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Chapter 1

Review of three-dimensional geometry

The geometric objects being discussed in this book all sit in three-dimensional real Euclidean space, which we refer to throughout as *space*. We can study space with various tools. The "synthetic" tools include notions of points, lines, planes, rays, segments, angles, triangles, congruence, interior/exterior, and betweenness. "Metric" geometry adds notions of distance and angle measurement. (See [Mo1963] for a study in the differences in the synthetic and metric approaches to plane geometry.) If desired, one may also impose a Cartesian system of coordinates on space and make use of algebraic formulas. Sometimes usage of the notion of vector is helpful.

In each section of this chapter we review key tools of three-dimensional Euclidean geometry.

We will understand a *set* as being a collection of objects. In geometry, the objects in the set are typically "points." The notion of set is perhaps the most basic structure in geometry.

If X and Y are sets of points, then the set of points which are in both X and Y is the *intersection* of X and Y, denoted $X \cap Y$. The set consisting of all points which lie in either X or Y is the *union* of X and Y, denoted $X \cup Y$.

1 Geometry in a plane

The notions of point, line, and plane are central to the geometry of space. We here summarize key properties of them that we will need. We should understand that a line and a plane are sets of points.

Given two distinct points A and B in space, there exists a unique line passing through them; the line through A and B is denoted by $\overleftrightarrow{AB}$. Points which lie on a single line are said to be *collinear*. The *distance* between A and B is denoted by AB. There exists a coordinate labeling of the line with real numbers such that the coordinate of A is zero, the coordinate of B is positive, and the absolute value of the difference between the coordinates of two points on the line is the distance between the points. The *ray* $\overrightarrow{AB}$ is the set of all points on $\overleftrightarrow{AB}$ whose coordinate is greater than or equal to zero. The point A is called the *endpoint* or *vertex* of the ray. If A, B, and C are distinct points on a line, we say that B and C are on the *same side* of A if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the same ray. If A, B, and C are distinct collinear points, then we say that A is between B and C if $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not the same ray (and then we say that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are *opposite* rays).

The set of all points between A and B taken together with the points A and B is called the *segment* $\overline{AB}$ (also referred to as the segment between A and B). The *length* of $\overline{AB}$ is the distance between A and B. Two segments are said to be *congruent* if they have the same length.

A set of points is said to be *convex* if for every pair of points in the set, the points on the segment between them also belong to the set.

An *angle* is the union of two rays which have the same endpoint, but which are not part of the same line. The endpoint is called the *vertex* of the angle and the two rays without the vertex are the *sides* of the angle. To every angle is associated a *measure* between 0 and π radians (or between 0 and 180 degrees). In this book we shall generally use radian measure as a default, but will also make use of degree measure when appropriate with certain applications. An angle measure given without units should be assumed to be a radian measure. If an angle is the union of two rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ then we use the notation $\angle ABC$ to denote the angle formed by these two rays, and $m\angle ABC$ to denote its measure. Two angles are said to be *congruent* if they have the same measure. A *right* angle is an angle with measure $\frac{\pi}{2}$ radians (90 degrees). Two lines that intersect are said to be *perpendicular* if the angles formed at their point of intersection are right angles. An angle with measure less than that of a right angle is said to be *acute*. An angle with measure greater than that of a right angle is said to be *obtuse*. If the sum of the measures of two angles equals $\frac{\pi}{2}$ radians (90 degrees), the two angles are said to be *complementary*, and are *complements* of each other. If the sum of the measures of two angles equals π radians (180 degrees), the angles are said to be *supplementary*, and are *supplements* of each other. Two angles are said to be in a *linear pair* if they have the same vertex, have one side in common, and the other two sides are opposite rays. The angles in a linear pair are supplementary.

Suppose that one angle is formed by the union of two rays $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$, and another angle is formed by the union of rays $\overrightarrow{r_3}$ and $\overrightarrow{r_4}$. If $\overrightarrow{r_1}$ is opposite to $\overrightarrow{r_3}$ and $\overrightarrow{r_2}$ is opposite to $\overrightarrow{r_4}$, then we say that the angles are *vertical* angles.

A pair of vertical angles must be congruent.

Given a line lying in a plane, every point in the plane belongs either to the line or to one of two convex sets called a *half-plane.* If a point in one half-plane and another point in the other half-plane are connected with a line segment, this segment intersects the line. The line is said to be the *edge* of each half-plane. If two points in the plane are in the same half-plane associated with the line, they are said to be on the *same side* of the line. If two points in the plane are in different half-planes associated with the line, they are said to be on *opposite sides* of the line.

Suppose that line $\overleftrightarrow{AB}$ is the edge of a half-plane h. Then given a real number m between 0 and π radians (0 and 180 degrees) there exists a unique ray r in h with vertex at A such that the angle formed by r and $\overrightarrow{AB}$ has measure equal to m.

Suppose that B is between A and C in a given plane, and that D and E lie in the plane on the same side of $\overleftrightarrow{AB}$. Then $\angle ABD$ and $\angle ACE$ are said to be *corresponding* angles.

Given a point P and a line ℓ lying in a single plane, there exists a unique line m passing through P perpendicular to the given line. If m meets ℓ at the point P^ℓ, then we say that P^ℓ is the *projection* of P to ℓ.

Two distinct lines in a plane intersect in either a single point, or in no point. If two lines in a plane do not meet, we say that the lines are *parallel* lines. If line ℓ_1 is parallel to line ℓ_2, we write $\ell_1 \| \ell_2$. If two distinct lines lying in a plane are both perpendicular to a third line, the first two lines are parallel. With the corresponding angles of the previous paragraph, if $\prec ABD \cong \angle ACE$ then we have that $\overleftrightarrow{BD} \parallel \overleftrightarrow{CE}$.

The (Euclidean) *parallel postulate* states that given a line and a point not on the line, there exists a unique line through the given point parallel to the given line.

The parallel postulate has a long and important history in geometry. Euclid took a similar statement as a basic proposition not to be proven, but many mathematicians in the millennia after him thought it important (and non-obvious) enough that it ought to be proven from other more basic propositions in plane geometry. In the nineteenth century it turned out that the uniqueness assertion in the parallel postulate above cannot be proven from other more generally accepted postulates in plane geometry. The basic reason is that there are other two-dimensional surfaces with natural notions of “point” and “line” satisfying all the accepted postulates about points and lines in the plane, except for the parallel postulate. The sphere will turn out to be a surface without parallelism.

Suppose that A, B, and C are three points which do not lie on a single line. Then the union of the three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ constitutes the *triangle* $\triangle ABC$. Each of the three points is said to be a *vertex* (plural: *vertices*) of the triangle, the three segments are the *sides* of the triangle, and the angles $\angle ABC$, $\angle ACB$, and $\angle BAC$ are the *(interior) angles* of the triangle. If one of the angles of a triangle is in a linear pair with a second angle, the second angle is said to be an *exterior angle* of the triangle.

If one of the angles of the triangle is a right angle, then the triangle is said to be a *right triangle*. The side opposite the right angle is called the *hypotenuse* and the other sides are called *legs*. If two sides of a triangle are congruent then the triangle is *isosceles*. In an isosceles triangle, the angles opposite the congruent sides are congruent. Conversely, if two angles of a triangle are congruent, the opposite sides are congruent.

