be the quantity $2E$, where $E = (m \prec A + m \prec B + m \prec C - \pi)/2$.*

Later on we will prove a number of formulas involving E; these may be seen as new formulas for the area of a spherical triangle. In particular, we will prove Cagnoli's formula ((4.68) of §19 and Theorem 34.6 of §34), Euler's formula (Theorem 34.5 of §34), and Lhuilier's formula ((8.47) of §34). Also of note are the formulas found in §34, Exercises 12, 13, and 15.

Exercises §14.

1. On a sphere with $r = 10$ find the area of a spherical triangle whose angles have measures

   (a) $A = \frac{\pi}{2}$, $B = \frac{\pi}{3}$, $C = \frac{\pi}{3}$.
   (b) $A = \frac{2\pi}{3}$, $B = \frac{2\pi}{3}$, $C = \frac{\pi}{2}$.
   (c) $A = \frac{4\pi}{5}$, $B = \frac{3\pi}{5}$, $C = \frac{3\pi}{5}$.
   (d) $A = \frac{5\pi}{7}$, $B = \frac{4\pi}{7}$, $C = \frac{3\pi}{7}$.
   (e) $A = \frac{7\pi}{9}$, $B = \frac{5\pi}{6}$, $C = \frac{2\pi}{3}$.

2. On a sphere with $r = 5$ find the area of a spherical triangle whose angles have measures

   (a) $A = 80°$, $B = 70°$, $C = 60°$.
   (b) $A = 110°$, $B = 90°$, $C = 75°$.
   (c) $A = 120°$, $B = 110°$, $C = 100°$.
   (d) $A = 145°$, $B = 125°$, $C = 115°$.
   (e) $A = 175°$, $B = 170°$, $C = 170°$.

3. Assume the earth has a radius of 3963 miles. Find the area in square miles of a triangle on the earth with angles 115°, 100° and 95°.

4. Prove that the area of the region enclosed by a spherically convex n-gon is found by taking the sum of the radian measures of its angles minus $(n-2)\pi$ and multiplying that total by r^2. What happens if the polygon is not convex?

5. Suppose on a sphere with $r = 5$ a spherically convex quadrilateral has four angles of measure $100°$. Find the area of the quadrilateral.

6. Suppose on a sphere with $r = 10$ a spherically convex hexagon has six angles of measure $130°$. Find the area of the hexagon.

7. Given $\overset{\frown}{BC}$ and constant k, determine the set of all points A on the sphere such that A, B, and C form a spherical triangle such that $A+B+C = k$. (Hint: see §12, Exercise 24. The answer to this question is a result known as *Lexell's Theorem.*)