Two triangles are said to be *congruent* if their vertices can be put into a one-to-one correspondence such that corresponding sides and angles are congruent. There are several important propositions concerning how to show triangles are congruent. First, *SAS congruence* states that if two pairs of corresponding sides and the angle between them are congruent, the triangles are congruent. Next, *SSS congruence* states that if all corresponding sides are congruent, then the triangles are congruent. Lastly, if two pairs of corresponding angles and a side between are congruent, the triangles are congruent (*ASA congruence*). In fact, the triangles are congruent even if the side is not between the angles (*SAA congruence* or *AAS congruence*). In general, an "SSA correspondence" between does not guarantee congruence of the triangles, unless the angle happens to be a right angle. (The so-called "hypotenuse-leg theorem" states that if two right triangles have congruent hypotenuses and one pair of corresponding legs is congruent, then the triangles are congruent.) The "hypotenuse-angle theorem" states that if two right triangles have congruent hypotenuses and one pair of corresponding acute angles is congruent, then the triangles are congruent. (The hypotenuse-angle theorem is simply the SAA congruence theorem applied to a right triangle.)

The SAA congruence theorem can be viewed as a consequence of the theorem that the sum of the measures of the angles of a triangle is π radians (180 degrees). Suppose that there is an SAA correspondence between $\triangle ABC$ and $\triangle DEF$: $\angle ABC \cong \angle DEF$ and $\angle BAC \cong \angle EDF$. Then $\angle BCA \cong \angle EFD$ also because the sum of the measures of the angles in each triangle is the same. Thus we must also have an ASA correspondence between the triangles; hence they are congruent.

However, this proof of SAA congruence suffers somewhat from the fact that the angle sum of a triangle in the plane is $180°$. This latter theorem depends on the parallel postulate. It turns out that SAA congruence can be proven without the assumption of the parallel postulate. In the exercises the reader will see how to do this.

There are also a number of theorems concerning inequalities in triangles. In any triangle, if two sides (respectively, angles) are unequal in measure, then the opposite angles (respectively, sides) are also unequal in measure, and in the same order. The *triangle inequality* states that the sum of the lengths of two sides of a triangle must be larger than the length of the third side. An exterior angle of a triangle is larger in measure than either of the two opposite interior angles. A consequence of this is the fact that in a right triangle, the non-right angles are acute.

Proposition 1.1 (Hinge Theorem) *Suppose that $\triangle ABC$ and $\triangle DEF$ are planar triangles with $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$. Then $m\angle ABC < m\angle DEF$ if and only if $\overline{AC}$ is shorter than $\overline{DF}$.*

Proposition 1.2 *If P is a point, ℓ is a line, and Q is the foot of the perpendicular from P to ℓ, then the distance PQ is less than PR for any point R in ℓ different from P.*

Suppose that the points of two geometric objects are in a one-to-one correspondence. Then the objects are said to be *similar* if the lengths of corresponding segments are all in the same proportion, and the measures of all corresponding angles are the same. For triangles, we have the AAA similarity property: if corresponding angles are congruent, the triangles are similar. Furthermore, we have the SAS similarity property: if two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\angle B \cong \angle E$ and $AB/BC = DE/EF$, then the triangles are similar. The SAS similarity theorem depends on the parallel postulate.

The most important consequence of the similarity properties of planar triangles is:

Theorem 1.3 (Pythagorean theorem) *If a right triangle has sides of length a, b, and c, where c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.*

With the Pythagorean theorem in the plane, the hypotenuse-leg congruence theorem for right triangles is not difficult to deduce. (See Exercise 9.) However, the Pythagorean theorem depends on the parallel postulate, and it turns out that the hypotenuse-leg theorem can be proven without the assumption of the parallel postulate. This is also discussed in the exercises.

In any triangle, the sum of the measures of the angles is π radians ($180°$). An immediate consequence is that the measure of an exterior angle is equal to the sum of the measures of the opposite interior angles.

Suppose that $A_1, A_2, \ldots, A_n$ is a set of n points in a plane such that no three consecutive points on the list are collinear. Consider the segments $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, $\ldots$, $\overline{A_{n-1}A_n}$, and $\overline{A_nA_1}$. If no two of them intersect anywhere but at their endpoints, their union is a *polygon* of *n sides*. A *quadrilateral* is a polygon of four sides and a *pentagon* is a polygon of five sides. A *parallelogram* is a quadrilateral where the non-intersecting (opposite) sides are parallel. If the opposite sides in a parallelogram are congruent, the parallelogram is called a *rhombus*. A rectangle is a quadrilateral such that the angle between any two sides with a common endpoint is a right angle. A *square* is a rectangle whose sides all have the same length. A *trapezoid* is a quadrilateral where one pair of two opposite sides is parallel (the *bases*) and the other pair of opposite sides is not parallel. The distance between the parallel sides is the *height* of the trapezoid. A trapezoid is said to be *isosceles* if the non-parallel sides are congruent. Given a trapezoid, there exists a line parallel to both bases halfway between the bases. This line intersects the non-parallel sides in one point each. The line segment between these two points is called the *midline* or *median* of the trapezoid. The length of the midline is the arithmetic mean (half of the sum) of the length of the bases.

Let O be a point in a plane and r a positive real number. The *circle* with *center* O and *radius* r is the set of all points in the plane at distance r from O.

We shall need the following lemma when we discuss areas in §7.

Lemma 1.4 *Suppose ℓ is a line in a plane, s is a line segment, and M is a point on s. Suppose that the line perpendicular to s at M intersects ℓ in a point N. Let a_1 be the length of s, b_1 the distance from M to ℓ, a_2 the length of the projection of s to ℓ, and b_2 the length of the segment from M to N. Then $a_1 b_1 = a_2 b_2$.*

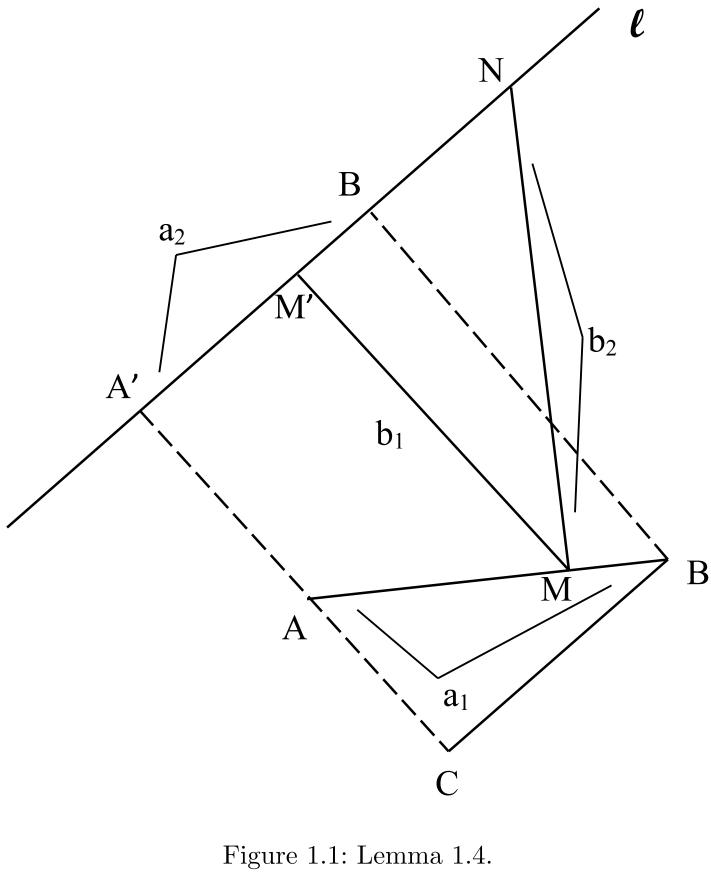

Figure 1.1: Lemma 1.4.

Proof. Because the point N of intersection exists, the line containing s cannot be perpendicular to ℓ. If s is parallel to ℓ, then $a_1 = a_2$ and $b_1 = b_2$ and so $a_1 b_1 = a_2 b_2$ by multiplying these equations. If s is not parallel to ℓ, then let A be the endpoint of s which is closer to ℓ and let B be the other endpoint. Let A', B', and M' be the projections of A, B, and M to ℓ and let C be the projection of B to $\overleftrightarrow{AA'}$. Since s is not parallel to ℓ, the points A, B, and C are not collinear; hence, they form a triangle. Similarly, M, M', and N form a triangle. We claim that $\triangle ABC \sim \triangle NMM'$. To see this, note that the triangles have a right angle at C and M'. We have that $\angle M'MN$ is complementary to $\angle M'MA$ since $\overleftrightarrow{MN}$ is perpendicular to $\overleftrightarrow{AB}$ at M. Also, $\angle M'MA$ is congruent to $\angle B'BA$ as corresponding angles of the parallel lines $\overleftrightarrow{BB'}$ and $\overleftrightarrow{MM'}$ (these lines being parallel since both are perpendicular to ℓ). Since C is the projection of B to $\overleftrightarrow{AA'}$, and A' and B' are the projections of A and B to ℓ, respectively, the quadrilateral $CA'B'B$ is a rectangle. Thus $\angle CBB'$ is right, and so $\angle B'BA$ and $\angle CBA$ are complementary. Summing up what we have found, $m\angle M'MN = \frac{\pi}{2} - m\angle M'MA = \frac{\pi}{2} - m\angle B'BA = m\angle CBA$. Thus in $\triangle ABC$ and $\triangle NMM'$, two pairs of corresponding angles are congruent. We conclude that these triangles are similar. Since the ratios of the lengths of corresponding sides are the same, $AB/MN = BC/MM'$, i.e., $a_1/b_2 = a_2/b_1$, or $a_1 b_1 = a_2 b_2$, as desired. $\diamondsuit$

The reader will find another approach to Lemma 1.4 in §4, Exercise 1.

The area enclosed by a rectangle is equal to the base times the height. The area enclosed by a triangle is half of the base times the height. For a circle of radius r, the area enclosed is πr^2.

Exercises §1

1. Prove: If $\angle ABC$ is acute (respectively, obtuse) then the projection of A to $\overleftrightarrow{BC}$ is on the same (respectively, opposite) side of B as C.

2. Let plane angle $\angle RST$ be an angle with vertex S. Let R' be chosen so that the ray $\overrightarrow{SR'}$ is perpendicular to the ray $\overrightarrow{ST}$ and R and R' are on the same side of the line $\overleftrightarrow{ST}$. Similarly suppose that T' is chosen so that $\overrightarrow{ST'}$ is perpendicular to $\overrightarrow{SR}$ and T and T' lie on the same side of line $\overleftrightarrow{RS}$. Prove that $\angle R'ST'$ and $\angle RST$ are supplementary angles (i.e., the sum of their measures is π).

3. In a parallelogram, show that the opposite sides are congruent and the diagonals bisect each other. In a rhombus, show that the diagonals also are perpendicular and bisect the angles.

4. In a trapezoid, let the height be h and let the length of the bases be b_1 and b_2. By using a formula for the area of a triangle, argue that the area of the trapezoid is $\frac{h}{2}(b_1 + b_2)$.

5. A *circular sector* is a region bounded by an arc of a circle and two radii of that circle. If the arc has radian measure θ and the circle has radius r, argue that the area is $\frac{1}{2}r^2\theta$.

6. An *annulus* is a region bounded by two circles with the same center but different radii. An *annular sector* is the portion of an annulus between two radii of the outer circle of the annulus. If the inner and outer arcs on the annulus have length s_1 and s_2, respectively, and the difference of the radii is ℓ, show that the area of the annular sector is $\frac{\ell}{2}(s_1 + s_2)$.

7. Prove Proposition 1.2.

8. In this exercise we indicate another proof of the SAA congruence theorem. Suppose that $\triangle ABC$ and $\triangle DEF$ satisfy $\overline{AB} \cong \overline{DE}$, $\angle C \cong \angle F$, and $\angle B \cong \angle E$. Without using the fact that the sum of the measures of the angles in the triangle is π (i.e., without using the parallel postulate), prove that $\triangle ABC \cong \triangle DEF$. To do this, note that if $BC < EF$ then there exists a point G in $\overline{EF}$ such that $\overline{BC} \cong \overline{EG}$. Proceed to a contradiction via the exterior angle theorem.

9. Explain how the hypotenuse-leg theorem follows quickly from two facts: (1) SSS congruence and (2) the Pythagorean theorem.

10. Explain how to prove the hypotenuse-leg theorem by using the exterior angle theorem.

2 Geometry in space

The following propositions are basic to our understanding of the way points, lines, and planes interact in space.

Proposition 2.1 *If a line contains two distinct points of a plane, the line lies entirely in the plane.*

Proposition 2.2 *Three noncollinear points determine a unique plane.*

Proposition 2.3 *A line and a point not on the line determine a unique plane which contains both the line and the point.*

Proposition 2.4 *If two distinct planes intersect, their intersection must be a line.*

If two lines in space do not intersect, then either they lie in the same plane (in which case they are parallel) or they do not lie in the same plane (in which case we say they are *skew*). A set of points which lies in some plane is said to be *planar*.

Proposition 2.5 *Given a plane in space, every point in space not on the plane belongs to one of two disjoint convex sets. Each of these two sets is called a* half-space. *If a point in one half-space and another point in the other half-space are connected with a line segment, this segment intersects the plane.*

If two points are chosen from the same half-space, then we say that the two points are on the *same side* of the plane. If two points are chosen such that one lies in one half-space and the other lies in the other half-space, we say that the two points lie on *opposite sides* of the plane.

Definition 2.6 *We say that a line ℓ is perpendicular to a plane p if there exists a point Q such that ℓ intersects p at Q and ℓ is perpendicular to all lines in p which pass through Q.*

Proposition 2.7 *Given a point P and a plane p in space, there exists a unique line ℓ through P perpendicular to p.*

In Proposition 2.7, the (unique) point P^p where the line meets the plane is called the *foot of the perpendicular* from P to p (or the *projection* of P to p) and PP^p is called the *distance* from the point P to the plane p. If $\overline{PQ}$ is a segment then the *projection* of $\overline{PQ}$ to p is the segment $\overline{P^pQ^p}$.

The following proposition is valuable in proving that a line is perpendicular to a plane.

Proposition 2.8 *If a line is perpendicular to each of two distinct intersecting lines where the latter two lines intersect, then the first line is perpendicular to the plane containing the other two lines.*

Proposition 2.9 *If line ℓ is perpendicular to plane p at Q, then any line containing the point Q which is perpendicular to ℓ must lie in p.*

Theorem 2.10 *If P is a point, p a plane, and P^p is the foot of the perpendicular from P to p, then the distance PP^p is less than PR for any point R in p different from P.*

Proof. Since PP^p is perpendicular to p, it is perpendicular to any line in plane p which passes through P^p. In particular, $\overleftrightarrow{PP^p}$ is perpendicular to $\overleftrightarrow{P^pR}$. Thus $\triangle PP^pR$ is a right triangle with a right angle at P^p. The angle opposite $\overline{PP^p}$ is acute. The angle opposite $\overline{PR}$ is a right angle. Since the longer side is opposite the larger angle, $PP^p < PR$. $\diamondsuit$

We have similar theorems and definitions concerning a point P and a line ℓ in space.

Proposition 2.11 *Given a point P and a line ℓ in space, there exists a unique plane p through P perpendicular to ℓ.*

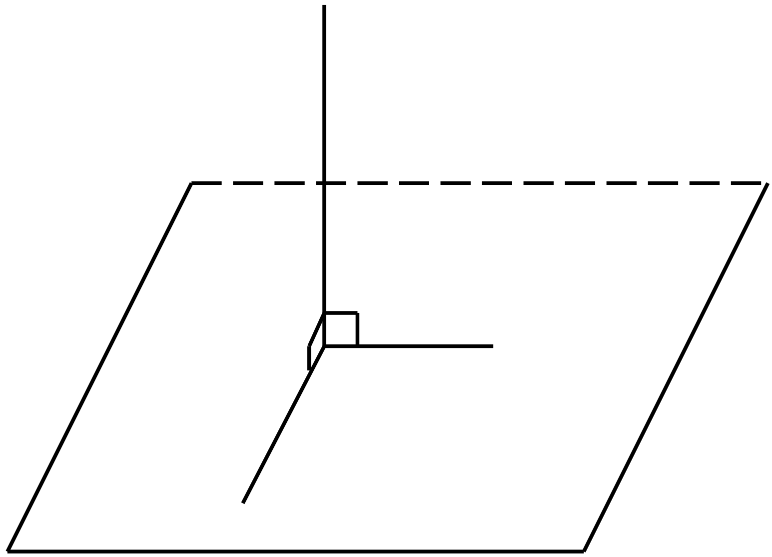

Figure 1.2: Propositions 2.8 and 2.9.

In Proposition 2.11, the point P^ℓ where the line meets the plane is called the *foot of the perpendicular* from P to ℓ (or the *projection* of P to ℓ) and PP^ℓ is called the *distance* from the point P to the line ℓ. If $\overline{PQ}$ is a segment, then its *projection* to ℓ is the segment $\overline{P^\ell Q^\ell}$.

We review the notion of dihedral angle in space. Recall that if a line lies in a plane, then the points of the plane not on the line consist of two pieces called *half-planes*.

Definition 2.12 *Suppose two half-planes in space have the same edge but are not in the same plane. Then the union of two such half-planes along with their common edge is called a* dihedral angle*; the two half-planes are called the* sides *of the dihedral angle and the line of intersection is the* edge *of the dihedral angle. If A is a point in one side of the dihedral angle, D lies in the other side and the edge is the line $\overleftrightarrow{BC}$, then the dihedral angle is denoted by $\angle A - BC - D$. The* interior *of $\angle A - BC - D$ consists of the set of all X such that X and A are on the same side of plane BCD and X and D are on the same side of plane ABC.*

Definition 2.13 *A* plane angle *of a dihedral angle is formed as follows: a point V in the edge of the dihedral angle is the vertex of the plane angle. Then in each side of the dihedral angle there exists a ray with vertex V perpendicular to the edge of the dihedral angle. The union of these two rays is the plane angle with vertex V.*

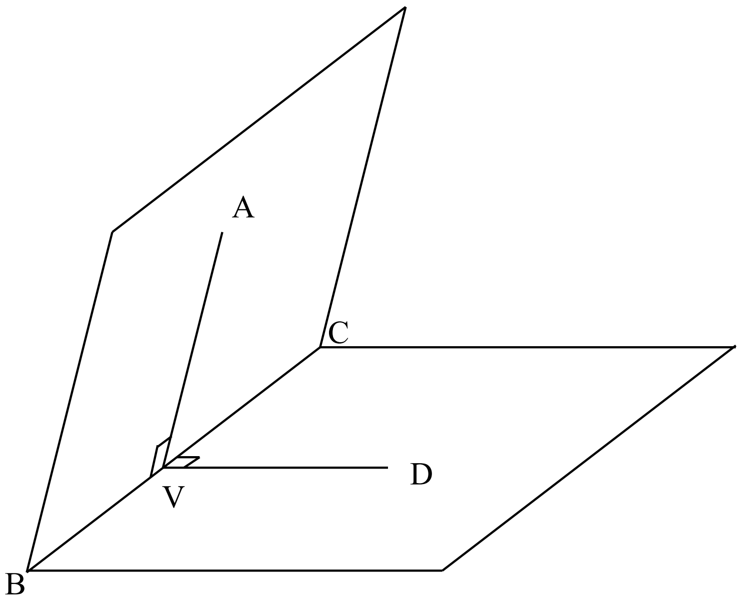

Figure 1.3: A dihedral angle $\angle A - BC - D$ and its plane angle $\angle AVD$.

Proposition 2.14 *An angle is a plane angle of a dihedral angle if and only if it is the intersection of the dihedral angle with a plane perpendicular to the edge of the dihedral angle.*

Proposition 2.15 *The plane angles of a dihedral angle are all congruent.*

Proposition 2.15 can be proven by observing that any two plane angles have corresponding sides which are parallel.

Definition 2.16 *The* measure *of a dihedral angle is given by the measure of any of its plane angles. We say that two planes are* perpendicular *if a dihedral angle formed between them is a right angle. Two dihedral angles are* congruent *if they have the same measure.*

Theorem 2.17 *If planes p and q are perpendicular, then the projection of any point of p to q must also lie in p.*

Theorem 2.18 *Suppose that a line is perpendicular to one plane and is contained in a second plane. Then the planes are perpendicular.*

Theorem 2.19 *Suppose that two intersecting planes are each perpendicular to a third plane. Then the third plane is perpendicular to the intersection of the first two planes.*

Exercises §2

1. If two distinct lines are perpendicular to the same plane, then the lines must be parallel.

2. Suppose that h is a half-plane whose edge lies in a plane p. If two dihedral angles are thus formed which are congruent, prove that h is perpendicular to p.

3. Suppose that planes p_1 and p_2 are orthogonal. Let P be a point in the intersection of p_1 and p_2 and let r_1 and r_2 be perpendicular rays based at P such that r_1 and r_2 lie in p_1 and p_2, respectively. Assume neither r_1 nor r_2 lies in the line of intersection of p_1 and p_2. Let p_3 be the plane containing r_1 and r_2. Prove that p_3 is perpendicular to either p_1 or p_2.

4. Suppose that P is a point in the interior of a dihedral angle and let O be a point in the edge of the angle. Prove that every point of $\overrightarrow{OP}$ is in the interior of the dihedral angle.

3 Plane trigonometry

Plane right triangle trigonometry is heavily based on the following fact: if $\triangle ABC$ and $\triangle DEF$ are plane right triangles with right angles at B and E, respectively, and $m\angle CAB = m\angle FDE$, then $AB/AC = DE/DF$, $BC/AC = EF/DF$, and $BC/AB = EF/DE$. (Verification of this via similar triangles is left to Exercise 1.) Then, given an angle measure θ we may define the cosine, sine, and tangent of angle θ as $\cos(\theta) = XY/XZ$, $\sin(\theta) = YZ/XZ$, and $\tan(\theta) = YZ/XY$, where $\triangle XYZ$ is some triangle with a right angle at Y and $m\angle ZXY = \theta$. Then Exercise 1 shows that these side ratios do not depend on the right triangle $\triangle XYZ$ chosen. Hence a mathematician would say that the trigonometric functions sin, cos, and tan are "well-defined."

A slightly different approach is needed when trigonometric functions of non-acute angles are to be defined. We assume the plane has an xy coordinate system with origin O. Let θ be an angle measure. Let r be any positive real number. We rotate the x-axis counterclockwise about the origin through an angle measure of θ and suppose that the point $(r, 0)$ rotates to a point with coordinates (x, y). Then $\cos(\theta)$ and $\sin(\theta)$ are defined to be the values x/r and y/r, respectively. It must be checked that this definition does not depend on the value of r. If θ is negative, we would rotate the x-axis clockwise through an angle measure equal to the absolute value of θ. Note that because the legs of a right triangle are always shorter than the hypotenuse, we must have for any angle θ, $-1 \leq \sin(\theta) \leq 1$ and $-1 \leq \cos(\theta) \leq 1$.

We then may define $\tan(\theta) = \sin(\theta)/\cos(\theta)$, $\cot(\theta) = \cos(\theta)/\sin(\theta)$, $\sec(\theta) = 1/\cos(\theta)$, and $\csc(\theta) = 1/\sin(\theta)$, except for a value of θ where a denominator is zero.

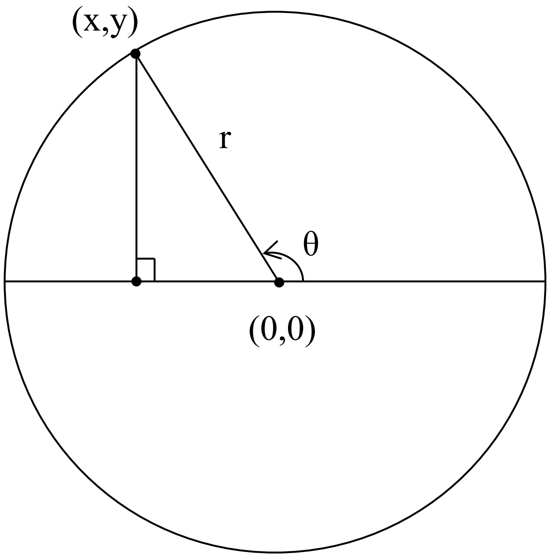

Figure 1.4: The sine and cosine: $\cos(\theta) = x/r, \sin(\theta) = y/r$.

Suppose that through a rotation of angle θ, the point $(r, 0)$ rotates to (x, y). Then a rotation through the supplementary angle $\pi - \theta$ moves $(r, 0)$ to $(-x, y)$. This leads to the identities:

$$\sin(\pi - \theta) = \sin(\theta) \tag{1.1}$$
$$\cos(\pi - \theta) = -\cos(\theta) \tag{1.2}$$

Similarly, if through a rotation of angle θ, the point $(r, 0)$ moves to (x, y), then a rotation through the complementary angle $\frac{\pi}{2} - \theta$ moves $(r, 0)$ to (y, x). This leads to the identities:

$$\sin(\frac{\pi}{2} - \theta) = \cos(\theta) \tag{1.3}$$
$$\cos(\frac{\pi}{2} - \theta) = \sin(\theta) \tag{1.4}$$

Similar arguments lead to

$$\sin(-\theta) = -\sin(\theta) \tag{1.5}$$
$$\cos(-\theta) = \cos(\theta) \tag{1.6}$$

Division of (1.5) and (1.6) leads to

$$\tan(-\theta) = -\tan(\theta) \tag{1.7}$$
$$\cot(-\theta) = -\cot(\theta). \tag{1.8}$$

The definitions of the sine and cosine together with the Pythagorean theorem lead immediately to the identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1 \tag{1.9}$$

We recall the sum and difference formulas for the sine and cosine

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y) \tag{1.10}$$

$$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y) \tag{1.11}$$

from which follow the double angle formulas

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \tag{1.12}$$

$$\sin(2x) = 2\sin(x)\cos(x) \tag{1.13}$$

and the half-angle formulas

$$\begin{aligned} \sin^2(\frac{x}{2}) &= \frac{1-\cos(x)}{2} \\ \cos^2(\frac{x}{2}) &= \frac{1+\cos(x)}{2} \\ \tan^2(\frac{x}{2}) &= \frac{1-\cos(x)}{1+\cos(x)}. \end{aligned} \tag{1.14}$$

If we add together the sum and difference formulas for the cosine we obtain $\cos(x+y)+\cos(x-y) = 2\cos(x)\cos(y)$. The variable substitutions $A = x+y$ and $B = x - y$ lead to the "sum-to-product" formula $\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$. Similar methods allow us to conclude a whole set of identities called the sum-to-product formulas:

$$\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2}) \tag{1.15}$$

$$\cos(A) - \cos(B) = 2\sin(\frac{A+B}{2})\sin(\frac{B-A}{2}) \tag{1.16}$$

$$\sin(A) + \sin(B) = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2}) \tag{1.17}$$

$$\sin(A) - \sin(B) = 2\cos(\frac{A+B}{2})\sin(\frac{A-B}{2}) \tag{1.18}$$

Let $\triangle ABC$ be a planar triangle with vertices at A, B, and C. We also use the letters A, B, and C to denote the measures of the angles at the vertices A, B, and C, respectively. We let a, b, and c denote the lengths of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$. Then we recall the planar law of sines

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} \tag{1.19}$$

and the planar law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos(C). \tag{1.20}$$

Given a real number L, we are occasionally confronted with the need to solve an equation $\cos(x) = L$ for x. Such an equation will have solutions if and only if $-1 \leq L \leq 1$. However, even then the solution is not unique unless we make an artificial restriction on what values are allowed for x. It is customary to demand that $0 \leq x \leq \pi$, in which there is a unique solution, and we write $x = \cos^{-1}(L)$ (or $x = \arccos(L)$). In the context of triangle trigonometry, this is pleasant if the x desired is the measure of an angle of a triangle because the measure of such an angle must be between 0 and π. Similarly, given any real number L, the equation $\cot(x) = L$ has a solution which is unique if we demand that $0 < x < \pi$. (We write $x = \cot^{-1}(L) = arccot(L)$.)

The situation is not quite so pleasant for the sine and tangent functions. If we need to solve $\sin(x) = L$ for x then there is a unique solution if we demand $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and we write $y = \sin^{-1}(L) = \arcsin(L)$.) Unfortunately there is usually not a unique solution for $0 \leq x \leq \pi$ because of (1.1): any acute angle whose measure x satisfies $\sin(x) = L$ has a corresponding supplement which is also a solution.

If $\tan(x) = L$, then this can always be solved for x (regardless of the value of L) and we write $x = \tan^{-1}(L) = \arctan(L)$ for $-\frac{\pi}{2} < x < -\frac{\pi}{2}$. It is not generally customary to seek x so that $0 < x < \pi$ since the tangent is not defined at $x = \frac{\pi}{2}$ but in particular cases this may be appropriate.

We recall from plane geometry that in a right triangle $\triangle ABC$ where $A = \frac{\pi}{6}$, $B = \frac{\pi}{2}$, $C = \frac{\pi}{3}$ and the hypotenuse has length 1, then $AB = \frac{\sqrt{3}}{2}$ and $BC = \frac{1}{2}$. This allows us to calculate $\cos(\frac{\pi}{6}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{\pi}{3}) = \sin(\frac{\pi}{6}) = \frac{1}{2}$.

In this book it will be useful to have the values of trigonometric functions for angles which are multiples of $\frac{\pi}{10} = 18°$. We derive some of these as follows. Note that if $x = \frac{\pi}{10}$, $2x + 3x = 5x = \frac{\pi}{2}$. Using (1.3), we get $\sin(2x) = \cos(3x)$, so using (1.13),(1.11), and (1.12) we obtain

$$\begin{aligned} 2\sin(x)\cos(x) &= \cos(2x)\cos(x) - \sin(2x)\sin(x) \\ 2\sin(x)\cos(x) &= (1 - 2\sin^2(x))\cos(x) - 2\sin^2(x)\cos(x), \end{aligned}$$

so gathering on the left and factoring we have $(4\sin^2(x)+2\sin(x)-1)\cos(x) = 0$. Now $\cos(x) \neq 0$ because x is strictly between 0 and $\frac{\pi}{2}$. Thus $4\sin^2(x) + 2\sin(x) - 1 = 0$. Solving for $\sin(x)$ by using the quadratic formula, $\sin(x) = (-2 \pm \sqrt{4 - 4(4)(-1)})/8 = (-2 \pm 2\sqrt{5})/8 = (-1 \pm \sqrt{5})/4$. Since $0 < x < \frac{\pi}{2}$, $\sin(x) > 0$, so $\sin(x) = \sin(\frac{\pi}{10}) = \sin(18°) = (-1 + \sqrt{5})/4$. It is left to the exercises to show that $\cos(\frac{\pi}{5}) = \cos(36°) = (1 + \sqrt{5})/4$. We also there present a different geometric approach to calculating these values.

Let us recall the customary approaches to "solving triangles" in the plane: that is, given some of the measures of sides and angles in a triangle, determine the measures of the remaining sides and angles. We suppose that the three angles have measures A, B, and C, and the opposite sides have measures a, b, and c, respectively.

Given the measures of all three sides of the triangle, the use of the planar law of cosines (1.20) delivers the value of $\cos(C)$, and hence that of C. Permutations of the values of a, b, c, A, B, C in (1.20) allows for the determination of A and B. A unique solution exists provided we have been given positive values of a, b, and c such that $a+b > c$, $a+c > b$, and $b+c > a$, as is required for any plane triangle.

Given the measures of two sides and the included angle (say a, b, and C) the law of cosines (1.20) delivers a unique value of c. Once the third side is found, the other two angles are found by the process in the previous paragraph. The only restrictions needed for a unique solution are that a, b be positive and $0 < C < \pi$.

Given the measures of two angles and the side between them, the value of the third angle is found from the fact that $A + B + C = \pi$. The planar law of sines (1.19) then delivers the values of the remaining two sides. The only restriction required for a unique solution is that the sum of the measures of the two given angles must be less than π.

The last scenario, where we are given the measures of two sides and an angle opposite one of them (say a, b, and A), is the most complex. This is the only case where we may have more than one solution (at most two) and the only case where, to most observers, the existence of a solution is not determined by a quick glance at the numbers given. Hence this scenario is dubbed the "ambiguous case." We here lay out a procedure by which one may find all solutions to a triangle in the ambiguous case.

Algorithm for solving the ambiguous case SSA.

Suppose that the known sides of a plane triangle are a and b and that the known angle is A.

(a) Determine all possible values for B: $\sin(B) = b\sin(A)/a$.

(b) If $\sin(B)$ is found to be larger than 1, there are no solutions.

(c) If $\sin(B)$ is found to be less than or equal to 1, then there are one (if $\sin(B) = 1$) or two (if $\sin(B) \leq 1$) possible values for B. We discard any value for B found where $A + B \geq \pi$.

(d) For each value of B emerging from part (c), we obtain one solution for the triangle, and find $C = \pi - A - B$ and $c = a\sin(C)/\sin(A)$.

Theorem 3.1 *Given real numbers $a, b > 0$ and A such that $0 < A < \pi$, the set of all possible solution triangles is the set of all triangles emerging from the above algorithm.*

Proof. First we prove that a known solution triangle with elements a, b, c, A, B, and C must emerge from this algorithm. Since the triangle satisfies the planar law of sines, B must satisfy the equation in part (a), where $\sin(B) \leq 1$ (in part (c)). Since the sum of the measures of the angles $A + B + C < \pi$, we must have $A + B < \pi$ in part (c). Furthermore, in part (d), C must equal

$\pi - A - B$ and by the law of sines, $c = a\sin(C)/\sin(A)$. So a known solution triangle must emerge from the algorithm.

Conversely, suppose that we are given some a, b, c, A, B, and C which emerge from the algorithm. We claim there exists a triangle having these values for the measures of its sides and angles. Since A, B, and c are all positive with $A + B < \pi$, there must be a unique triangle $\triangle ABC$ with elements A, B, and c, but where the other elements are possibly different from a, b, and C: we call them $\tilde{a}$, $\tilde{b}$, and $\tilde{C}$. From the algorithm part (d), we know that $A + B + C = \pi$; we also know that in $\triangle ABC$, $A + B + \tilde{C} = \pi$. Thus $C = \tilde{C}$. From the algorithm part (d), $c = a\sin(C)/\sin(A)$. Applying the law of sines to $\triangle ABC$, $c = \tilde{a}\sin(\tilde{C})/\sin(A)$, which we now know equals $\tilde{a}\sin(C)/\sin(A)$ since $C = \tilde{C}$. Thus $a = \tilde{a}$. From the algorithm part (a), $\sin(B) = b\sin(A)/a$. Applying the law of sines to $\triangle ABC$, $\sin(B) = \tilde{b}\sin(A)/\tilde{a}$, which we now know is $\tilde{b}\sin(A)/a$, since $a = \tilde{a}$. But then $b\sin(A)/a = \tilde{b}\sin(A)/a$, so $b = \tilde{b}$, as desired. So the elements a, b, c, A, B, and C which emerge from the algorithm form a triangle with these side and angle measures. $\diamondsuit$

Key facts from the study of areas: given two sides of a triangle and an included angle C, the area is $\frac{1}{2}ab\sin(C)$. A parallelogram with two sides of length a and b, and angle C between has area $ab\sin(C)$.

Exercises §3

1. Show (using similar triangles) that if $\triangle ABC$ and $\triangle DEF$ are plane right triangles with right angles at B and E, respectively, and $m\angle CAB = m\angle FDE$, then $AB/AC = DE/DF$, $BC/AC = EF/DF$, and $BC/AB = EF/DE$. Conclude that the notions of $\sin(\theta)$ and $\cos(\theta)$ for general angle θ are well-defined. (That is, the value of r used in the definition does not matter.)

2. Following the method of the text, prove the formulas (1.16),(1.17),(1.18).

3. Explain why formulas (1.5) and (1.6) are true.

4. Calculate the following in terms of radicals.

   (a) $\cos(36°)$ (b) $\cos(18°)$ (c) $\sin(36°)$ (d) $\sin(54°)$ (e) $\sin(72°)$ (f) $\cos(162°)$ (g) $\cos(108°)$ (h) $\sin(126°)$ (i) $\sin(144°)$ (j) $\cos(216°)$ (k) $\cos(288°)$ (l) $\sin(234°)$ (m) $\sin(342°)$ (n) $\csc(18°)$ (o) $\sec(18°)$ (p) $\tan(18°)$ (q) $\csc(18°)$ (r) $\sec(36°)$ (s) $\csc(36°)$ (t) $\tan(36°)$ (u) $\sin(66°)$ (v) $\cos(63°)$ (w) $\sin(99°)$ (x) $\cos(9°)$ (y) $\sin(3°)$ (z) $\cos(3°)$

5. Let $\triangle ABC$ be an isosceles triangle with vertex A: i.e., length AB is the same as length AC. Suppose the measure of the angle at A is $\frac{\pi}{5}$ (i.e., 36°). Suppose $AB = 1$ and let $x = BC$. Let D be a point in side $\overline{AC}$ such that $\overrightarrow{BD}$ bisects $\angle ABC$.

   (a) Prove that $AD = BD$.
   (b) Prove that $BD = BC$.

(c) Explain why $\frac{1}{x} = \frac{x}{1-x}$.

(d) Solve the equation in (c) for x.

(e) Use part (d) to conclude that $\sin(18^\circ) = (-1+\sqrt{5})/4$.

6. Solve the following triangles using the algorithm for solving the ambiguous case. Draw a picture to illustrate the situation.

   (a) $a = 1, A = 40^\circ, b = 5$

   (b) $a = 4, A = 40^\circ, b = 5$

   (c) $a = 7, A = 40^\circ, b = 5$

   (d) $a = 1, A = 130^\circ, b = 6$

   (e) $a = 5, A = 130^\circ, b = 6$

   (f) $a = 8, A = 130^\circ, b = 6$

7. Consider the following step in the algorithm for solving the ambiguous case: "We discard any value for B found where $A + B \geq \pi$." Prove that if $\angle A$ is acute or $a \neq b$ this is the same as discarding any value of B found where $a - b$ and $A - B$ are not both positive, both zero, or both negative. What happens in the other cases?

8. Let A, B, and C be the measures of the angles in a triangle and let a, b, and c be the lengths of the opposite sides. Use the law of sines to transform the quantity $\frac{a-b}{c}$ into an expression involving angles. Then conclude that

$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}. \tag{1.21}$$

9. Use the technique of Exercise 8 to prove that

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}. \tag{1.22}$$

(Equations (1.21) and (1.22) are called Mollweide's equations.)

10. Prove that

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b}. \tag{1.23}$$

when both fractions are defined. (This is the planar *law of tangents*.)

11. Suppose we write $s = (a+b+c)/2$. Use (1.20) to show that

$$\sin(\frac{C}{2}) = \sqrt{\frac{(s-a)(s-b)}{ab}} \quad (1.24)$$

$$\cos(\frac{C}{2}) = \sqrt{\frac{s(s-c)}{ab}}. \quad (1.25)$$

12. Prove that

$$\tan(\frac{C}{2}) = \sqrt{\frac{(s-b)(s-a)}{s(s-c)}}. \quad (1.26)$$

where $s = (a+b+c)/2$.

13. Using Exercise 12, prove that given three positive real numbers a, b, and c such that $a+b > c$, $a+c > b$, and $b+c > a$, there exists a plane triangle whose side lengths are a, b, and c.

14. Using Exercise 12, show that

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad (1.27)$$

where r is the radius of the inscribed circle of the triangle.

15. Using Exercise 14, prove that the area of a triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}. \quad (1.28)$$

(This is Hero's formula.)

16. Suppose that x and y are positive numbers such that $x+y < \pi$. Prove that

$$\frac{\cot(x)+\cot(y)}{2} \geq \cot(\frac{x+y}{2}), \quad (1.29)$$

where equality occurs if and only if $x = y$.

4 Coordinates and vectors

In this section we briefly review ideas from rectangular coordinates and vectors in space which we will need. Every point in space can be written as an ordered triple (x, y, z), where x, y, and z are real numbers. These are the so-called *Cartesian* coordinates of the point.

Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in space. Then the *distance* between them is calculated via the *distance formula*

$$d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Since a sphere of radius r and center (x_1, y_1, z_1) consists of all points (x, y, z) such that $\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} = r$, i.e.,

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = r^2, \tag{1.30}$$

we shall say that (1.30) is the equation of the sphere with center (x_1, y_1, z_1) and radius r. For convenience, most of the time we will assume the sphere has center at the origin $(0, 0, 0)$ so the equation would be

$$x^2+y^2+z^2 = r^2.$$

We will also need the notation of vectors. A *vector* in space may be thought of as an arrow in space pointing from one point to another. The arrow has an *initial* point P_1 and a *terminal* point P_2 and is denoted by $\vec{P_1P_2}$. Vectors may also be expressed with coordinates $\langle x, y, z\rangle$ which may be determined as follows: if P_1 and P_2 have coordinates as given above, then $\vec{P_1P_2}$ has coordinates $\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$ found by subtracting the Cartesian coordinates of P_1 from the Cartesian coordinates of P_2 (terminal coordinates minus initial coordinates). Another vector pointing in the same direction for the same distance as $\vec{P_1P_2}$ will be identified as the same vector as $\vec{P_1P_2}$ and hence has the same coordinates. (See Figure 1.5.)

The *length* of a vector $\mathbf{r}$ with coordinates $\langle x, y, z\rangle$ is defined to be

$$|\mathbf{r}| = \sqrt{x^2+y^2+z^2}. \tag{1.31}$$

The motivation behind this definition is as follows. Suppose a vector $\mathbf{r}$ has initial point $P_1 = (x_1, y_1, z_1)$ and terminal point $P_2 = (x_2, y_2, z_2)$. Then $\mathbf{r}$ has coordinates $\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$ and hence by (1.31) has length $|\mathbf{r}| = \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$. But by the distance formula, this is the distance between P_1 and P_2 in space. So the length of a vector is the length of its associated arrow. The length is also sometimes called the *norm*, the *absolute value*, or the *modulus*.

Two vectors $\mathbf{r}_1 = \vec{P_1Q_1} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{r}_2 = \vec{P_2Q_2} = \langle x_2, y_2, z_2\rangle$ can be *added* to obtain the vector $\mathbf{r}_1+\mathbf{r}_2 = \langle x_1+x_2, y_1+y_2, z_1+z_2\rangle$, the *sum* of $\mathbf{r}_1$ and $\mathbf{r}_2$. The arrow associated with the sum is found as follows: move the arrow for $\mathbf{r_2}$ without changing its length or direction so that its initial point coincides with the terminal point of $\mathbf{r}_1$. Then the vector $\mathbf{r}_1+\mathbf{r}_2$ has an arrow which points from the initial point of $\mathbf{r}_1$ to the terminal point of $\mathbf{r}_2$. (See Figure 1.5.)

A vector $\mathbf{r} = \langle x, y, z\rangle$ may be multiplied by a real number c called a *scalar* to obtain another vector we denote by $c\mathbf{r}$ with coordinates $\langle cx, cy, cz\rangle$. This *scalar multiplication* of a real number with a vector has a geometric interpretation: if $c > 0$, the vector $c\mathbf{r}$ has an arrow with the same initial point and direction as $\mathbf{r}$ but whose length is $c|\mathbf{r}|$, i.e., $c\mathbf{r}$ is c times as long as $\mathbf{r}$. If $c < 0$, $c\mathbf{r}$ is a vector $|c|$ times as long as $\mathbf{r}$ pointing in the opposite direction as $\mathbf{r}$. If we write $\mathbf{i} = \langle 1, 0, 0\rangle$, $\mathbf{j} = \langle 0, 1, 0\rangle$ and $\mathbf{k} = \langle 0, 0, 1\rangle$ then $\langle x, y, z\rangle$ can be written as $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

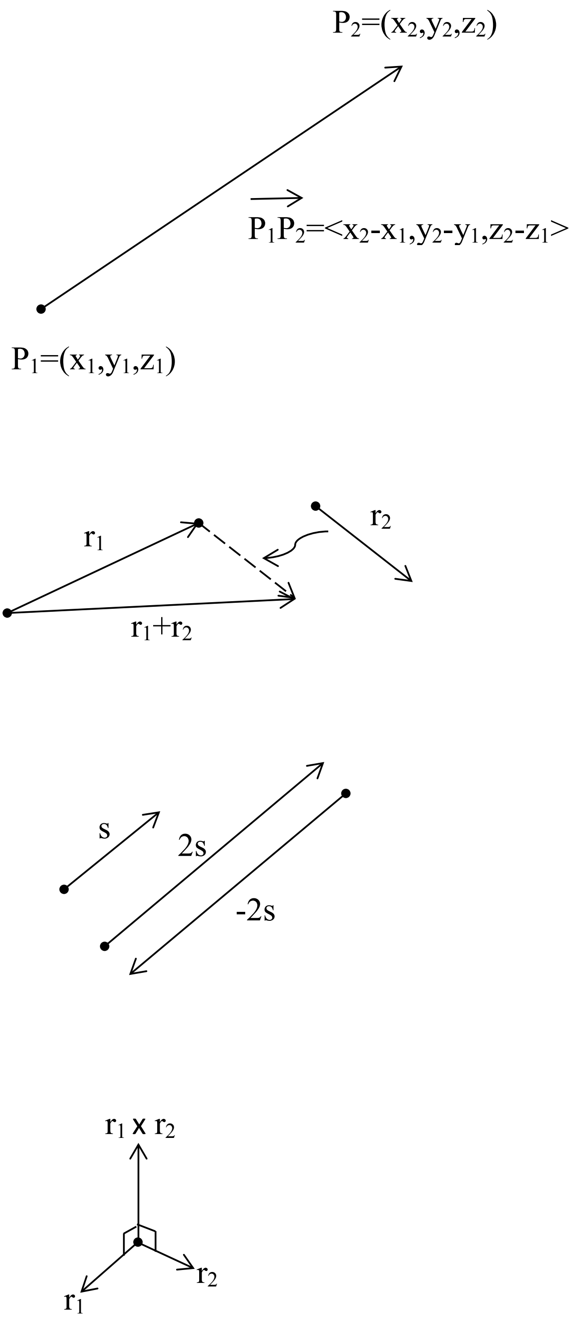

Figure 1.5: Vectors: coordinates, sum, scalar multiplication, and cross product.

The *angle* between two vectors $\vec{OP}$ and $\vec{OQ}$ with the same initial point is the angle $\angle POQ$. If two vectors do not have the same initial point, then we may form an angle between them by moving one without changing its length or direction so that they do have the same initial point.

Another operation with vectors we shall need is called the *dot product* or *scalar product*. Given vector $\mathbf{r_1} = \langle x_1, y_1, z_1\rangle$ and $\mathbf{r}_2 = \langle x_2, y_2, z_2\rangle$ the dot product $\mathbf{r}_1 \cdot \mathbf{r}_2$ is the real number

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1x_2 + y_1y_2 + z_1z_2. \tag{1.32}$$

For the dot product, we recall the following key theorem.

Theorem 4.1 *If θ is the angle between the vectors* $\mathbf{r}$ *and* $\mathbf{s}$ *then*

$$\mathbf{r} \cdot \mathbf{s} = |\mathbf{r}||\mathbf{s}| \cos(\theta). \tag{1.33}$$

Note that the angle between the vectors is a right angle when their dot product is zero; in this case we say that the vectors are perpendicular.

We also have an operation called the *cross product* or *vector product*. If $\mathbf{r_1}$ and $\mathbf{r_2}$ are as above then

$$\begin{aligned} \mathbf{r_1} \times \mathbf{r_2} &= \langle y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1\rangle & (1.34)\\ &= (y_1z_2 - y_2z_1)\mathbf{i} + (x_2z_1 - x_1z_2)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}. & (1.35) \end{aligned}$$

Note that the cross product can be expressed with the determinant

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

(See Figure 1.5.) One key property of the cross product $\mathbf{r} \equiv \mathbf{r_1} \times \mathbf{r_2}$ is that it is perpendicular to both of $\mathbf{r_1}$ and $\mathbf{r_2}$, and the direction of $\mathbf{r}$ is found from $\mathbf{r_1}$ and $\mathbf{r_2}$ via the right hand rule. That is, if one turns $\mathbf{r_1}$ into $\mathbf{r_2}$ with the right hand, the thumb will point in the direction of $\mathbf{r_1} \times \mathbf{r_2}$. We also have

$$|\mathbf{r_1} \times \mathbf{r_2}| = |\mathbf{r_1}||\mathbf{r_2}| \sin(\theta), \tag{1.36}$$

where θ is the angle between $\mathbf{r_1}$ and $\mathbf{r_2}$. A consequence of this is that $|\mathbf{r_1} \times \mathbf{r_2}|$ is the area of the parallelogram determined by the vectors $\mathbf{r_1}$ and $\mathbf{r_2}$.

If $\mathbf{r_3} = (x_3, y_3, z_3)$ then properties of determinants (or algebraic manipulation) show that $\mathbf{r_1} \cdot \mathbf{r_2} \times \mathbf{r_3} = \mathbf{r_2} \cdot \mathbf{r_3} \times \mathbf{r_1} = \mathbf{r_3} \cdot \mathbf{r_1} \times \mathbf{r_2}$. The absolute value of each of these gives the volume of the parallelepiped with three edges $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$. (See Figure 1.6.) This is because $|\mathbf{r_1} \cdot \mathbf{r_2} \times \mathbf{r_3}| = |\mathbf{r_1}||\mathbf{r_2} \times \mathbf{r_3}|| \cos(\theta)|$, where θ is the angle between $\mathbf{r_1}$ and the normal to the plane containing $\mathbf{r_2}$ and $\mathbf{r_3}$. But $|\mathbf{r_1}|| \cos(\theta)|$ is the length of the perpendicular from the terminal point of $\mathbf{r_1}$ to the plane containing $\mathbf{r_2}$ and $\mathbf{r_3}$. This is the height of the parallelepiped. Since $|\mathbf{r_2} \times \mathbf{r_3}|$ is the area of a base parallelogram, we obtain the volume of the appropriate parallelepiped.

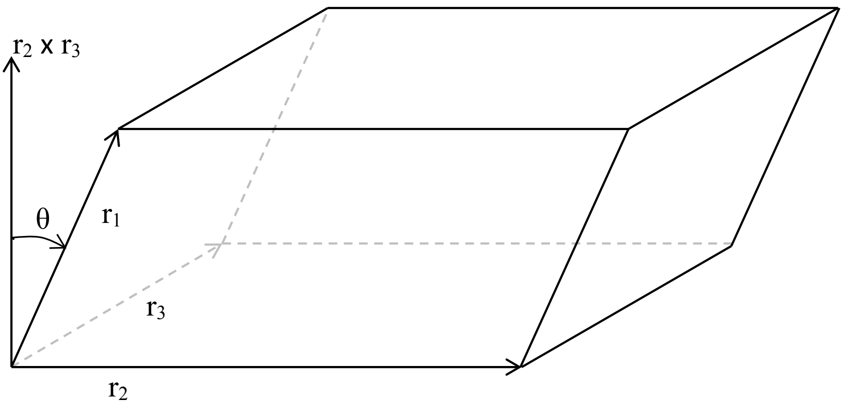

Figure 1.6: The solid parallelepiped formed by $\mathbf{r_1}, \mathbf{r_2}$, and $\mathbf{r_3}$.

Exercises §4

1. Use vectors to prove Lemma 1.4 with vectors as follows. Choose vectors pointing along the four line segments and prove that appropriately chosen pairs of cross products are equal.